Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point theory for nonlinear mappings in Banach spaces and applications

Atid Kangtunyakarn[*]

[*]Correspondence:
beawrock@hotmail.com
Department of Mathematics,
Faculty of Science, King Mongkut's
Institute of Technology Ladkrabang,
Bangkok 10520, Thailand

Abstract

The purpose of this research is to study a finite family of the set of solutions of variational inequality problems and to prove a convergence theorem for the set of such problems and the sets of fixed points of nonexpansive and strictly pseudo-contractive mappings in a uniformly convex and 2-uniformly smooth Banach space. We also prove a fixed point theorem for finite families of nonexpansive and strictly pseudo-contractive mappings in the last section.

Keywords: nonexpansive mapping; strictly pseudo-contractive mapping; inverse-strongly monotone; variational inequality problems; uniformly convex; 2-uniformly smooth

1 Introduction

Let E and E^* be a Banach space and the dual space of E, respectively, and let C be a nonempty closed convex subset of E. Throughout this paper, we use '$\to$' and '$\rightharpoonup$' to denote strong and weak convergence, respectively. The duality mapping $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \|x\| = \|x^*\|\}$ for all $x \in E$.

Definition 1.1 Let E be a Banach space. Then a function $\delta_X : [0,2] \to [0,1]$ is said to be the modulus of convexity of E if

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}.$$

If $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$, then E is uniformly convex.

The function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the modulus of smoothness of E if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}, \quad t \ge 0.$$

If $\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0$, then E is uniformly smooth. It is well known that every uniformly smooth Banach space is smooth and if E is smooth, then J is single-valued which is denoted by j. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. If E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth.

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

T is called η-strictly pseudo-contractive if there exists a constant $\eta \in (0,1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \eta \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. It is clear that (1.1) is equivalent to the following:

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \eta \|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny (see [1]) provided $P(x + t(x - P(x))) = P(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retract of C (see [2]) if there exists a sunny nonexpansive retraction from C onto D.

An operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be *α-inverse strongly accretive* if there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is called γ-strongly accretive if there exist $j(x - y) \in J(x - y)$ and a constant $\gamma > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \gamma \|x - y\|^2$$

for all $x, y \in C$.

In 2006, Aoyama *et al.* [3] studied the variational inequality problem in Banach spaces. Such a problem is to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\langle Ax^*, j(x^* - x)\rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

The set of solutions of (1.3) in Banach spaces is denoted by $S(C, A)$, that is,

$$S(C, A) = \{u \in C : \langle Au, J(v - u)\rangle \geq 0, \forall v \in C\}. \tag{1.4}$$

They introduced the strong convergence theorem involving the variational inequality problem in Banach spaces as follows.

Theorem 1.1 *Let E be a uniformly convex and 2-uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$, and let A be an α-inverse strongly accretive operator of C into E with $S(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \ldots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b, c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E.

Many authors have studied the variational inequality problem; see, for example, [4–8]. The variational inequality problem is an important tool for studying fixed point theory, equilibrium problems, optimization problems and partial differential equations with applications principally drawn from mechanics; see, *e.g.*, [9, 10].

Recently, Kangtunyakarn [11] introduced a new mapping in uniformly convex and 2-smooth Banach spaces to prove a strong convergence theorem for finding a common element of the set of fixed points of finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems as follows.

Theorem 1.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let A, B be α and β-inverse strongly accretive mappings of C into E, respectively. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\mathcal{F} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap S(C,A) \cap S(C,B) \neq \emptyset$ and $\kappa = \min\{\kappa_i : i = 1, 2, \ldots, N\}$ with $K^2 \leq \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1, 2, \ldots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $\{x_n\}$ be the sequence generated by x_1, $u \in C$ and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \in [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$;
(iii) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n|, \sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n|, \sum_{n=1}^\infty |\delta_{n+1} - \delta_n|, \sum_{n=1}^\infty |\eta_{n+1} - \eta_n|, \sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(v) $a \in (0, \frac{\alpha}{K^2})$ *and* $b \in (0, \frac{\beta}{K^2})$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

For every $i = 1, 2, \ldots, N$, let $A_i : C \to H$ be a mapping. From (1.3), we introduce the combination of variational inequality problems in Banach spaces as follows: to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\left\langle \sum_{i=1}^{N} a_i A_i x^*, j\bigl(x^* - x\bigr) \right\rangle \geq 0 \tag{1.6}$$

for all $x \in C$ and a_i is a positive real number for all $i = 1, 2, \ldots, N$ with $\sum_{i=1}^{N} a_i = 1$. The set of solutions of (1.6) in Banach spaces is denoted by $S(C, \sum_{i=1}^{N} a_i A_i)$, that is,

$$S\left(C, \sum_{i=1}^{N} a_i A_i\right) = \left\{ u \in C : \left\langle \sum_{i=1}^{N} a_i A_i u, J(v - u) \right\rangle \geq 0, \forall v \in C \right\}. \tag{1.7}$$

By using (1.6) we prove the convergence theorem for a finite family of the set of solutions of variational inequality problems and two sets of fixed points of nonlinear mappings in a Banach space.

2 Preliminaries

The following lemmas are important tools to prove our main results in the next section.

Lemma 2.1 (See [12]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\bigl\langle y, J(x)\bigr\rangle + 2\|Ky\|^2$$

for any $x, y \in E$.

Lemma 2.2 (See [13]) *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty] \to [0, \infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g\bigl(\|x - y\|\bigr)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

Remark 2.3 For every $i = 1, 2, \ldots, N$, if $x_i \in B_r(0)$, from Lemma 2.2, we have $\| \sum_{i=1}^{N} a_i x_i\|^2 \leq \sum_{i=1}^{N} a_i \|x_i\|^2$, where $a_i \in [0, 1]$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.4 (See [3]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$S(C, A) = F\bigl(Q_C(I - \lambda A)\bigr).$$

Lemma 2.5 (See [12]) *Let $r > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and for any $\alpha \in [0, 1]$, we have $\|\alpha x + (1 - \alpha)y\|^2 \leq \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)g(\|x - y\|)$.*

Lemma 2.6 (See [14]) *Let C be a closed and convex subset of a real uniformly smooth Banach space E, and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty} \langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u)\rangle \le 0$$

for any given $u \in C$.

Lemma 2.7 (See [15]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} \le (1-\alpha_n)s_n + \delta_n$, $\forall n \ge 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8 [11] *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1, T_2 and T_3 be three nonexpansive mappings from C into itself with $F(T_1)\cap F(T_2)\cap F(T_3) \neq \emptyset$. Define a mapping S by*

$$Sx = \alpha T_1 x + \beta T_2 x + \gamma T_3 x, \quad \forall x \in C,$$

where α, β, γ is a constant in $(0,1)$ and $\alpha + \beta + \gamma = 1$. Then S is a nonexpansive mapping and $F(S) = F(T_1) \cap F(T_2) \cap F(T_3)$.

Lemma 2.9 *Let C be a nonempty closed convex subset of a real smooth Banach space E. For every $i = 1,2,\dots,N$, let $A_i : C \to E$ be an α_i-strongly accretive mapping with $\overline{\alpha} = \min_{i=1,2,\dots,N}\{\alpha_i\}$ and $\bigcap_{i=1}^N S(C,A_i) \neq \emptyset$. Then $S(C, \sum_{i=1}^N a_iA_i) = \bigcap_{i=1}^N S(C,A_i)$, where $a_i \in [0,1]$ and $\sum_{i=1}^N a_i = 1$.*

Proof It is easy to see that $\bigcap_{i=1}^N S(C,A_i) \subseteq S(C, \sum_{i=1}^N a_iA_i)$. Let $x_0 \in S(C, \sum_{i=1}^N a_iA_i)$ and $x^* \in \bigcap_{i=1}^N S(C,A_i)$. Then there exist $j(y-x^*) \in J(y-x^*)$ and $j(y-x_0) \in J(y-x_0)$ such that

$$\left\langle \sum_{i=1}^N a_iA_ix_0, j(y-x_0)\right\rangle \ge 0, \quad \forall y \in C \tag{2.1}$$

and

$$\left\langle \sum_{i=1}^N a_iA_ix^*, j\bigl(y-x^*\bigr)\right\rangle \ge 0, \quad \forall y \in C. \tag{2.2}$$

From (2.1), (2.2) and $x_0, x^* \in C$, we have

$$\left\langle \sum_{i=1}^N a_iA_ix_0, j\bigl(x^*-x_0\bigr)\right\rangle \ge 0 \tag{2.3}$$

and

$$\left\langle \sum_{i=1}^N a_iA_ix^*, j\bigl(x_0-x^*\bigr)\right\rangle \ge 0. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}
0 &\le \left\langle \sum_{i=1}^{N} a_i A_i x_0 - \sum_{i=1}^{N} a_i A_i x^*, j\bigl(x^* - x_0\bigr) \right\rangle \\
&= -\left\langle \sum_{i=1}^{N} a_i A_i x^* - \sum_{i=1}^{N} a_i A_i x_0, j\bigl(x^* - x_0\bigr) \right\rangle \\
&= -\sum_{i=1}^{N} a_i \bigl\langle A_i x^* - A_i x_0, j\bigl(x^* - x_0\bigr)\bigr\rangle \le -\sum_{i=1}^{N} a_i \alpha_i \bigl\| x^* - x_0 \bigr\|^2 \\
&\le -\sum_{i=1}^{N} a_i \overline{\alpha} \bigl\| x^* - x_0 \bigr\|^2 = -\overline{\alpha} \bigl\| x^* - x_0 \bigr\|^2.
\end{aligned}$$

It implies that $x^* = x_0$, that is, $x_0 \in \bigcap_{i=1}^N S(C, A_i)$. Therefore $S(C, \sum_{i=1}^N a_i A_i) \subseteq \bigcap_{i=1}^N S(C, A_i)$. □

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \ldots, N$, let $A_i : C \to E$ be α_i-strongly accretive and L_i-Lipschitz continuous with $\overline{\alpha} = \min_{i=1,2,\ldots,N} \alpha_i$ and $\overline{L} = \max_{i=1,2,\ldots,N} L_i$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $K^2 \le \eta$, where K is the 2-uniformly smooth constant of E. Assume that $\mathcal{F} = F(T) \cap F(S) \cap \bigcap_{i=1}^N S(C, A_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) S x_n, \\ y_n = b_n x_n + (1 - b_n) T z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C (I - \lambda \sum_{i=1}^N a_i A_i) y_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $a_i \in [0,1]$ for all $i = 1, 2, \ldots, N$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $0 < a \le \beta_n, \gamma_n, c_n, b_n \le b < 1$ *for some* $a, b > 0$, $\forall n \in \mathbb{N}$ *and* $\sum_{i=1}^N a_i = 1$;
(iii) $0 \le \lambda K^2 \le \frac{\overline{\alpha}}{\overline{L}^2}$;
(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n|, \sum_{n=1}^\infty |b_{n+1} - b_n|, \sum_{n=1}^\infty |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof First, we show that $\sum_{i=1}^N a_i A_i$ is an $\frac{\overline{\alpha}}{\overline{L}^2}$-inverse strongly monotone mapping.

Let $x, y \in C$, there exists $j(x - y) \in J(x - y)$ and

$$\begin{aligned}
\left\langle \sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y, j(x - y) \right\rangle &= \sum_{i=1}^{N} a_i \bigl\langle A_i x - A_i y, j(x - y)\bigr\rangle \\
&\ge \sum_{i=1}^{N} a_i \alpha_i \|x - y\|^2
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=1}^{N} a_i \frac{\alpha_i}{L_i^2} \|A_i x - A_i y\|^2 \\
&\geq \frac{\overline{\alpha}}{\overline{L}^2} \sum_{i=1}^{N} a_i \|A_i x - A_i y\|^2 \\
&\geq \frac{\overline{\alpha}}{\overline{L}^2} \left\| \sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y \right\|^2. \qquad (3.2)
\end{aligned}$$

Next, we show that $Q_C(I - \lambda \sum_{i=1}^{N} a_i A_i)$ is a nonexpansive mapping. From (3.2), we have

$$\begin{aligned}
&\left\| Q_C\left(I - \lambda \sum_{i=1}^{N} a_i A_i\right)x - Q_C\left(I - \lambda \sum_{i=1}^{N} a_i A_i\right)y \right\|^2 \\
&\quad \leq \left\| x - y - \lambda \left(\sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y \right) \right\|^2 \\
&\quad \leq \|x - y\|^2 - 2\lambda \left\langle \sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y, j(x - y) \right\rangle \\
&\qquad + 2K^2\lambda^2 \left\| \sum_{i=1}^{N} a_i (A_i x - A_i y) \right\|^2 \\
&\quad \leq \|x - y\|^2 - 2\lambda \frac{\overline{\alpha}}{\overline{L}^2} \left\| \sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y \right\|^2 \\
&\qquad + 2K^2\lambda^2 \left\| \sum_{i=1}^{N} a_i (A_i x - A_i y) \right\|^2 \\
&\quad = \|x - y\|^2 \\
&\qquad - 2\lambda \left(\frac{\overline{\alpha}}{\overline{L}^2} - K^2\lambda \right) \left\| \sum_{i=1}^{N} a_i A_i x - \sum_{i=1}^{N} a_i A_i y \right\|^2 \\
&\quad \leq \|x - y\|^2
\end{aligned}$$

for all $x, y \in C$. Let $x^* \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\| \alpha_n(u - x^*) + \beta_n(x_n - x^*) + \gamma_n \left(Q_C\left(I - \lambda \sum_{i=1}^{N} a_i A_i\right) y_n - x^* \right) \right\| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|y_n - x^*\| \\
&= \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \| b_n(x_n - p) + (1 - b_n)(Tz_n - x^*) \| \\
&\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \big(b_n \|x_n - p\| + (1 - b_n) \|z_n - x^*\| \big) \\
&= \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \big(b_n \|x_n - p\| \\
&\quad + (1 - b_n) \| c_n(x_n - x^*) + (1 - c_n)(Sx_n - x^*) \| \big). \qquad (3.3)
\end{aligned}$$

Since S is a strictly pseudo-contractive mapping, we have

$$\begin{aligned}
\|c_n(x_n - x^*) + (1-c_n)(Sx_n - x^*)\|^2 &= \|x_n - x^* + (1-c_n)(Sx_n - x_n)\|^2 \\
&\leq \|x_n - x^*\|^2 + 2(1-c_n)\langle Sx_n - x_n, j(x_n - x^*)\rangle \\
&\quad + 2K^2(1-c_n)^2\|Sx_n - x_n\|^2 \\
&= \|x_n - x^*\|^2 - 2(1-c_n)\langle (I-S)x_n, j(x_n - x^*)\rangle \\
&\quad + 2K^2(1-c_n)^2\|(I-S)x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2(1-c_n)\eta\|(I-S)x_n\|^2 \\
&\quad + 2K^2(1-c_n)^2\|(I-S)x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2(1-c_n)^2\eta\|(I-S)x_n\|^2 \\
&\quad + 2K^2(1-c_n)^2\|(I-S)x_n\|^2 \\
&\leq \|x_n - x^*\|^2 - 2(1-c_n)^2(\eta - K^2)\|(I-S)x_n\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\big(b_n\|x_n - p\| \\
&\quad + (1-b_n)\|c_n(x_n - x^*) + (1-c_n)(Sx_n - x^*)\|\big) \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.
\end{aligned}$$

From induction we can conclude that $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{z_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \Bigg\|\alpha_n u + \beta_n x_n + \gamma_n Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_n \\
&\quad - \alpha_{n-1}u - \beta_{n-1}x_{n-1} - \gamma_{n-1}Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_{n-1}\Bigg\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&\quad + \gamma_n\Bigg\|Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_n - Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_{n-1}\Bigg\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\Bigg\|Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_{n-1}\Bigg\| \\
&\leq |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&\quad + \gamma_n\|y_n - y_{n-1}\| + |\gamma_n - \gamma_{n-1}|\Bigg\|Q_C\Bigg(I - \lambda\sum_{i=1}^N a_i A_i\Bigg)y_{n-1}\Bigg\|.
\end{aligned} \tag{3.5}$$

From the definition of y_n, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \big\|b_n x_n + (1-b_n)Tz_n - b_{n-1}x_{n-1} - (1-b_{n-1})Tz_{n-1}\big\| \\ &\le b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + (1-b_n)\|Tz_n - Tz_{n-1}\| \\ &\quad + |b_n - b_{n-1}|\|Tz_{n-1}\| \\ &\le b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + (1-b_n)\|z_n - z_{n-1}\| \\ &\quad + |b_n - b_{n-1}|\|Tz_{n-1}\|. \end{aligned} \tag{3.6}$$

From the definition of z_n, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &= \big\|c_n x_n + (1-c_n)Sx_n - c_{n-1}x_{n-1} - (1-c_{n-1})Sx_{n-1}\big\| \\ &= \big\|c_n(x_n - x_{n-1}) + (c_n - c_{n-1})x_{n-1} + (1-c_n)(Sx_n - Sx_{n-1}) \\ &\quad + (c_{n-1} - c_n)Sx_{n-1}\big\| \\ &\le \big\|c_n(x_n - x_{n-1}) + (1-c_n)(Sx_n - Sx_{n-1})\big\| + |c_n - c_{n-1}|\|x_{n-1}\| \\ &\quad + |c_n - c_{n-1}|\|Sx_{n-1}\|. \end{aligned} \tag{3.7}$$

Since S is an η-strictly pseudo-contractive mapping, we have

$$\begin{aligned}&\big\|c_n(x_n - x_{n-1}) + (1-c_n)(Sx_n - Sx_{n-1})\big\|^2 \\ &\quad = \big\|x_n - x_{n-1} - (1-c_n)\big((I-S)x_n - (I-S)x_{n-1}\big)\big\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 - 2(1-c_n)\big\langle (I-S)x_n - (I-S)x_{n-1}, j(x_n - x_{n-1})\big\rangle \\ &\qquad + 2K^2(1-c_n)^2\big\|(I-S)x_n - (I-S)x_{n-1}\big\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 \\ &\qquad - 2(1-c_n)\eta\big\|(I-S)x_n - (I-S)x_{n-1}\big\|^2 \\ &\qquad + 2K^2(1-c_n)^2\big\|(I-S)x_n - (I-S)x_{n-1}\big\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2 \\ &\qquad - 2(1-c_n)^2\big(\eta - K^2\big)\big\|(I-S)x_n - (I-S)x_{n-1}\big\|^2 \\ &\quad \le \|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &\le b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + (1-b_n)\|z_n - z_{n-1}\| \\ &\quad + |b_n - b_{n-1}|\|Tz_{n-1}\| \\ &\le b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + (1-b_n)\big(\big\|c_n(x_n - x_{n-1}) \\ &\quad + (1-c_n)(Sx_n - Sx_{n-1})\big\| + |c_n - c_{n-1}|\|x_{n-1}\| \\ &\quad + |c_n - c_{n-1}|\|Sx_{n-1}\|\big) + |b_n - b_{n-1}|\|Tz_{n-1}\| \\ &\le b_n\|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| \end{aligned}$$

$$\begin{aligned}
&+ (1-b_n)\bigl(\|x_n - x_{n-1}\| + |c_n - c_{n-1}|\|x_{n-1}\| \\
&+ |c_n - c_{n-1}|\|Sx_{n-1}\|\bigr) + |b_n - b_{n-1}|\|Tz_{n-1}\| \\
\le{}& \|x_n - x_{n-1}\| + |b_n - b_{n-1}|\|x_{n-1}\| + |c_n - c_{n-1}|\|x_{n-1}\| \\
&+ |c_n - c_{n-1}|\|Sx_{n-1}\| + |b_n - b_{n-1}|\|Tz_{n-1}\|.
\end{aligned} \tag{3.9}$$

From (3.9) and (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| \le{}& |\alpha_n - \alpha_{n-1}|\|u\| + \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| \\
&+ \gamma_n\|y_n - y_{n-1}\| + |\gamma_n - \gamma_{n-1}|\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_{n-1}\right\| \\
\le{}& (1-\alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u\| \\
&+ |\gamma_n - \gamma_{n-1}|\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_{n-1}\right\| \\
&+ |b_n - b_{n-1}|\|x_{n-1}\| + |c_n - c_{n-1}|\|x_{n-1}\| \\
&+ |c_n - c_{n-1}|\|Sx_{n-1}\| + |b_n - b_{n-1}|\|Tz_{n-1}\| \\
\le{}& (1-\alpha_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M + |\alpha_n - \alpha_{n-1}|M \\
&+ |\gamma_n - \gamma_{n-1}|M + 2|b_n - b_{n-1}|M + 2|c_n - c_{n-1}|M,
\end{aligned}$$

where $M = \max_{n\in\mathbb{N}}\{\|x_n\|, \|u\|, \|Q_C(I - \lambda\sum_{i=1}^N a_iA_i)y_n\|, \|Sx_n\|, \|Tz_n\|\}$. Applying Lemma 2.7, conditions (i) and (iv), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Next, we show that

$$\lim_{n\to\infty}\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)x_n - x_n\right\| = \lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 ={}& \left\|\alpha_n(u - x^*) + \beta_n(x_n - x^*) + \gamma_n\left(Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_n - x^*\right)\right\|^2 \\
\le{}& \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_n - x^*\right\|^2 \\
&- \beta_n\gamma_n g_1\left(\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_n - x_n\right\|\right) \\
\le{}& \alpha_n\|u - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2 \\
&- \beta_n\gamma_n g_1\left(\left\|Q_C\left(I - \lambda\sum_{i=1}^N a_iA_i\right)y_n - x_n\right\|\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|b_n(x_n - x^*) + (1 - b_n)(Tz_n - x^*)\|^2 \\
&\quad - \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \bigl(b_n \|x_n - x^*\|^2 + (1 - b_n) \|z_n - x^*\|^2 - b_n(1 - b_n) g_2 (\|x_n - Tz_n\|) \bigr) \\
&\quad - \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \bigl(b_n \|x_n - x^*\|^2 + (1 - b_n) \bigl(\|x_n - x^*\|^2 \\
&\quad - 2(1 - c_n)^2 (\eta - K^2) \|Sx_n - x_n\|^2 \bigr) - b_n(1 - b_n) g_2 (\|x_n - Tz_n\|) \bigr) \\
&\quad - \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&\le \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 \\
&\quad + \gamma_n \|x_n - x^*\|^2 - 2\gamma_n (1 - b_n)(1 - c_n)^2 (\eta - K^2) \|Sx_n - x_n\|^2 \\
&\quad - \gamma_n b_n (1 - b_n) g_2 (\|x_n - Tz_n\|) \\
&\quad - \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&\le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - 2\gamma_n (1 - b_n)(1 - c_n)^2 (\eta - K^2) \|Sx_n - x_n\|^2 \\
&\quad - \gamma_n b_n (1 - b_n) g_2 (\|x_n - Tz_n\|) \\
&\quad - \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&2\gamma_n (1 - b_n)(1 - c_n)^2 (\eta - K^2) \|Sx_n - x_n\|^2 + \gamma_n b_n (1 - b_n) g_2 (\|x_n - Tz_n\|) \\
&\quad + \beta_n \gamma_n g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&\quad \le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad \le \alpha_n \|u - x^*\|^2 + \bigl(\|x_n - x^*\| + \|x_{n+1} - x^*\| \bigr) \|x_{n+1} - x_n\|.
\end{aligned}
$$

From (3.10), conditions (i) and (ii), we have

$$
\begin{aligned}
\lim_{n\to\infty} \|Sx_n - x_n\| &= \lim_{n\to\infty} g_1 \left(\left\| Q_C \left(I - \lambda \sum_{i=1}^{N} a_i A_i \right) y_n - x_n \right\| \right) \\
&= \lim_{n\to\infty} g_2 (\|x_n - Tz_n\|) = 0.
\end{aligned}
\tag{3.11}
$$

From the properties of g_1 and g_2, we have

$$\lim_{n\to\infty}\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)y_n-x_n\right\|=\lim_{n\to\infty}\|x_n-Tz_n\|=0. \tag{3.12}$$

From (3.11) and the definition of z_n, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.13}$$

From (3.12) and the definition of y_n, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{3.15}$$

From (3.12) and (3.14), we have

$$\lim_{n\to\infty}\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)y_n-y_n\right\|=0. \tag{3.16}$$

Then

$$\begin{aligned}\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)x_n-x_n\right\| &\le \left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)x_n-Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)y_n\right\|\\ &\quad+\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)y_n-x_n\right\|\\ &\le \|x_n-y_n\|+\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)y_n-x_n\right\|.\end{aligned}$$

From (3.12) and (3.14), we have

$$\lim_{n\to\infty}\left\|Q_C\left(I-\lambda\sum_{i=1}^{N}a_iA_i\right)x_n-x_n\right\|=0. \tag{3.17}$$

Since

$$\begin{aligned}\|Tx_n-x_n\| &\le \|Tx_n-Tz_n\|+\|Tz_n-x_n\|\\ &\le \|x_n-z_n\|+\|Tz_n-x_n\|,\end{aligned}$$

from (3.12) and (3.13), we have

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{3.18}$$

Define the mapping $G: C \to C$ by $Gx = \alpha Q_C(I - \lambda \sum_{i=1}^N a_i A_i)x + \beta Tx + \gamma Wx$, where $Wx = cx + (1-c)Sx$ for all $x \in C$ and $\alpha, \beta, \gamma, c \in [0,1]$ with $\alpha + \beta + \gamma = 1$. We show that W is a nonexpansive mapping. Let $x, y \in C$, we have

$$\begin{aligned}
\|Wx - Wy\|^2 &= \bigl\|c(x-y) + (1-c)(Sx - Sy)\bigr\|^2 \\
&= \bigl\|x - y - (1-c)\bigl((I-S)x - (I-S)y\bigr)\bigr\|^2 \\
&\le \|x-y\|^2 - 2(1-c)\bigl\langle (I-S)x - (I-S)y, j(x-y)\bigr\rangle \\
&\quad + 2K^2(1-c)^2\bigl\|(I-S)x - (I-S)y\bigr\|^2 \\
&\le \|x-y\|^2 - 2(1-c)\eta\bigl\|(I-S)x - (I-S)y\bigr\|^2 \\
&\quad + 2K^2(1-c)^2\bigl\|(I-S)x - (I-S)y\bigr\|^2 \\
&\le \|x-y\|^2 - 2(1-c)^2\bigl(\eta - K^2\bigr)\bigl\|(I-S)x - (I-S)y\bigr\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

Then W is a nonexpansive mapping. It is easy to see that the mapping G is nonexpansive. From the definition of W, we have

$$F(S) = F(W). \tag{3.19}$$

From (3.19) Lemmas 2.8, 2.9 and the definition of G, we have $F(G) = F(T) \cap F(S) \cap \bigcap_{i=1}^N S(C, A_i) = \mathcal{F}$. Since

$$\begin{aligned}
\|Gx_n - x_n\| &\le \alpha \left\| Q_C\left(I - \lambda \sum_{i=1}^N a_i A_i\right)x_n - x_n\right\| + \beta\|Tx_n - x_n\| + \gamma\|Wx_n - x_n\| \\
&= \alpha \left\| Q_C\left(I - \lambda \sum_{i=1}^N a_i A_i\right)x_n - x_n\right\| + \beta\|Tx_n - x_n\| + \gamma(1-c)\|Sx_n - x_n\|,
\end{aligned}$$

and (3.11), (3.17) and (3.18), we have

$$\lim_{n\to\infty} \|Gx_n - x_n\| = 0. \tag{3.20}$$

From Lemma 2.6, we have

$$\limsup_{n\to\infty} \bigl\langle u - z_0, j(x_n - z_0)\bigr\rangle \le 0, \tag{3.21}$$

where $z_0 = Q_{\mathcal{F}}u$.

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \left\| \alpha_n(u - z_0) + \beta_n(x_n - z_0) + \gamma_n\left(Q_C\left(I - \lambda\sum_{i=1}^N a_i A_i\right)y_n - z_0\right)\right\|^2 \\
&\le \left\| \beta_n(x_n - z_0) + \gamma_n\left(Q_C\left(I - \lambda\sum_{i=1}^N a_i A_i\right)y_n - z_0\right)\right\|^2
\end{aligned}$$

$$\begin{aligned}&+ 2\alpha_n\langle u - z_0, j(x_{n+1} - z_0)\rangle\\ &\leq (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, j(x_{n+1} - z_0)\rangle.\end{aligned}$$

From Lemma 2.7 and condition (i), we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$. This completes the proof. □

The following corollary is a direct sequel of Theorem 3.1. Therefore, we omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be α-strongly accretive and L-Lipschitz continuous. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $K^2 \leq \eta$, where K is the 2-uniformly smooth constant of E. Assume that $\mathcal{F} = F(T) \cap F(S) \cap S(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} z_n = c_n x_n + (1-c_n)Sx_n,\\ y_n = b_n x_n + (1-b_n)Tz_n,\\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda A)y_n, \quad \forall n \geq 1,\end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < a \leq \beta_n, \gamma_n, c_n, b_n \leq b < 1$, *for some* $a, b > 0$, $\forall n \in \mathbb{N}$;
(iii) $0 \leq \lambda K^2 \leq \frac{\alpha}{L^2}$;
(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \sum_{n=1}^{\infty} |b_{n+1} - b_n|, \sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

4 Applications

Using the concepts of the S^A-mapping and Theorem 3.1, we prove the strong convergence theorem for the set of fixed points of two finite families of nonlinear mappings. We need the following definition and lemma to prove our result.

Definition 4.1 [11] Let C be a nonempty convex subset of a real Banach space. Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of the mappings of C into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S^A : C \to C$ as follows:

$$\begin{aligned}&U_0 = T_1 = I,\\ &U_1 = T_1\big(\alpha_1^1 S_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I\big),\\ &U_2 = T_2\big(\alpha_1^2 S_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I\big),\\ &U_3 = T_3\big(\alpha_1^3 S_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I\big),\\ &\vdots\end{aligned}$$

$$U_{N-1} = T_{N-1}\big(\alpha_1^{N-1} S_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I\big),$$
$$S^A = U_N = T_N\big(\alpha_1^N S_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I\big).$$

This mapping is called the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 4.1 [11] *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \min\{\kappa_i : i = 1, 2, \dots, N\}$ with $K^2 \leq \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S^A) = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i)$ and S^A is a nonexpansive mapping.*

Theorem 4.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. For every $i = 1, 2, \dots, N$, let $A_i : C \to E$ be α_i-strongly accretive and L_i-Lipschitz continuous with $\overline{\alpha} = \min_{i=1,2,\dots,N} \alpha_i$ and $\overline{L} = \max_{i=1,2,\dots,N} L_i$. Let $S : C \to C$ be an η-strictly pseudo-contractive mapping with $K^2 \leq \eta$, where K is the 2-uniformly smooth constant of E. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\kappa = \min\{\kappa_i : i = 1, 2, \dots, N\}$ with $K^2 \leq \kappa$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Assume that $\mathcal{F} = F(S) \cap \bigcap_{i=1}^N S(C, A_i) \cap \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} z_n = c_n x_n + (1 - c_n) S x_n, \\ y_n = b_n x_n + (1 - b_n) S^A z_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - \lambda \sum_{i=1}^N a_i A_i) y_n, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $a_i \in [0,1]$ for all $i = 1, 2, \dots, N$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $0 < a \leq \beta_n, \gamma_n, c_n, b_n \leq b < 1$ *for some* $a, b > 0$, $\forall n \in \mathbb{N}$ *and* $\sum_{i=1}^N a_i = 1$;
(iii) $0 \leq \lambda K^2 \leq \frac{\overline{\alpha}}{\overline{L}^2}$;
(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n|, \sum_{n=1}^\infty |\beta_{n+1} - \beta_n|, \sum_{n=1}^\infty |b_{n+1} - b_n|, \sum_{n=1}^\infty |c_{n+1} - c_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof From Lemma 4.1 and Theorem 3.1, we can reach the desired conclusion. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This research was supported by the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang.

Received: 16 December 2013 Accepted: 21 April 2014 Published: 06 May 2014

References

1. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57-70 (1973)
2. Kopecka, E, Reich, S: Nonexpansive retracts in Banach spaces. Banach Cent. Publ. **77**, 161-174 (2007)
3. Aoyama, K, Iiduka, H, Takahashi, W: Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. **2006**, Article ID 35390 (2006). doi:10.1155/FPTA/2006/35390
4. Ceng, LC, Wang, CY, Yao, JC: Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities. Math. Methods Oper. Res. **67**, 375-390 (2008)
5. Yao, Y, Liou, YC, Kang, SM: Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. Comput. Appl. Math. **59**, 3472-3480 (2010)
6. Qin, X, Kang, SM: Convergence theorems on an iterative method for variational inequality problems and fixed point problems. Bull. Malays. Math. Soc. **33**, 155-167 (2010)
7. Censor, Y, Gibali, A, Reich, S: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**, 318-335 (2011)
8. Kassay, G, Reich, S, Sabach, S: Iterative methods for solving systems of variational inequalities in reflexive Banach spaces. SIAM J. Optim. **21**, 1319-1344 (2011)
9. Yao, Y, Cho, YJ, Chen, R: An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems. Nonlinear Anal. **71**, 3363-3373 (2009)
10. Qin, X, Shang, M, Su, Y: Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems. Math. Comput. Model. **48**, 1033-1046 (2008)
11. Kangtunyakarn, A: A new mapping for finding a common element of the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of variational inequalities in uniformly convex and 2-smooth Banach spaces. Fixed Point Theory Appl. **2013**, Article ID 157 (2013)
12. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127-1138 (1991)
13. Cho, YJ, Zhou, HY, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **47**, 707-717 (2004)
14. Zhou, H: Convergence theorems for κ-strict pseudocontractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. **69**, 3160-3173 (2008)
15. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)

10.1186/1687-1812-2014-108
Cite this article as: Kangtunyakarn: Fixed point theory for nonlinear mappings in Banach spaces and applications. *Fixed Point Theory and Applications* 2014, 2014:108