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The contraction-proximal point algorithm with square-summable errors

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Abstract
In this paper, we study the contraction-proximal point algorithm for approximating a zero of a maximal monotone mapping. The norm convergence of such an algorithm has been established under two new conditions. This extends a recent result obtained by Ceng, Wu and Yao to a more general case.
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1 Introduction

We consider the problem of finding $\hat{x} \in \mathcal{H}$ so that

$$0 \in A\hat{x},$$

where $\mathcal{H}$ is a Hilbert space and $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a given maximal monotone mapping. This problem is essential due to its various application in some concrete disciplines, including convex programming and variational inequalities. A classical way to solve such a problem is the proximal point algorithm (PPA) [1]. For any initial guess $x_0 \in \mathcal{H}$, the PPA generates an iterative sequence as

$$x_{n+1} = J_{c_n}(x_n + e_n), \tag{1}$$

where J_{c_n} stands for the resolvent of A and (e_n) is the error sequence. In general, the following accuracy criterion on the error sequence:

$$\|e_n\| \le \epsilon_n \quad \text{with} \sum_{n=0}^{\infty} \epsilon_n < \infty, \tag{I}$$

is needed to ensure the convergence of PPA. In [1], Rockefeller also presented another accuracy criterion on the error sequence:

$$\|e_n\| \le \eta_n \|\tilde{x}_n - x_n\| \quad \text{with} \sum_{n=0}^{\infty} \eta_n < \infty,$$

where

$$\tilde{x}_n = J_{c_n}(x_n + e_n).$$

This criterion was then improved by Han and He [2] as

$$\|e_n\| \le \eta_n \|\tilde{x}_n - x_n\| \quad \text{with} \sum_{n=0}^{\infty} \eta_n^2 < \infty. \tag{II}$$

It is well known that the PPA does not necessarily converge strongly [3]. Then how to modify the PPA so that the strong convergence is guaranteed attracts serious attention of many researchers (see, *e.g.*, [4–8]). In particular, one method for doing this has the following scheme:

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) J_{c_n}(x_n + e_n), \tag{2}$$

where $u \in \mathcal{H}$ is fixed and (λ_n) is a real sequence. This algorithm, introduced independently by Xu [8] and Kamimura-Takahashi [5], is known as the contraction-proximal point algorithm (CPPA) [9], which is indeed a combination of Halpern's iteration and the PPA. There are various conditions that ensure the norm convergence of the CPPA with criterion (I) (*cf.* [7, 10–12]) and the weakest one so far may be the following [13]:

(i) $c_n \ge c > 0$;
(ii) $\lim_n \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\|e_n\| \le \eta_n$, $\sum_{n=0}^{\infty} \eta_n < \infty$.

Let us now turn our attention to the CPPA under criterion (II). In this situation, Ceng, Wu and Yao [14] obtained the norm convergence under the following conditions:

(i) $\lim_n c_n = \infty$;
(ii) $\lim_n \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\|e_n\| \le \eta_n \|\tilde{x}_n - x_n\|$ with $\sum_{n=0}^{\infty} \eta_n^2 < \infty$.

In the hypothesis mentioned above, the sequence (c_n) is assumed to tend to infinity, so it is natural to ask whether the norm convergence is still guaranteed for bounded (c_n), especially for constant sequence. In the present paper, we shall answer this question affirmatively and relax condition $\lim_n c_n = \infty$ to a more general case:

$$c_n \ge c > 0,$$

that is, we only need to assume the sequence (c_n) is bounded below away from zero. The paper is organized as follows. In Section 2, we prove two useful lemmas that are very useful for proving the boundedness of the iteration. In Section 3, we establish norm convergence of the CPPA under two different conditions. As a result, we extend the corresponding result obtained in [14].

2 Some lemmas

We denote by '$\to$' strong convergence, and '$\rightharpoonup$' weak convergence. An operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is called monotone if

$$\langle u - v, x - y \rangle \ge 0,$$

for any $u \in Ax$, $v \in Ay$; maximal monotone if its graph

$$\mathcal{G}(A) = \{(x, y) : x \in \mathcal{D}(A), y \in Ax\}$$

is not properly contained in the graph of any other monotone operator.

Let C be a nonempty, closed and convex subset of $\mathcal{H}$. We use P_C to denote the projection from $\mathcal{H}$ onto C; namely, for $x \in \mathcal{H}$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is well known that $P_C x$ is characterized by

$$\langle x - P_C x, z - P_C x \rangle \leq 0, \quad \forall z \in C. \tag{3}$$

A mapping $T : \mathcal{H} \to \mathcal{H}$ is called *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in \mathcal{H}$. Here and hereafter, we denote by

$$J_c = (I + cA)^{-1}$$

the resolvent of A, where $c > 0$ and I is the identity operator. The zero set of A is denoted by $S := \{x \in \mathcal{D}(A) : 0 \in Ax\}$. The resolvent operator has the following properties (see [15]).

Lemma 1 *Let A be a maximal monotone operator. Then*
- (i) $\mathcal{D}(J_c) = \mathcal{H}$;
- (ii) *J_c is single-valued and firmly nonexpansive*;
- (iii) $\mathrm{Fix}(J_c) = S$, *where* $\mathrm{Fix}(J_c)$ *denotes the fixed point set of* J_c;
- (iv) *its graph $\mathcal{G}(A)$ is weak-to-strong closed in $\mathcal{H} \times \mathcal{H}$.*

Since J_c is firmly nonexpansive, this implies that

$$\|J_c x - z\|^2 \leq \|x - z\|^2 - \|(I - J_c)x\|^2 \tag{4}$$

for all $x \in \mathcal{H}$ and all $z \in S$. In what follows, we present two lemmas that are very useful for proving the boundedness of the iterative sequence.

Lemma 2 *Given $\beta > 0$, let (s_n) be a nonnegative real sequence satisfying*

$$s_{n+1} \leq (1 - \lambda_n)(1 + \epsilon_n)s_n + \lambda_n \beta,$$

where $(\lambda_n) \subset (0, 1)$ and $(\epsilon_n) \in \ell_1$ are real sequences. Then (s_n) is bounded; more precisely,

$$s_n \leq \max\{\beta, s_0\} \exp\left(\sum_{n=0}^{\infty} \epsilon_n\right) < \infty.$$

Proof We first show the following estimates:

$$s_{n+1} \le \max\{\beta, s_0\} \prod_{k=0}^{n} (1+\epsilon_k), \quad \forall n \ge 0. \tag{5}$$

For $n = 0$, we have

$$\begin{aligned} s_1 &\le (1-\lambda_0)(1+\epsilon_0)s_0 + \lambda_0\beta \\ &\le (1+\epsilon_0)\big[(1-\lambda_0)s_0 + \lambda_0\beta\big] \\ &\le \max\{\beta, s_0\}(1+\epsilon_0). \end{aligned}$$

Assume $s_n \le \max\{\beta, s_0\}\prod_{k=0}^{n-1}(1+\epsilon_k)$. Since $\max\{\beta, s_0\}\prod_{k=0}^{n-1}(1+\epsilon_k) \ge \beta$, we have

$$\begin{aligned} s_{n+1} &\le (1+\epsilon_n)(1-\lambda_n)s_n + \lambda_n\beta \\ &\le (1+\epsilon_n)\big[(1-\lambda_n)s_n + \lambda_n\beta\big] \\ &\le (1+\epsilon_n)\max\{s_0, \beta\}\prod_{k=0}^{n-1}(1+\epsilon_k) \\ &= \max\{\beta, s_0\}\prod_{k=0}^{n}(1+\epsilon_k). \end{aligned}$$

We thus verify inequality (5) by induction. Hence,

$$\begin{aligned} s_{n+1} &\le \max\{\beta, s_0\}\prod_{k=0}^{n}(1+\epsilon_k) \\ &= \max\{\beta, s_0\}\exp\left(\sum_{k=0}^{n}\ln(1+\epsilon_k)\right) \\ &\le \max\{\beta, s_0\}\exp\left(\sum_{k=0}^{\infty}\epsilon_k\right) < \infty, \end{aligned}$$

where the last inequality follows from the basic inequality: $\ln(1+x) < x$ for all $x > 0$. □

Lemma 3 *Given $\beta > 0$, let (s_n) be a nonnegative real sequence satisfying*

$$s_{n+1} \le (1-\lambda_n)(1+\epsilon_n)s_n + \lambda_n\beta,$$

where $(\lambda_n) \subset (0,1)$ and $(\epsilon_n) \subseteq [0,\infty)$ are real sequences. If $2\epsilon_n(1-\lambda_n) \le \lambda_n$, then (s_n) is bounded; more precisely, $s_n \le \max\{2\beta, s_0\} < \infty$.

Proof Let $\tau_n = \lambda_n - \epsilon_n(1-\lambda_n)$. Then $\tau_n \in (0,1)$. It follows that

$$\begin{aligned} s_{n+1} &\le (1+\epsilon_n)(1-\lambda_n)s_n + \lambda_n\beta \\ &= (1-\tau_n)s_n + \tau_n(\lambda_n\beta/\tau_n) \\ &\le \max\{\lambda_n\beta/\tau_n, s_n\}. \end{aligned}$$

Since $2\epsilon_n(1-\lambda_n) \le \lambda_n$, we have

$$\frac{\lambda_n}{\tau_n} = \frac{\lambda_n}{\lambda_n - \epsilon_n(1-\lambda_n)} \le 2,$$

which implies that

$$s_{n+1} \le \max\{2\beta, s_n\}.$$

By induction, we can show the result as desired. □

We end this section by two useful lemmas. The first one is due to Maingé [16] and the second one is due to Xu [8].

Lemma 4 *Let (s_n) be a real sequence that does not decrease at infinity, in the sense that there exists a subsequence (s_{n_k}) so that*

$$s_{n_k} \le s_{n_k+1} \quad \textit{for all } k \ge 0.$$

For every $n > n_0$ define an integer sequence $(\tau(n))$ as

$$\tau(n) = \max\{n_0 \le k \le n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n > n_0$

$$\max(s_{\tau(n)}, s_n) \le s_{\tau(n)+1}. \tag{6}$$

Lemma 5 *Let $\{s_n\}$, $\{c_n\} \subset \mathbb{R}^+$, $\{\lambda_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$s_{n+1} \le (1-\lambda_n)s_n + b_n + c_n \quad \textit{for all } n \ge 0.$$

If $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\overline{\lim}_n b_n/\lambda_n \le 0$, then $\lim_n s_n = 0$.

3 Convergence analysis

In what follows, we assume that A is a maximal monotone mapping and its zero set S is nonempty. To establish the convergence, we need the following lemma, which is indeed proved in [2]. We present here a different proof that is mainly based on property of firmly nonexpansive mappings.

Lemma 6 *Let $\eta \in (0,1/2)$, $x, e \in \mathcal{H}$ and $\tilde{x} := J_c(x+e)$. If $\|e\| \le \eta\|x-\tilde{x}\|$, then*

$$\|\tilde{x} - z\|^2 \le \bigl(1 + (2\eta)^2\bigr)\|x - z\|^2 - \frac{1}{2}\|\tilde{x} - x\|^2, \quad \forall z \in S. \tag{7}$$

Proof Since $z \in \operatorname{Fix}(J_c)$, it follows from (4) that

$$\begin{aligned}
\|\tilde{x} - z\|^2 &\le \|x + e - z\|^2 - \bigl\|x + e - J_c(x+e)\bigr\|^2 \\
&= \bigl\|(x-z) + e\bigr\|^2 - \bigl\|(x-\tilde{x}) + e\bigr\|^2 \\
&= \|x - z\|^2 + 2\langle \tilde{x} - z, e\rangle - \|\tilde{x} - x\|^2.
\end{aligned} \tag{8}$$

By using inequality $2\langle a,b\rangle \le 2\eta^2\|a\|^2 + \|b\|^2/2\eta^2$, we have

$$2\langle \tilde{x}-z,e\rangle \le 2\eta^2\|\tilde{x}-z\|^2 + \frac{1}{2\eta^2}\|e\|^2.$$

Subsisting this into (8) and noting $\|e\| \le \eta\|x-\tilde{x}\|$, we see that

$$\|\tilde{x}-z\|^2 \le \|x-z\|^2 + 2\eta^2\|\tilde{x}-z\|^2 - \frac{1}{2}\|\tilde{x}-x\|^2,$$

from which it follows that

$$\|\tilde{x}-z\|^2 \le \left(1+\frac{2\eta^2}{1-2\eta^2}\right)\|x-z\|^2 - \frac{1}{2(1-2\eta^2)}\|\tilde{x}-x\|^2.$$

Consequently, the desired inequality (7) follows from the fact $\eta \in (0,1/2)$. □

We now are ready to prove our main results.

Theorem 1 *For any $x_0 \in \mathcal{H}$, the sequence (x_n) generated by*

$$\begin{cases} \tilde{x}_n = J_{c_n}(x_n + e_n), \\ x_{n+1} = \lambda_n u + (1-\lambda_n)\tilde{x}_n, \end{cases} \tag{9}$$

converges strongly to $P_S(u)$, provided that

(i) $c_n \ge c > 0$;
(ii) $\lim_n \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\|e_n\| \le \eta_n\|\tilde{x}_n - x_n\|$, $\sum_{n=0}^{\infty} \eta_n^2 < \infty$.

Proof Let $z = P_S(u)$. By our hypothesis, we may assume without loss of generality that $\eta_n \in (0,1/2)$. Then by Lemma 6, we have

$$\|\tilde{x}_n - z\|^2 \le (1+\epsilon_n)\|x_n - z\|^2 - \frac{1}{2}\|\tilde{x}_n - x_n\|^2, \tag{10}$$

where $\epsilon_n := (2\eta_n)^2$ satisfying $\sum_{n=0}^{\infty} \epsilon_n < \infty$. It then follows from (9) that

$$\begin{aligned} \|x_{n+1}-z\|^2 &= \big\|(1-\lambda_n)(\tilde{x}_n - z)^2 + \lambda_n(u-z)\big\|^2 \\ &\le (1-\lambda_n)\|\tilde{x}_n - z\|^2 + \lambda_n\|u-z\|^2, \end{aligned}$$

which together with (10) yields

$$\|x_{n+1}-z\|^2 \le (1-\lambda_n)(1+\epsilon_n)\|x_n - z\|^2 + \lambda_n\|u-z\|^2.$$

Applying Lemma 2 to the last inequality, we conclude that (x_n) is bounded.

It follows from the subdifferential inequality that

$$\begin{aligned} \|x_{n+1}-z\|^2 &= \big\|(1-\lambda_n)(\tilde{x}_n - z) + \lambda_n(u-z)\big\|^2 \\ &\le (1-\lambda_n)\|\tilde{x}_n - z\|^2 + 2\lambda_n\langle u-z, x_{n+1}-z\rangle. \end{aligned}$$

Combining this with (10) yields

$$\|x_{n+1}-z\|^2 \le (1-\lambda_n)\|x_n-z\|^2 - \frac{1-\lambda_n}{2}\|\tilde{x}_n - x_n\|^2 + 2\lambda_n\langle u-z, x_{n+1}-z\rangle + M\epsilon_n, \tag{11}$$

where $M > 0$ is a sufficiently large number. Since $(\epsilon_n) \in \ell_1$, we assume that

$$s := \lim_{n\to\infty}\sum_{k=0}^{n}\epsilon_k < \infty$$

and define $t_n := s - \sum_{k=0}^{n-1}\epsilon_k$. Setting $s_n = \|x_n - z\|^2 + Mt_n$, we rewrite (11) as

$$s_{n+1} - s_n + \lambda_n\|x_n-z\|^2 + \frac{1-\lambda_n}{2}\|\tilde{x}_n - x_n\|^2 \le 2\lambda_n\langle u-z, x_{n+1}-z\rangle. \tag{12}$$

It is obvious that $s_n \to 0 \Leftrightarrow \|x_n - z\| \to 0$.

We next consider two possible cases on the sequence (s_n).

Case 1. (s_n) is eventually decreasing (*i.e.*, there exists $N \ge 0$ such that (s_n) is decreasing for $n \ge N$). In this case, (s_n) must be convergent, and from (12) it follows

$$\frac{1-\lambda_n}{2}\|\tilde{x}_n - x_n\|^2 \le (s_n - s_{n+1}) + M\lambda_n \to 0,$$

from which we have $\|\tilde{x}_n - x_n\| \to 0$. Extract a subsequence (x_{n_k}) from (x_n) so that (x_{n_k}) converges weakly to $\hat{x}$ and

$$\overline{\lim_{n\to\infty}}\langle u-z, x_{n+1}-z\rangle = \lim_{k\to\infty}\langle u-z, x_{n_k}-z\rangle.$$

By noting the fact that $\tilde{x}_n = J_n(x_n + e_n)$, this implies

$$0 \leftarrow \frac{x_{n_k} + e_{n_k} - \tilde{x}_{n_k}}{c_{n_k}} \in A(\tilde{x}_{n_k})$$

and $\tilde{x}_{n_k} = x_{n_k} + (\tilde{x}_{n_k} - x_{n_k}) \rightharpoonup \hat{x}$. Hence, the weak-to-strong closedness of $\mathcal{G}(A)$ implies $0 \in A(\hat{x})$, *i.e.*, $\hat{x} \in S$. Consequently, we have

$$\overline{\lim_{n\to\infty}}\langle u-z, x_{n+1}-z\rangle = \langle u-z, \hat{x}-z\rangle \le 0,$$

where the inequality follows from (3). Again it follows from (12) that

$$\|x_{n+1}-z\|^2 \le (1-\lambda_n)\|x_n-z\|^2 + 2\lambda_n\langle u-z, x_{n+1}-z\rangle + M\epsilon_n.$$

By using Lemma 5, we conclude that $\|x_n - z\| \to 0$.

Case 2. (s_n) is not eventually decreasing. Hence, we can find a subsequence (s_{n_k}) so that $s_{n_k} \le s_{n_k+1}$ for all $k \ge 0$. In this case, we may define an integer sequence $(\tau(n))$ as in Lemma 4. Since $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$, it follows again from (12) that

$$\frac{1-\lambda_{\tau(n)}}{2}\|\tilde{x}_{\tau(n)} - x_{\tau(n)}\|^2 \le M\lambda_{\tau(n)} \to 0,$$

so that $\|\tilde{x}_{\tau(n)} - x_{\tau(n)}\| \to 0$ as $n \to \infty$. Analogously,

$$\overline{\lim_{n\to\infty}} \langle u - z, x_{\tau(n)} - z\rangle \le 0. \tag{13}$$

On the other hand, we deduce from (9) that

$$\|x_{\tau(n)} - x_{\tau(n)+1}\| \le \lambda_{\tau(n)}\|u - x_{\tau(n)}\| + \|\tilde{x}_{\tau(n)} - x_{\tau(n)}\| \to 0,$$

which together with (13) gets

$$\overline{\lim_{n\to\infty}} \langle u - z, x_{\tau(n)+1} - z\rangle \le 0. \tag{14}$$

Noting $s_{\tau(n)+1} - s_{\tau(n)} \ge 0$ and dividing by $\lambda_{\tau(n)}$ in (12), we arrive at

$$\|x_{\tau(n)} - z\|^2 \le 2\langle u - z, x_{\tau(n)+1} - z\rangle$$

for all $n > n_0$, which together with (14) yields

$$\overline{\lim_{n\to\infty}} \|x_{\tau(n)} - z\| \le 0.$$

In view of (6), we have

$$\|x_n - z\|^2 + Mt_n \le \|x_{\tau(n)+1} - z\|^2 + Mt_{\tau(n)+1}.$$

Since $\|x_{\tau(n)} - z\| \to 0$ and $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$ implies $\|x_{\tau(n)+1} - z\| \to 0$, this together with the fact $t_n \to 0$ immediately yields $x_n \to z$. □

For criterion (I), Boikanyo and Morosanu [10] introduced a new condition:

$$\|e_n\| \le \eta_n \quad \text{with } \lim_{n\to\infty} \frac{\eta_n}{\lambda_n} = 0$$

to ensure the convergence of the CPPA. In the following theorem, we shall present a similar condition under the accuracy criterion (II).

Theorem 2 *For any $x_0 \in \mathcal{H}$, the sequence (x_n) generated by*

$$\begin{cases} \tilde{x}_n = J_{c_n}(x_n + e_n), \\ x_{n+1} = \lambda_n u + (1 - \lambda_n)\tilde{x}_n, \end{cases} \tag{15}$$

converges strongly to $P_S(u)$, provided that

(i) $c_n \ge c > 0$;
(ii) $\lim_n \lambda_n = 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) $\|e_n\| \le \eta_n\|\tilde{x}_n - x_n\|$, $\lim_n \eta_n^2/\lambda_n = 0$.

Proof Let $z = P_S(u)$. Similarly, we have

$$\|x_{n+1} - z\|^2 \le (1 - \lambda_n)(1 + \epsilon_n)\|x_n - z\|^2 + \lambda_n\|u - z\|^2, \tag{16}$$

where $\epsilon_n := (2\eta_n)^2$ satisfying $\epsilon_n/\lambda_n \to 0$, so we assume without loss of generality that $2\epsilon_n(1-\lambda_n) \le \lambda_n$. Applying Lemma 3, we conclude that (x_n) is bounded.

From inequality (11), we also obtain

$$s_{n+1} - s_n + \lambda_n s_n + \frac{1-\lambda_n}{2}\|\tilde{x}_n - x_n\|^2 \le 2\lambda_n\langle u - z, x_{n+1} - z\rangle + s_n\epsilon_n, \tag{17}$$

where we define $s_n := \|x_n - z\|^2$.

To show $s_n \to 0$, we consider two possible cases for (s_n).

Case 1. (s_n) is eventually decreasing (*i.e.*, there exists $N \ge 0$ such that (s_n) is decreasing for $n \ge N$). In this case, (s_n) must be convergent, and from (12) it follows

$$\frac{1-\lambda_n}{2}\|\tilde{x}_n - x_n\|^2 \le (s_n - s_{n+1}) + M(\epsilon_n + \lambda_n) \to 0,$$

where $M > 0$ a sufficiently large number. Analogous to the previous theorem,

$$\overline{\lim_{n\to\infty}}\,\langle u - z, x_{n+1} - z\rangle \le 0.$$

Rearranging terms in (17) yields that

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n M\bigl(2\langle u - z, x_{n+1} - z\rangle + \epsilon_n/\lambda_n\bigr).$$

We note that by our hypothesis ϵ_n/λ_n goes to zero, and thus apply Lemma 5 to the previous inequality to conclude that $s_n \to 0$.

Case 2. (s_n) is not eventually decreasing. In this case, we may define an integer sequence $(\tau(n))$ as in Lemma 4. Since $s_{\tau(n)} \le s_{\tau(n)+1}$ for all $n > n_0$, it follows again from (17) that

$$\frac{1-\lambda_{\tau(n)}}{2}\|\tilde{x}_{\tau(n)} - x_{\tau(n)}\|^2 \le M(\lambda_{\tau(n)} + \epsilon_{\tau(n)}) \to 0,$$

so that $\|\tilde{x}_{\tau(n)} - x_{\tau(n)}\| \to 0$, and furthermore $\|x_{\tau(n)} - x_{\tau(n)+1}\| \to 0$. Analogously,

$$\overline{\lim_{n\to\infty}}\,\langle u - z, x_{\tau(n)+1} - z\rangle \le 0.$$

It follows from (17) that for all $n > n_0$

$$s_{\tau(n)} \le 2\langle u - z, x_{\tau(n)+1} - z\rangle + \frac{M\epsilon_{\tau(n)}}{\lambda_{\tau(n)}}.$$

By combining the last two inequalities, we have

$$\overline{\lim_{n\to\infty}}\, s_{\tau(n)} \le 0,$$

from which we arrive at

$$\begin{aligned}\sqrt{s_{\tau(n)+1}} &= \bigl\|(x_{\tau(n)} - z) - (x_{\tau(n)} - x_{\tau(n)+1})\bigr\| \\ &\le \sqrt{s_{\tau(n)}} + \|x_{\tau(n)} - x_{\tau(n)+1}\| \to 0.\end{aligned}$$

Consequently, $s_n \to 0$ follows from (6) immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly to writing this manuscript. Both authors read and approved the manuscript.

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