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Certain sufficient conditions for strongly starlikeness and convexity

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Abstract
The object of the present paper is to derive some sufficient conditions for strongly starlikeness and convexity.

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1 Introduction

Let $A(n)$ $(n \geq 2)$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. We write $A = A(2)$. Let S^* and K be the subclasses of $A(n)$ consisting of all starlike functions $f(z)$ in U and of all convex functions $f(z)$ in U, respectively.

If $f(z) \in A(n)$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\gamma \quad (z \in U) \tag{1.2}$$

for some γ $(0 < \gamma \leq 1)$, then $f(z)$ is said to be strongly starlike of order γ in U, and denoted by $f(z) \in \widetilde{S}^*(\gamma)$. If $f(z) \in A(n)$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\gamma \quad (z \in U) \tag{1.3}$$

for some γ $(0 < \gamma \leq 1)$, then we say that $f(z)$ is strongly convex of order γ in U, and we denote by $\widetilde{K}(\gamma)$ the class of all such functions. It is obvious that $f(z) \in A(n)$ belongs to $\widetilde{K}(\gamma)$ if and only if $zf'(z) \in \widetilde{S}^*(\gamma)$. Further, we note that $\widetilde{S}^*(1) = S^*$ and $\widetilde{K}(1) = K$.

The strongly starlike and convex functions have been extensively studied by several authors (see, *e.g.*, [1–11]). The object of the present paper is to derive some sufficient conditions for strongly starlikeness and strongly convexity. Some previous results are extended.

For our purpose, we have to recall here the following results.

Lemma 1 (see [5]) *Let a function* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* U *and* $p(z) \neq 0$ $(z \in U)$. *If there exists a point* $z_0 \in U$ *such that*

$$\left|\arg p(z)\right| < \frac{\pi}{2}\beta \quad \left(|z| < |z_0|\right)$$

and

$$\left|\arg p(z_0)\right| = \frac{\pi}{2}\beta \quad (0 < \beta \leq 1),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = \frac{\pi}{2}\beta\right),$$
$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \left(\text{when } \arg p(z_0) = -\frac{\pi}{2}\beta\right),$$

and $p(z_0)^{1/\beta} = \pm ia$ $(a > 0)$.

Lemma 2 (see [4]) *If* $f(z) \in A$ *satisfies*

$$\left|f'(z) - 1\right| < \frac{\sqrt{20}}{5} \quad (z \in U),$$

then $f(z) \in S^*$.

2 Starlikeness and convexity

Our first result is contained in the following.

Theorem 1 *Let* $0 < \alpha \leq \frac{1}{1+\frac{2}{\pi}\int_0^1 \sin^{-1}(\frac{2\rho}{1+\rho^2})\,d\rho}$. *If* $f(z) \in A(n)$ $(n \geq 2)$ *satisfies*

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha \quad (z \in U), \tag{2.1}$$

then $f(z) \in \widetilde{S}^*(\beta)$, *where*

$$\beta = \left(1 + \frac{2}{\pi}\int_0^1 \sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho\right)\alpha.$$

Proof Note that

$$\arg f'(z) = \arg\left(\frac{zf'(z)}{f(z)}\right) + \arg\left(\frac{f(z)}{z}\right)$$

and

$$\begin{aligned}\arg\left(\frac{f(z)}{z}\right) &= \arg\left(\frac{1}{z}\int_0^z f'(t)\,dt\right)\\ &= \arg\left(\frac{1}{z}\int_0^r f'\left(\rho e^{i\theta}\right)e^{i\theta}\,d\rho\right) \quad \left(z = re^{i\theta}, t = \rho e^{i\theta}\right)\\ &= \arg\left(\int_0^r f'\left(\rho e^{i\theta}\right)d\rho\right). \end{aligned} \tag{2.2}$$

Let

$$0 = \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_{m-1} < \rho_m = r,$$

and

$$\rho_j - \rho_{j-1} = \delta_m \quad (j = 1, 2, \ldots, m).$$

Then, by using (2.2), we have that

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| = \left|\arg\left(\lim_{m\to\infty}\sum_{j=1}^m \delta_m f'\left(\rho_j e^{i\theta}\right)\right)\right| \le \lim_{m\to\infty}\sum_{j=1}^m \delta_m \left|\arg f'\left(\rho_j e^{i\theta}\right)\right|.$$

Since the condition (2.1) implies that

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in U),$$

we obtain that

$$\begin{aligned}\left|\arg\left(\frac{f(z)}{z}\right)\right| &\le \lim_{m\to\infty}\sum_{j=1}^m \delta_m \left|\arg\left(\frac{1+\rho_j e^{i\theta}}{1-\rho_j e^{i\theta}}\right)^\alpha\right| < \alpha\int_0^r \sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho\\ &< \alpha\int_0^1 \sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho\\ &= \frac{\pi}{2}\alpha\left(\frac{2}{\pi}\int_0^1 \sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho\right). \end{aligned} \tag{2.3}$$

Furthermore, since

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| - \left|\arg\left(\frac{f(z)}{z}\right)\right| \le \left|\arg f'(z)\right| \quad (z \in U),$$

we conclude from (2.1) and (2.3) that

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &\le \left|\arg f'(z)\right| + \left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\alpha + \frac{\pi}{2}\alpha\left(\frac{2}{\pi}\int_0^1 \sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho\right)\\ &= \frac{\pi}{2}\beta, \end{aligned}$$

which shows that $f(z) \in \widetilde{S}^*(\beta)$. □

Theorem 2 *Let $0 < \alpha \leq 1$. If $f(z) \in A(n)$ $(n \geq 2)$ satisfies*

$$\left|\arg\bigl(f'(z) + zf''(z)\bigr)\right| < \frac{\pi}{2}\alpha\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right) \quad (z \in U), \tag{2.4}$$

then $f(z) \in \widetilde{K}(\alpha)$, where $\alpha_1 = 0.3834\ldots$ is the root of the equation

$$2\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1 = 1.$$

Proof Note that

$$\arg\bigl(f'(z) + zf''(z)\bigr) = \arg f'(z) + \arg\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

If there exists a point $z_0 \in U$ such that

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha_1\alpha \quad \bigl(|z| < |z_0|\bigr)$$

and

$$\left|\arg f'(z_0)\right| = \frac{\pi}{2}\alpha_1\alpha,$$

then by Lemma 1, we have

$$\frac{z_0 f''(z_0)}{f'(z_0)} = i\alpha_1\alpha k.$$

Therefore, if $\arg f'(z_0) = \frac{\pi}{2}\alpha_1\alpha$, then we have

$$\begin{aligned}\arg f'(z_0) + \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \frac{\pi}{2}\alpha_1\alpha + \arg(1 + i\alpha_1\alpha k)\\ &= \frac{\pi}{2}\alpha_1\alpha + \tan^{-1}(\alpha_1\alpha k)\\ &\geq \frac{\pi}{2}\alpha_1\alpha + \alpha\tan^{-1}\alpha_1\\ &= \frac{\pi}{2}\alpha\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right),\end{aligned}$$

which contradicts (2.4). If $\arg f'(z_0) = -\frac{\pi}{2}\alpha_1\alpha$, then applying the same method for the previous case, we also have

$$\arg f'(z_0) + \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \leq -\frac{\pi}{2}\alpha\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right),$$

which contradicts (2.4). Therefore, there exists no $z_0 \in U$ such that $|\arg f'(z_0)| = \frac{\pi}{2}\alpha_1\alpha$. This implies that

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha_1\alpha \quad (z \in U).$$

Furthermore, since

$$\begin{aligned}\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| - \left|\arg f'(z)\right| &\leq \left|\arg\bigl(f'(z)+zf''(z)\bigr)\right| \\ &< \frac{\pi}{2}\alpha\left(\alpha_1+\frac{2}{\pi}\tan^{-1}\alpha_1\right) \quad (z\in U),\end{aligned}$$

we conclude that

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha\left(2\alpha_1+\frac{2}{\pi}\tan^{-1}\alpha_1\right) = \frac{\pi}{2}\alpha \quad (z\in U),$$

which shows that $f(z)\in\widetilde{K}(\alpha)$. □

Theorem 3 *If* $f(z)=z+a_nz^n+\cdots\in A(n)$ $(n\geq 2)$ *satisfies*

$$\left|f^{(n)}(z)\right| \leq \frac{\sqrt{20}}{5} \quad (z\in U), \tag{2.5}$$

then $f(z)\in S^*$.

Proof From (2.5), one can see that

$$\begin{aligned}\left|f^{(n-1)}(z)\right| &= \left|\int_0^z f^{(n)}(t)\,dt\right| \\ &\leq \int_0^{|z|}\left|f^{(n)}(t)\right|\left|dt\right| \\ &\leq \frac{\sqrt{20}}{5}|z| < \frac{\sqrt{20}}{5} \quad (z\in U), \\ &\cdots \\ \left|f''(z)\right| &\leq \frac{\sqrt{20}}{5} \quad (z\in U).\end{aligned}$$

Noting that

$$\begin{aligned}\left|f'(z)-1\right| &= \left|\int_0^z f''(t)\,dt\right| \\ &\leq \int_0^{|z|}\left|f''(t)\right|\left|dt\right| \\ &\leq \frac{\sqrt{20}}{5}|z| < \frac{\sqrt{20}}{5} \quad (z\in U).\end{aligned}$$

By Lemma 2, we have $f(z)\in S^*$. □

Theorem 4 *If* $f(z)=z+a_nz^n+\cdots\in A(n)$ $(n\geq 2)$ *satisfies*

$$\left|f^{(n)}(z)\right| \leq \frac{\sqrt{5}}{5} \quad (z\in U), \tag{2.6}$$

then $f(z)\in K$.

Proof By using the same method as in the proof of Theorem 3, we have

$$\left|f''(z)\right| \le \frac{\sqrt{5}}{5} \quad (z \in U).$$

It follows that

$$\begin{aligned}\left|\left(zf'(z)\right)' - 1\right| &= \left|f'(z) + zf''(z) - 1\right| \\ &\le \left|f'(z) - 1\right| + \left|zf''(z)\right| \\ &\le \left|\int_0^z f''(t)\,dt\right| + \left|zf''(z)\right| \\ &\le \int_0^{|z|} \left|f''(t)\right| |\,dt| + \frac{\sqrt{5}}{5}|z| \\ &\le \frac{2\sqrt{5}}{5}|z| < \frac{\sqrt{20}}{5} \quad (z \in U).\end{aligned}$$

Therefore, using Lemma 2, we see that $zf'(z) \in S^*$, or $f(z) \in K$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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