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Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings

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Abstract
A Krasnoselskii-type algorithm is constructed and the sequence of the algorithm is proved to be an approximate fixed point sequence for a common fixed point of a suitable finite family of multi-valued strictly pseudo-contractive mappings in a real Hilbert space. Under some mild additional compactness-type condition on the operators, the sequence is proved to converge strongly to a common fixed point of the family.

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Keywords: *k*-strictly pseudo-contractive mappings; multi-valued mappings; Hilbert spaces

1 Introduction

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [1], Chang [2], Chidume *et al.* [3], Denavari and Frigon [4], Yingtaweesittikul [5], Kakutani [6], Nash [7, 8], Geanakoplos [9], Nadler [10], Downing and Kirk [11]).

Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in *Game Theory* and *Market Economy* and in other areas of mathematics, such as in *Non-Smooth Differential Equations* (see *e.g.*, [12]).

Game theory is perhaps the most successful area of application of fixed point theory for multi-valued mappings. However, it has been remarked that the applications of this theory to *equilibrium problems* in game theory are mostly static in the sense that while they enhance the understanding of conditions under which equilibrium may be achieved, they do not indicate how to construct a process starting from a non-equilibrium point that will converge to an equilibrium solution. Iterative methods for fixed points of multi-valued mappings are designed to address this problem. For more details, one may consult [12–14].

Let K be a nonempty subset of a normed space E. The set K is called *proximinal* (see *e.g.*, [15–17]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in K\} = d(x,K),$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal.

Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets, and of nonempty, proximinal and bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by

$$D(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\} \quad \text{for all } A, B \in CB(K).$$

Let $T : D(T) \subseteq E \to CB(E)$ be a *multi-valued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$D(Tx, Ty) \leq L\|x-y\| \quad \forall x, y \in D(T). \tag{1.1}$$

When $L \in (0,1)$ in (1.1), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Several papers deal with the problem of approximating fixed points of *multi-valued nonexpansive* mappings (see, for example [15–20] and the references therein) and their generalizations (see *e.g.*, [21, 22]).

Recently, Abbas *et al.* [18], introduced a one-step iterative process as follows, $x_1 \in K$:

$$x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \quad n \geq 1. \tag{1.2}$$

Using (1.2), Abbas *et al.* proved weak and strong convergence theorems for approximation of common fixed point of *two multi-valued nonexpansive mappings* in real Banach spaces.

Very recently, Chidume *et al.* [12], introduced the class of *multi-valued k-strictly pseudo-contractive* maps defined on a real Hilbert space H as follows.

Definition 1.1 A multi-valued map $T : D(T) \subset H \to CB(H)$ is called *k-strictly pseudo-contractive* if there exists $k \in (0,1)$ such that for all $x, y \in D(T)$,

$$\big(D(Tx, Ty)\big)^2 \leq \|x-y\|^2 + k\big\|x-y-(u-v)\big\|^2 \quad \forall u \in Tx, v \in Ty. \tag{1.3}$$

In the case that T is single-valued, definition 1.1 reduces to the definition introduced and studied by Browder and Petryshn [23] as an important generalization of the class of nonexpansive mappings. Chidume *et al.* [12], proved strong convergence theorems for approximating fixed points of this class of mappings using a *Krasnoselskii-type algorithm*, [24] which is well known to be superior to the recursion formula of Mann [25] or Ishikawa [26].

In this paper, motivated by the results of Chidume *et al.* [12], Abbas *et al.* [18], Khastan [27], Eslamian [28], Shahzad and Zegeye [29] and Song and Wong [17], we introduce a new *Krasnoselskii-type* algorithm and prove strong convergence theorems for the sequence of the algorithm for approximating a common fixed point of a *finite family of multi-valued strictly pseudo-contractive mappings in a real Hilbert space.* Our results, under the setting of our theorems, generalize those of Abbas *et al.* [18] and Chidume *et al.* [12], which are themselves generalizations of many important results, from *two multi-valued nonexpansive mappings*, and a single *multi-valued strictly pseudo-contractive mapping*, respectively, to *a finite family* of *multi-valued strictly pseudo-contractive mappings*.

2 Main result

In the sequel, we shall need the following lemma whose proof can be found in Eslamian [28]. We reproduce the proof here for completeness.

Lemma 2.1 *Let H be a real Hilbert space. Let $\{x_i, i = 1, \dots, m\} \subset H$. For $\alpha_i \in (0,1)$, $i = 1, \dots, m$ such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds*:

$$\left\| \sum_{i=1}^m \alpha_i x_i \right\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i,j=1,i\neq j}^m \alpha_i\alpha_j \|x_i - x_j\|^2. \tag{2.1}$$

Proof The proof is by induction. For $m = 2$, (2.1) reduces to the standard identity in Hilbert spaces. Assume that (2.1) is true for some $k \geq 2$, that is,

$$\left\| \sum_{i=1}^k \alpha_i x_i \right\|^2 = \sum_{i=1}^k \alpha_i \|x_i\|^2 - \sum_{i,j=1,i\neq j}^k \alpha_i\alpha_j \|x_i - x_j\|^2. \tag{2.2}$$

Then

$$\begin{aligned} \left\| \sum_{i=1}^{k+1} \alpha_i x_i \right\|^2 &= \alpha_1 \|x_1\|^2 + (1-\alpha_1) \left\| \sum_{i=2}^{k+1} \frac{\alpha_i}{(1-\alpha_1)} x_i \right\|^2 \\ &\quad - \alpha_1(1-\alpha_1) \left\| \sum_{i=2}^{k+1} \frac{\alpha_i}{(1-\alpha_1)} (x_1 - x_i) \right\|^2 \\ &= \alpha_1 \|x_1\|^2 + (1-\alpha_1) \left\| \sum_{i=1}^{k} \alpha'_{i+1} x_{i+1} \right\|^2 \\ &\quad - \alpha_1(1-\alpha_1) \left\| \sum_{i=1}^{k} \alpha'_{i+1} (x_1 - x_{i+1}) \right\|^2, \quad \alpha'_i := \frac{\alpha_i}{(1-\alpha_1)}. \end{aligned} \tag{2.3}$$

Using (2.2) in (2.3), we obtain

$$\begin{aligned} \left\| \sum_{i=1}^{k+1} \alpha_i x_i \right\|^2 &= \alpha_1 \|x_1\|^2 + (1-\alpha_1) \left[\sum_{i=1}^{k} \alpha'_{i+1} \|x_{i+1}\|^2 \right. \\ &\quad \left. - \sum_{i,j=1,i\neq j}^{k} \alpha'_{i+1}\alpha'_{j+1} \|x_{i+1} - x_{j+1}\|^2 \right] \\ &\quad - \alpha_1(1-\alpha_1) \left[\sum_{i=1}^{k} \alpha'_{i+1} \|(x_1 - x_{i+1})\|^2 \right. \\ &\quad \left. - \sum_{i,j=1,i\neq j}^{k} \alpha'_{i+1}\alpha'_{j+1} \|x_{i+1} - x_{j+1}\|^2 \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{k+1} \alpha_i \|x_i\|^2 - \sum_{i=2}^{k+1} \alpha_1 \alpha_i \big\|(x_1 - x_i)\big\|^2 \\
&\quad - \sum_{i,j=2,i\neq j}^{k+1} \alpha_i \alpha_j \|x_i - x_j\|^2 \\
&= \sum_{i=1}^{k+1} \alpha_i \|x_i\|^2 - \sum_{i,j=1,i\neq j}^{k+1} \alpha_i \alpha_j \|x_i - x_j\|^2. \qquad (2.4)
\end{aligned}$$

Therefore, by induction, we find that (2.1) is true. This completes the proof. □

We now prove the following theorem.

Theorem 2.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T_i : K \to CB(K)$ be a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1)$, $i = 1,\dots,m$ such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \lambda_2 y_n^2 + \cdots + \lambda_m y_n^m, \qquad (2.5)$$

where $y_n^i \in T_i x_n$, $n \geq 1$ and $\lambda_i \in (k,1)$, $i = 0,1,\dots,m$, such that $\sum_{i=0}^m \lambda_i = 1$ and $k := \max\{k_i, i = 1,\dots,m\}$. Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ $\forall i = 0,\dots,m$.

Proof Let $p \in \bigcap_{i=1}^m F(T_i)$. Then

$$\|x_{n+1} - p\|^2 = \left\| \lambda_0 (x_n - p) + \sum_{i=1}^m \lambda_i \big(y_n^i - p\big) \right\|^2 := \left\| \sum_{i=0}^m \lambda_i z_i \right\|^2,$$

where $z_0 = (x_n - p)$, $z_i = (y_n^i - p)$, $i = 1,\dots,m$, and $\sum_{i=1}^m \lambda_i z_i = \sum_{i=1}^m \lambda_i (y_n^i - p)$.

Using Lemma 2.3 of [12] and the identity (2.1), we obtain the following estimates:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^m \lambda_i \big\|y_n^i - p\big\|^2 \\
&\quad - \sum_{i=1}^m \lambda_i \lambda_0 \big\|x_n - y_n^i\big\|^2 - \sum_{i,j=1,i\neq j}^m \lambda_i \lambda_j \big\|y_n^i - y_n^j\big\|^2 \\
&\leq \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^m \lambda_i \big\|y_n^i - p\big\|^2 - \sum_{i=1}^m \lambda_i \lambda_0 \big\|x_n - y_n^i\big\|^2 \\
&\leq \lambda_0 \|x_n - p\|^2 + \sum_{i=1}^m \lambda_i \big(D(T_i x_n, T_i p)\big)^2 - \sum_{i=1}^m \lambda_i \lambda_0 \big\|x_n - y_n^i\big\|^2 \\
&= \|x_n - p\|^2 - \sum_{i=1}^m \lambda_i (\lambda_0 - k) \big\|x_n - y_n^i\big\|^2. \qquad (2.6)
\end{aligned}$$

Thus, since $\lambda_i \in (k,1)$, we obtain

$$\sum_{i=1}^m \big\|x_n - y_n^i\big\|^2 \leq \frac{1}{(\lambda_0 - k)k} \big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big) \quad \forall n \geq 1.$$

This implies that, for each $i = 1, 2, \dots, m$, $\sum_{n=1}^{\infty} \|x_n - y_n^i\|^2 < \infty$. Hence, $\lim_{n\to\infty} \|x_n - y_n^i\| = 0$, for each $i = 1, \dots, m$. Since $y_n^i \in T_i x_n$, $i = 1, \dots, m$, we have $0 \le d(x_n, T_i x_n) \le \|x_n - y_n^i\|$ and so

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0.$$

This completes the proof. □

Definition 2.3 A mapping $T : K \to CB(K)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. We note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Theorem 2.4 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T_i : K \to CB(K)$ be a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, \dots, m$ such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ and T_i, $i = 1, \dots, m$ is hemicompact and continuous. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \lambda_2 y_n^2 + \cdots + \lambda_m y_n^m, \tag{2.7}$$

where $y_n^i \in T_i x_n$, $n \ge 1$ and $\lambda_i \in (k, 1)$, $i = 0, 1, \dots, m$ such that $\sum_{0=1}^m \lambda_i = 1$ with $k := \max\{k_i, i = 1, \dots, m\}$. Then the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^m F(T_i)$.

Proof From Theorem 2.2, we have $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, $i = 1, \dots, m$. Since T_i, $i = 1, \dots, m$, is hemicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$ for some $q \in K$. Moreover, by continuity of T_i, $i = 1, \dots, m$, we also have $d(x_{n_k}, T_i x_{n_k}) \to d(q, T_i q)$, $i = 1, \dots, m$ as $k \to \infty$. Therefore, $d(q, T_i q) = 0$, $i = 1, \dots, m$ and so $q \in F(T_i)$. Setting $p = q$ in the proof of Theorem 2.2, it follows from inequality (2.6) that $\lim_{n\to\infty} \|x_n - q\|$ exists. So, $\{x_n\}$ converges strongly to $q \in \bigcap_{i=1}^m F(T_i)$. This completes the proof. □

The following is an immediate corollary of Theorem 2.4. The proof basically follows as the proof of its analog for single multi-valued strictly pseudo-contractive map in Chidume *et al.* [12]. The proof is therefore omitted.

Corollary 2.5 *Let K be a nonempty, compact and convex subset of a real Hilbert space H and $T_i : K \to CB(K)$ be a finite family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, \dots, m$ such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Assume that for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ and T_i, $i = 1, \dots, m$ is continuous. Let $\{x_n\}$ be a sequence defined by $x_0 \in K$,*

$$x_{n+1} = \lambda_0 x_n + \lambda_1 y_n^1 + \lambda_2 y_n^2 + \cdots + \lambda_m y_n^m, \tag{2.8}$$

where $y_n^i \in T_i x_n$, $n \ge 1$ and $\lambda_i \in (k, 1)$, $i = 0, 1, \dots, m$ such that $\sum_{0=1}^m \lambda_i = 1$ with $k := \max\{k_i, i = 1, \dots, m\}$. Then the sequence $\{x_n\}$ converges strongly to an element of $\bigcap_{i=1}^m F(T_i)$.

Remark 2.6 In Theorem 2.4, the continuity assumption on T_i, $i = 1, \dots, m$ can be dispensed with if we assume that for every $x \in K$, $T_i x$, $i = 1, \dots, m$ is proximinal and weakly closed.

Remark 2.7 If we set $i = 1$ in all the results obtained in this paper, we recover the results of Chidume *et al.* [12].

Remark 2.8 The recursion formulas studied in this paper are of the Krasnoselkii type (see *e.g.* [24]) which is well known to be superior to the recursion formula of either the Mann algorithm or the *so-called* Ishikawa-type algorithm.

Remark 2.9 Our theorems and corollary improve the results of Chidume *et al.* [12] from *single* multi-valued strictly pseudo-contractive mapping to *finite family* of multi-valued strictly pseudo-contractive mappings. Furthermore, under the setting of Hilbert space, our theorems and corollary improve the convergence theorems for multi-valued *nonexpansive mappings* to the more general class of multi-valued *strictly pseudo-contractive mappings* studied in Sastry and Babu [15], Panyanak [16], Song and Wang [17], Shahzad and Zegeye [29] and Abbas *et al.* [18]. Also, in all our algorithms, $y_n \in Tx_n$ is arbitrary and is not required to satisfy the very restrictive condition, '$y_n \in T(x_n)$ such that $\|y_n - x^*\| = d(x^*, Tx_n)$' imposed in [15–18, 20, 29].

For examples of multi-valued maps such that, for each $x \in K$, the set Tx is proximinal and weakly closed, the reader may consult Chidume *et al.* [12].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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