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Monotone type operators in nonreflexive Banach spaces

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Abstract

Let E be a real Banach space, E^* be the dual space of E, E^{**} be the dual space of E^*. Let $T: D(T) \subseteq E^{**} \to 2^{E^*}$ be a monotone type mapping. In this paper, first, we introduce the special case when T is the weak* sub-differential $\partial^*\phi$ of a convex function ϕ and obtain a surjective result for the mapping $\partial^*(\phi + \epsilon\|\cdot\|^2)$, where $\epsilon > 0$. Second, we show the existence of solutions of the variational inequality problems for strictly quasi-monotone operators and semi-monotone operators. Finally, we construct a degree theory for mappings of the class (S_+) and then construct a generalized degree for the weak* sub-differential of a convex function.

1 Introduction

Monotone operators in reflexive Banach spaces has many applications in nonlinear partial differential equations, nonlinear semi-group theory, variational inequality and so on (see [1–4]). The theory for monotone operators in reflexive Banach spaces has been well developed. In recent years, many authors have generalized the monotone operator theory to nonreflexive Banach spaces. For example, maximal monotone operators in nonreflexive Banach spaces has been studied in [5–8] and variational inequality problems related to monotone type mappings in nonreflexive Banach spaces have been studied in [9–14]. For more references on variational inequality problems, see [15–24] and [25]. Also, degree theory for monotone type mappings in nonreflexive separable Banach spaces has been studied in [26, 27]. Also, see [3, 28–36] for more references on degree theory of monotone type operators.

In this paper, we study variational inequality problems and degree theory for monotone type mappings in nonreflexive spaces. This paper is organized as follows:

Let E be a real Banach space, E^* be the dual space of E and E^{**} be the dual space of E^*. In Section 2, we introduce the weak* sub-differential $\partial^*\phi$ of a convex function $\phi: E^{**} \to R \cup \{+\infty\}$, which is a subset of the classical sub-differential, and we obtain $\partial^*(\phi + \epsilon\|\cdot\|^2) = E^*$ for the sum of a lower semi-continuous convex function $\phi: E^{**} \to R \cup \{+\infty\}$ in the weak* topology and $\epsilon\|x\|^2$, where $\epsilon > 0$. In Section 3, we show the existence of solutions of variational inequality problems related to strictly quasi-monotone operators and semi-monotone operators. In Section 4, we construct a degree theory for mappings of class (S_+) and then construct a generalized degree for the weak* sub-differential of a convex function and obtain some degree results.

Through this paper, we use $\rightharpoonup^*$ to represent the convergence in the weak* topology, $\rightharpoonup$ to represent the convergence in the weak topology and $\to$ represent the convergence in norm topology.

2 The weak* sub-differential of convex functions

In this section, let E be a real Banach space, E^* be the dual space of E and E^{**} be the dual space of E^*.

Now, we introduce the weak* sub-differential of a convex function and study the solvability problems related this mapping.

First, we recall that the classical sub-differential of a convex function $\phi : E \to R \cup \{+\infty\}$ at y is defined by

$$\partial\phi(y) = \bigl\{f \in E^* : \phi(x) - \phi(y) \geq (f, x - y), \forall x \in D(\phi)\bigr\}.$$

It is well known (Rockfellar [8]) that $\partial\phi$ is a maximal monotone mapping.

Definition 2.1 Let $\phi : E^{**} \to R \cup \{+\infty\}$ be a convex function. Then

$$\partial^*\phi(y) = \bigl\{f \in E^* : \phi(x) - \phi(y) \geq (f, x - y), \forall x \in D(\phi)\bigr\}$$

is called the *weak* sub-differential* of ϕ at y.

It is obvious that $\partial^*\phi(y) \subseteq \partial\phi(y)$, but $\partial^*\phi(y) = \partial\phi(y)$ when E is reflexive.

The following result is obvious.

Proposition 2.2 *Let $\phi : E^{**} \to R \cup \{+\infty\}$ be a convex function. Then we have the following*:

(1) $\partial^*\phi(y)$ *is a weak closed convex subset of* E^*;

(2) $0 \in \partial^*\phi(y_0)$ *if and only if* $\phi(y_0) = \inf_{y \in D(\phi)} \phi(y)$;

(3) $\partial^*\phi : E^{**} \to E^*$ *is monotone*.

Definition 2.3 (see [37]) Let X be a topological space. A function $f : X \to R$ is said to be *sequentially lower semi-continuous from above* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \leq f(x_n)$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_n)$.

Similarly, f is said to be *sequentially upper semi-continuous from below* at x_0 if, for any sequence $\{x_n\}$ with $x_n \to x_0$, $f(x_{n+1}) \geq f(x_n)$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_0)$.

Remark 1 It is well known that a lower semi-continuous function is a lower semi-continuous from above function, but the converse is not true and a lower semi-continuous from above and convex function with the coercive condition in a reflexive Banach space attains its minimum (see [37]). Also, it is well known that, for a convex function in a reflexive Banach space, lower semi-continuity in the strong topology is equivalent to lower semi-continuity in the weak topology, but this is not true for lower semi-continuity from above (see [38]). For more on lower semi-continuous from above functions with its generalizations and applications in nonconvex equilibrium problems, variational problems and fixed point problems, see [38–50] and [51].

Proposition 2.4 *Let $\phi : E^{**} \to R \cup \{+\infty\}$ be a convex function which is sequentially lower semi-continuous from above in the weak* topology and $\lim_{\|x\| \to +\infty} \phi(x) = +\infty$, then there exists $x_0 \in E^{**}$ such that $\phi(x_0) = \inf_{y \in D(\phi)} \phi(y)$.*

Proof We take a sequence $\{x_n\}$ in E^{**} such that

$$\phi(x_1) \geq \phi(x_2) \geq \cdots \geq \phi(x_n) \geq \cdots, \quad \phi(x_n) \to \inf_{x \in D(\phi)} \phi(x).$$

Since $\lim_{\|x\| \to +\infty} \phi(x) = +\infty$ and $\{x_n\}$ is a bounded sequence in E^{**}, it follows that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \rightharpoonup^* x_0$ in E^{**}. By the assumption, since ϕ is sequentially lower semi-continuous from above, we have $\phi(x_0) \leq \lim_{n \to \infty} \phi(x_n)$ and so it follows that

$$\phi(x_0) = \inf_{y \in D(\phi)} \phi(y).$$

This completes the proof. □

Proposition 2.5 *The function $\phi : E^{**} \to R$ defined by $\phi(x) = \|x\|^2$ is sequentially lower semi-continuous in the weak* topology.*

Proof Suppose $x_n \rightharpoonup^* x_0$. Then $x_0(f) = \lim_{n \to \infty} x_n(f)$ for all $f \in E^*$ and so

$$\left|x_0(f)\right| \leq \liminf_{n \to \infty} \|x_n\| \|f\|$$

for all $f \in E^*$. Thus we have

$$\|x_0\| = \sup_{\|f\|=1} \left|x_0(f)\right| \leq \liminf_{n \to \infty} \|x_n\|$$

and so $\|x_0\|^2 \leq \liminf_{n \to \infty} \|x_n\|^2$. This completes the proof. □

Theorem 2.6 *Let $\phi : E^{**} \to R \cup \{+\infty\}$ be a convex function which is sequentially lower semi-continuous in the weak* topology. Then we have*

$$\partial^*\left(\phi + \epsilon\| \cdot \|^2\right)\left(E^{**}\right) = E^*$$

for all $\epsilon > 0$.

Proof For any $f \in E^*$, we set $\psi(x) = \phi(x) + \epsilon\|x\|^2 - x(f)$ for all $x \in D(\phi)$. It is obvious that ψ is sequentially lower semi-continuous in the weak* topology. Thus ψ is sequentially lower semi-continuous from above in the weak* topology and

$$\lim_{\|x\| \to +\infty} \psi(x) = +\infty.$$

By Proposition 2.4, there exists $x_0 \in E^{**}$ such that $\phi(x_0) = \inf_{x \in D(\psi)} \psi(x)$. By (2) of Proposition 2.2, $0 \in \partial^*(\phi + \epsilon\| \cdot \|^2) - x(f))(x_0)$, which is equivalent to $f \in \partial^*(\phi + \epsilon\| \cdot \|^2)(x_0)$. This completes the proof. □

3 Existence of variational inequality problems

In this section, we study variational inequality problems related to monotone type operators in nonreflexive Banach spaces.

First, we recall the following.

Definition 3.1 ([11]) A mapping $A(u,v): E^{**} \times E^{**} \to E^*$ is said to be *semi-monotone* if it satisfies the following conditions:

(1) for each $u \in E^{**}$, $A(u,\cdot)$ is monotone, *i.e.*, $(A(u,v) - A(u,w), v - w) \geq 0$ for all $v, w \in E^{**}$;

(2) for each fixed $v \in E^{**}$, $A(\cdot, v)$ is completely continuous, *i.e.*, if $u_j \rightharpoonup u_0$ in weak* topology of E^{**}, then $A(u_j, v)$ has a subsequence $A(u_{j_k}, v)$ with $A(u_{j_k}, v) \to A(u_0, v)$ in norm topology of E^*.

Definition 3.2 ([15]) Let E be a real Banach space and $T : D \subseteq E^{**} \to 2^{E^*}$ be a mapping. T is said to be *strictly quasi-monotone* if $(g, u - v) > 0$ for all $u, v \in D$ and for some $g \in Tv$ implies that $(f, u - v) > 0$ for all $f \in Tu$.

Remark 2 For quasi-monotone mappings, see [21].

Lemma 3.3 *Let E be a real Banach space and C be a nonempty bounded closed convex subset of E^{**}. If $A : C \to 2^{E^*}$ is a finite dimensional weak* upper semi-continuous* (*i.e. for each finite dimensional subspace F of E^{**} with $F \cap C \neq \emptyset$, $A : C \cap F \to 2^{E^*}$ is upper semi-continuous in the weak topology*) *and strictly quasi-monotone mapping with bounded closed convex values, then $(f_v, u_0 - v) \leq 0$ for all $v \in C$ and for some $f_v \in Tu_0$ if and only if $(g, u_0 - v) \leq 0$ for all $v \in C$ and $g \in Tv$.*

Proof The proof is similar to Lemma 2.3 in [15], we omit the details. □

Remark 3 For the results of Lemma 3.3 in monotone case, we refer to [10].

Theorem 3.4 *Let E be a real Banach space and C be a nonempty weak* closed convex bounded subset of E^{**}. If $A : C \to 2^{E^*}$ is a finite dimensional weakly upper semi-continuous and strictly quasi-monotone mapping with bounded closed convex values, then there exists $u_0 \in C$ such that*

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and for some $f_v \in Tu_0$.

Proof For any finite dimensional subspace F of E with $F \cap C \neq \emptyset$, let $j_F : F \to E$ be the natural inclusion and j_F^* be the conjugate mapping of j_F. Consider the following variational inequality problem:

Find $u \in F \cap C$ such that

$$\left(j_F^* f_v, u - v\right) \leq 0$$

for all $v \in C \cap F$ and for some $f_v \in Tu$.

Since T is finite dimensional weakly upper semi-continuous and j_F^*T is upper semi-continuous on $F \cap C$, there exists $u_F \in F \cap C$ such that

$$\left(j_F^* f_v, u_F - v\right) \leq 0$$

for all $v \in C \cap F$ and for some $f_v \in Tu_F$, *i.e.*, $(f_v, u_F - v) \leq 0$ for all $v \in C \cap F$ and for some $f_v \in Tu_F$. By Lemma 3.3, we get

$$(g, u_F - v) \leq 0$$

for all $v \in C \cap F$ and $g \in Tv$. Now, we put

$$W_F = \left\{u \in C : (g, u - v) \leq 0, \forall v \in F \cap C, g \in Tv\right\}.$$

It is obvious that W_F is weak* closed convex. One can easily check that

$$W_{\bigcup_{i=1}^n F_i} \subseteq W_{F_i}, \qquad \dim(F_i) < +\infty, \qquad F_i \cap C \neq \emptyset$$

for $i = 1, 2, \ldots, n$. Hence $\bigcap_{F \in \mathcal{F}} W_F \neq \emptyset$, where

$$\mathcal{F} = \left\{F \subset E : F \cap C \neq \emptyset, \dim(F) < +\infty\right\}.$$

Take $u_0 \in \bigcap_{F \in \mathcal{F}} W_F$. We claim that u_0 satisfies the conclusion of Theorem 3.4. In fact, $(g, u_0 - v) \leq 0$ for all $v \in C$ and $g \in Tv$. By Lemma 3.3, it follows that

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and for some $f_v \in Tu_0$. This completes the proof. □

From Theorem 3.4, we have the following.

Corollary 3.5 *Let E be a real Banach space and C be a nonempty weak* closed convex unbounded subset of E^{**}. If $A : C \to 2^{E^*}$ is a finite dimensional weakly upper semi-continuous and strictly quasi-monotone mapping with bounded closed convex values and there exist $v_0 \in C$ and $r > 0$ such that*

$$(f, u - v_0) > 0$$

for all $f \in Tu$ and $u \in C$ with $\|u\| > r$, then there exists $u_0 \in C$ such that

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and for some $f_v \in Tu_0$.

Proof If $C_n = C \cap B(0, n)$, then, by Theorem 3.4, there exists $u_n \in C_n$ such that

$$(f, u_n - v) \leq 0$$

for all $v \in C_n$ and for some $f_v \in Tu_n$. By Lemma 3.3, we know that

$$(g, u_n - v) \leq 0$$

for all $v \in C_n$ and for some $g \in Tv$. By the assumption, we know that $\|u_n\| \leq r$ for each $n = 1, 2, \ldots$ and thus we may assume that $u_n \rightharpoonup^* u_0$ as $n \to \infty$. Otherwise, we take a subsequence. Consequently, it follows that

$$(g, u_0 - v) \leq 0$$

for all $v \in C$ and $g \in Tv$. Again, if we use Lemma 3.3, we get the conclusion. This completes the proof. □

Corollary 3.6 *Let E be a real Banach space, $B(0,R) = \{\|x\| < R : x \in X^{**}\} \subset E^{**}$ is the open ball centered at 0 with radius R. If $A : \overline{B(0,R)} \to E^*$ is a finite dimensional weakly continuous and strictly quasi-monotone mapping and*

$$(Au, u) > -\|Au\| \|u\|$$

for all $u \in \partial B(0,R)$, then there exists $u_0 \in B(0,r)$ such that $Au_0 = 0$.

Proof It is obvious that $\overline{B(0,R)}$ is weak* closed and convex. By Theorem 3.4, there exists $u_0 \in \overline{B(0,R)}$ such that

$$(Au_0, u_0 - v) \leq 0$$

for all $v \in B(0,R)$. Now, we claim that $Au_0 = 0$. First, we prove that $\|u_0\| < R$. In fact, if $\|u_0\| = R$, then, by the assumption, $\|Au_0\| \neq 0$ and thus there exists $v_0 \in \partial B(0,R)$ such that $(Au_0, v_0) = -\|Au_0\| \|v_0\|$. But we have

$$-\|Au_0\| \|u_0\| < (Au_0, u_0) \leq (Au_0, v_0) = -\|Au_0\| \|v_0\|,$$

which is a contradiction. Therefore, we have $\|u_0\| < R$. Since there exists $r > 0$ such that $u_0 + v \in B(0,R)$ for all $v \in E^{**}$ with $\|v\| \leq r$, we have

$$(Au_0, v) \geq 0$$

for all $v \in B(0,r)$ and so $Au_0 = 0$. This completes the proof. □

Theorem 3.7 *Let $K \subset E^{**}$ be a bounded weak* closed convex subset. Suppose that $\phi : E^{**} \to R \cup \{+\infty\}$ is a lower semi-continuous convex function in the weak* topology $K \subseteq D(\phi)$, $A : K \times K \to E^*$ is semi-monotone, and $A(u, \cdot)$ is finite dimensional continuous for each $u \in K$. Then there exists $w_0 \in K$ such that*

$$\big(A(w_0, w_0), u - w_0\big) + \phi(u) - \phi(w_0) \geq 0$$

for all $u \in K$.

Proof For each finite dimensional subspace F of E^{**} with $F \cap K \neq \emptyset$, set $K_F = K \cap F$ and $\phi_F(x) = \phi(x)$ for $x \in F \cap D(\phi)$. By Theorem 2.5 in [11], there exists $u_F \in K_F$ such that

$$\big(A(u_F, u_F), u - u_F\big) + \phi_F(u) - \phi_F(u_F) \geq 0 \tag{3.1}$$

for all $u \in K_F$. Let

$$\mathcal{F} = \big\{F \subset E^{**} : F \text{ is finite dimensional subspace with } F \cap K \neq \emptyset\big\}$$

and

$$W_F = \big\{w \in K : \big(A(w, u), u - w\big) + \phi(u) - \phi(w) \geq 0\big\}.$$

By (3.1) and the monotonicity of $A(u_F, \cdot)$, W_F is a nonempty bounded subset. Denote by $\overline{W_F}^*$ the weak* closure of W_F. For any $F_i \in \mathcal{F}$ for each $i = 1, 2, \ldots, n$, it is easy to see that $W_{\bigcup_i F_i} \subset W_{F_i}$ for each $i = 1, 2, \ldots, n$. So, we have

$$\bigcap_{F \in \mathcal{F}} \overline{W_F}^* \neq \emptyset.$$

Let $w_0 \in \bigcap_{F \in \mathcal{F}} \overline{W_F}^*$. Now, we prove that

$$\big(A(w_0, w_0), u - w_0\big) + \phi(u) - \phi(w_0) \geq 0$$

for all $u \in K$. For each $u \in K$, take $F \in \mathcal{F}$ such that $w_0 \in K_F$ and $u \in K_F$. There exists $w_j \in W_F$ such that $w_j \rightharpoonup^* w_0$ and

$$\big(A(w_j, u), u - w_j\big) + \phi(u) - \phi(w_j) \geq 0$$

for each $j = 1, 2, \ldots$. By letting $j \to \infty$, the complete continuity of $A(\cdot, u)$ and weak* lower semi-continuity of ϕ imply that

$$\big(A(w_0, u), u - w_0\big) + \phi(u) - \phi(w_0) \geq 0.$$

Set $u = tw_0 + (1 - t)v$ for all $t \in (0, 1)$ and $v \in K$, by using the convexity of ϕ and letting $t \to 1$, we get

$$\big(A(w_0, w_0), v - w_0\big) + \phi(v) - \phi(w_0) \geq 0.$$

This completes the proof. □

4 Degree theory for monotone type mapping

In this section, assume that E is always a real Banach space, E^* is the dual space of E and E^{**} is the dual space of E^*.

Definition 4.1 A set-valued operator $T : D(T) \subseteq E^{**} \to 2^{E^*}$ is said to be *strong to weak upper semi-continuous* at $x_0 \in D(T)$ if, for each weak open neighborhood V of 0 in E^* (*i.e.*, open in the weak topology of E^*), there exists an open neighborhood W of 0 in E^{**} such that $Ty \cap (Tx_0 + V) \neq \emptyset$ for all $y \in x_0 + W$.

Definition 4.2 A set-valued operator $T : D(T) \subseteq E^{**} \to 2^{E^*}$ is said to be a *mapping of class* (S_+) if the following conditions are satisfied:

(1) for each $x \in D(T)$, Tx is a bounded closed convex subset;
(2) T is strong to weak upper semi-continuous;
(3) if $x_n \in D(T)$, $f_n \in Tx_n$ for each $n \geq 1$ and $x_j \rightharpoonup^* x_0$ such that

$$\overline{\lim_{n\to\infty}} (f_n, x_n - x_0) \leq 0,$$

then $x_n \to x_0 \in D(T)$ and $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ with $f_{n_k} \rightharpoonup f_0 \in Tx_0$.

Definition 4.3 A family of set-valued operators $T_t : D \subseteq E^{**} \to 2^{E^*}$ for all $t \in [0,1]$ is said to be a *homotopy of mappings of class* (S_+) if the following conditions are satisfied:

(1) for each $t \in [0,1]$, $x \in D$, $T_t x$ is a bounded closed convex subset;
(2) $T_t x : [0,1] \times D \to E^*$ is strong to weak upper semi-continuous;
(3) if $x_n \in D(T)$, $t_n \in [0,1]$, $f_n \in T_{t_n} x_n$ for each $n \geq 1$, $t_n \to t_0$ and $x_j \rightharpoonup^* x_0$ such that

$$\overline{\lim_{n\to\infty}} (f_n, x_n - x_0) \leq 0,$$

then $x_n \to x_0 \in D$ and $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ with $f_{n_k} \rightharpoonup f_0 \in T_{t_0} x_0$.

Definition 4.4 Let $T : D(T) \subseteq E^{**} \to 2^{E^*}$ be a mapping satisfying the conditions (1) and (2) in Definition 4.1. Let $\{x_j\} \subset D(T)$ with $x_j \rightharpoonup^* x_0 \in D(T)$ and $f_j \in Tx_j$ with $f_j \rightharpoonup f_0$. If $\limsup_{j\to\infty} (f_j, x_j - x_0) \leq 0$ implies that

$$f_0 \in Tx_0, \quad (f_0, x_0) = \lim_{j\to\infty} (f_j, x_j),$$

then T is called a *generalized pseudo-monotone mapping*.

Proposition 4.5 *Let $T : D(T) \subseteq E^{**} \to 2^{E^*}$ be a mapping of class (S_+) and $S : E^{**} \to E^*$ be a mapping with closed convex values. Then the following conclusions hold*:

(1) *if S is an upper semi-continuous and compact mapping, then $T + S$ is a mapping of class (S_+)*;
(2) *if S is a generalized pseudo-monotone mapping and weak compact, i.e., S maps bounded subsets in E^{**} to weak compact subsets in E^*, then $T + S$ is a mapping of class (S_+)*.

For any subspace F of E^{**}, let $J_F : F \to E^{**}$ be the natural inclusion and $J_F^* : E^{***} \to F^*$ be the conjugate mapping of j_F. Note that, under the canonical injection mapping $J : E^* \to E^{***}$, *i.e.*, $Jx(f) = f(x)$ for all $f \in E^{**}$ and $x \in E^*$, E^* can be injected as a subspace of E^{***} and so, in the following, we always regard E^* as a subspace of E^{***}.

First, we need the following result from [36] (also, see [3]).

Lemma 4.6 *Let F be a finite dimensional subspace, $\Omega \subset F$ be an open bounded subset and let $0 \in \Omega$. Let $T : \overline{\Omega} \to 2^{F^*}$ be an upper semi-continuous mapping with compact convex values, F_0 be a proper subspace of F, $\Omega_{F_0} = \Omega \cap F_0 \neq \emptyset$ and $T_{F_0} = j_{F_0}^* T : \overline{\Omega_{F_0}} \to 2^{F_0^*}$ be the Galerkin approximation of T, where $j_{F_0}^*$ is the adjoint mapping of the natural inclusion $j_{F_0} : F_0 \to F$. If $d(T, \Omega, 0) \neq d(T_{F_0}, \Omega_{F_0}, 0)$, then there exist $x \in \partial\Omega$ and $f \in Tx$ such that $(f, x) \leq 0$*

and $(f,v)=0$ *for all* $v\in F_0$, *where* $d(\cdot,\cdot,\cdot)$ *is the topological degree for upper semi-continuous mappings with compact convex values in finite dimensional spaces* (*see Ma* [52]).

Remark See [53, 54] for more references on degree theory of multivalued mappings.

Lemma 4.7 *Let* $T:\overline{\Omega}\to 2^{E^*}$ *be a bounded mapping of* (S_+) *and let* $0\notin T(\partial\Omega)$. *Then there exists a finite dimensional subspace* F_0 *of* E^{**} *such that*

$$0\notin T_F(\partial\Omega\cap F)$$

for all finite dimensional subspace F *of* E^{**} *with* $F_0\subseteq F$, *where* $T_F=j_F^*T$.

Under the condition of Lemma 4.7, we know that $\deg(T_F,\Omega\cap F,0)$ is well defined for the whole finite dimensional subspace F of E^{**} with $F_0\subseteq F$, where F_0 is the same as in Lemma 4.7.

Lemma 4.8 *Under the condition of Lemma* 4.7, *there exists a finite dimensional subspace* F_0 *of* E^{**} *such that* $\deg(T_F,\Omega\cap F,0)$ *does not depend on* F.

Now, let $\Omega\subset E^{**}$ be a nonempty open bounded subset and $T:\overline{\Omega}\to 2^{E^*}$ be a mapping of class (S_+). Suppose that $0\notin T(\partial\Omega)$. In view of Lemmas 4.6 and 4.8, we may define the topological degree as follows:

$$\deg\bigl(T,\Omega\cap D(T),0\bigr)=\deg(T_F,\Omega\cap F,0), \tag{4.1}$$

where F is a finite dimensional subspace of E^{**} such that $F_0\subset F$ and F_0 is the same as in Lemma 4.8.

Theorem 4.9 *If* $\deg(T,\Omega,0)\neq 0$, *then* $0\in Tx$ *has a solution in* Ω.

Proof The proof can be seen from the following proof of Theorem 4.10. □

Theorem 4.10 *Let* $\{T_t\}_{t\in[0,1]}$ *be a homotopy of mappings of class* (S_+). *If* $0\notin T_t(\partial\Omega)$ *for all* $t\in[0,1]$, *then* $\deg(T_t,\Omega,0)$ *does not depends on* $t\in[0,1]$.

Proof First, we claim that there exist finite dimensional subspaces F_0 of E^{**} such that $0\notin j_F^*T_t(\partial\Omega\cap F)$ for all finite dimensional subspaces F with $F_0\subset F$. Suppose that this is not true. For any finite dimensional subspaces F, we define a set W_F as follows:

$$\begin{aligned}W_F=\bigl\{(t,x)\in[0,1]\times\partial\Omega:\ &\text{there exists } f\in T_tx\\ &\text{such that } (f,x)\le 0 \text{ and } (f,v)=0,\forall v\in F\bigr\}.\end{aligned}$$

Then W_F is nonempty. Let $\overline{W_F}$ be the closure of W_F in $[0,1]\times E^{**}$ with E^{**} endowed with weak* topology. Consider the following family of sets:

$$\mathcal{F}=\bigl\{\overline{W_F}:F_0\subset F,\dim(F)\le\infty\bigr\}.$$

It is easy to show that $\bigcap_{F\in\mathcal{F}}\overline{W_F}\neq\emptyset$. Let $(t_0,x_0)\in\bigcap_{F\in\mathcal{F}}\overline{W_F}$. If, for each $v\in E^{**}$, we take a finite dimensional subspace F such that $v\in F$ and $x_0\in F$, then there exist $(t_j^v,x_j^v)\in W_F$ and $f_j^v\in T_{t_j^v}x_j^v$ such that

$$t_j^v\to t_0,\qquad x_j^v\rightharpoonup x_0,$$
$$\bigl(f_j^v,x_j^v\bigr)\le 0,\qquad \bigl(f_j^v,v\bigr)=0$$

for each $j\ge 0$. Hence we have

$$\limsup_{j\to\infty}\bigl(f_j^v,x_j^v-x_0\bigr)\le 0.$$

But, since $\{T_t:t\in[0,1]\}$ is a homotopy of mappings of class (S_+), it follows that $x_j^v\to x_0\in\partial\Omega$ and $\{f_j^v\}$ has a subsequence $\{f_{j_k}^v\}$ that converges weakly to $f_0^v\in T_{t_0}x_0$. Therefore, we have $(f_0^v,v)=0$. By Mazur's separation theorem (see [55]), we get $0\in T_{t_0}x_0$, which is a contradiction. The claim is completed. So, it follows that $\deg(T_{t,F},\Omega_F,0)$ is well defined for the whole finite dimensional subspace F with $F_0\subset F$.

Next, we prove that there exist a finite dimensional subspace F_1 and $F_0\subset F_1$ such that $\deg(T_{t,F},\Omega_F,0)$ does not depend on $t\in[0,1]$ for all finite dimensional subspace F of E^{**} with $F_1\subset F$.

Suppose that this is not true. For any finite dimensional subspace F with $F_0\subset F$, we define

$$W_F=\bigl\{(t,x)\in[0,1]\times\partial\Omega:\text{ there exists } f\in T_tx \text{ such that } (f,x)\le 0 \text{ and } (f,v)=0,\forall v\in F\bigr\}.$$

Then W_F is nonempty by Lemma 4.6. Let $\overline{W_F}$ be the closure of W_F in $[0,1]\times E^{**}$ with E^{**} endowed with the weak** topology. Consider again the following family of sets:

$$\mathcal{F}=\bigl\{\overline{W_F}:F_0\subset F\text{ with }\dim(F)\le\infty\bigr\}.$$

It is easy to show that $\bigcap_{F\in\mathcal{F}}\overline{W_F}\neq\emptyset$. Let $(t_0,x_0)\in\bigcap_{F\in\mathcal{F}}\overline{W_F}$. Then, for each $v\in E^{**}$, we take a finite dimensional subspace F such that $F_0\subset F$, $v\in F$ and $x_0\in F$. Then there exist $(t_j^v,x_j^v)\in W_F$ and $f_j^v\in T_{t_j^v}x_j^v$ such that

$$t_j^v\to t_0,\qquad x_j^v\rightharpoonup x_0,$$
$$\bigl(f_j^v,x_j^v\bigr)\le 0,\qquad \bigl(f_j^v,v\bigr)=0$$

for $j\ge 0$. Hence we have

$$\lim_{j\to\infty}\bigl(f_j^v,x_j^v-x_0\bigr)\le 0.$$

But, since $\{T_t:t\in[0,1]\}$ is a homotopy of mappings of class (S_+), we have $x_j^v\to x_0\in\partial\Omega$ and f_j^v has a subsequence $\{f_{j_k}^v\}$ which converges weakly to $f_0^v\in T_{t_0}x_0$. Therefore, we have $(f_0^v,v)=0$. Again, by Mazur's separation theorem, $0\in T_{t_0}x_0$, which is a contradiction. This completes the proof. □

Theorem 4.11 *Let* $T : \overline{\Omega} \to 2^{E^*}$ *be a mapping of class* (S_+), *where* $\Omega \subset E^{**}$ *is an open bounded subset. If* $0 \in \Omega$ *and* $(f, x) > 0$ *for all* $x \in \partial\Omega \cap D(T)$ *and* $f \in Tx$, *then*

$$\deg(T, \Omega, 0) = 1.$$

Proof Assume that F is a finite dimensional subspaces of E^{**}. It is straightforward to check that

$$\left(j_F^* f, x\right) > 0$$

for all $x \in \partial\Omega \cap F$ and $f \in Tx$. Therefore, we have $\deg(T_F, \Omega_F, 0) = 1$ and so, by (4.1),

$$\deg(T, \Omega, 0) = 1. \qquad \square$$

Theorem 4.12 *Let* $T : E^{**} \to 2^{E^*}$ *be a bounded mapping of class* (S_+). *If*

$$\lim_{\|x\| \to \infty} \inf_{f \in Tx} \frac{(f, x)}{\|x\|} = +\infty,$$

then $TE^{**} = E^*$.

Proof For each $p \in E^*$, we set $T_1 x = Tx - p$ for all $x \in E^{**}$. Then it is easy to see that T_1 is a mapping of class (S_+). One can easily see that $(f, x) > 0$ for all $x \in \partial B(0, R)$, $f \in T_1 x$ and sufficiently large R. Thus, by Theorem 4.11, $\deg(T_1, B(0, R), 0) = 1$ and so, by Theorem 4.9, $0 \in T_1 x$ has a solution in $B(0, R)$, *i.e.*, $p \in Tx$ has a solution in $B(0, R)$. This completes the proof. $\square$

In the following, we assume that E^{**} is separable and so we take any sequence $\{F_n\}$ of finite dimensional subspaces of E^{**} such that

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots, \qquad \overline{\bigcup_{n=1}^{\infty} F_n} = E^{**}. \tag{4.2}$$

Lemma 4.13 *Let* $\phi : D(\phi) \subseteq E^{**} \to R \cup \{+\infty\}$ *be a lower semi-continuous convex function in the weak* topology,* $\Omega \subset E^{**}$ *be open bounded and let* $x_1 \in D(\phi)$. *Suppose that* $\phi(x_1) < \phi(x)$ *for all* $x \in \partial\Omega \cap D(\phi)$. *Then there exists a positive integer* N *such that*

$$0 \notin \partial\phi_n\left(\partial\Omega \cap F_n' \cap D(\partial\phi_n)\right),$$

where $\phi_n : F_n' \to R \cup \{+\infty\}$ *is a mapping defined by* $\phi_n(x) = \phi(x)$ *for all* $x \in F_n'$ *and* $F_n' = \operatorname{span}(F_n \cup \{x_1\})$ *for all* $n > N$.

Proof Suppose that the conclusion is not true. There exists $x_n \in D(\phi)$ such that $0 \in \partial\phi_n(x_n)$ and so we have $\phi(x) - \phi(x_n) \geq 0$ for all $x \in F_n' \cap D(\phi)$, which contradicts $\phi(x_1) < \phi(x)$ for all $x \in \partial\Omega \cap D(\phi)$.

Under the assumption of Lemma 4.13, we know that there exists a positive integer N such that

$$0 \notin \partial\phi_n\left(\partial\Omega \cap F_n' \cap D(\partial\phi_n)\right)$$

for all $n > N$ and so, by [32], $\deg(\partial\phi_n, \Omega \cap F'_n, 0)$ is well defined. Now, we define a generalized degree as follows:

$$\begin{aligned}&\operatorname{Deg}\bigl(\partial^*\phi, \Omega \cap D\bigl(\partial^*\phi\bigr), 0\bigr)\\&\quad= \bigl\{k: \text{ there exists } F_n, n \geq 1, \text{ satisfying (4.2)}\\&\qquad \text{such that } \deg\bigl(\partial\phi_{n_j}, \Omega \cap F'_{n_j}, 0\bigr) \to k\bigr\}.\end{aligned}$$

□

Remark For generalized degree theory, see [56].

Theorem 4.14 *Let $\phi : D(\phi) \subseteq E^{**} \to R \cup \{+\infty\}$ be a lower semi-continuous convex function in the weak* topology. If $\lim_{\|x\|\to+\infty} \phi(x) = +\infty$, then*

$$\operatorname{Deg}\bigl(\partial^*\phi, B(0,r) \cap D\bigl(\partial^*\phi\bigr), 0\bigr) = \{1\}$$

for sufficiently large r.

Proof By the assumption $\lim_{\|x\|\to+\infty} \phi(x) = +\infty$, it follows from Proposition 2.4 that there exists $x_0 \in D(\phi)$ such that $\phi(x_0) = \inf_{x\in D(\phi)} \phi(x)$ if we take a large enough r such that $\phi(x_0) < \phi(x)$ for all $x \in D(\phi) \cap \partial B(0,r)$.

For any F_n ($n \geq 1$) satisfying (4.2), we put $F'_n = \operatorname{span}(F_n \cup \{x_0\})$. We may easily see that

$$\phi_n(x_0) = \inf_{x\in F_n\cap D(\phi)} \phi(x_n)$$

and so we have

$$(f, x) \geq 0$$

for all $x \in \partial B(0,r) \cap F'_n \cap D(\partial\phi_n)$. Thus we have

$$\deg\bigl(\partial\phi_n, B(0,r) \cap F'_n \cap D(\phi_n), 0\bigr) = 1$$

and, consequently, we have

$$\operatorname{Deg}\bigl(\partial^*\phi, B(0,r) \cap D\bigl(\partial^*\phi\bigr), 0\bigr) = \{1\}.$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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