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A note on 'Coupled fixed point theorems for mixed *g*-monotone mappings in partially ordered metric spaces'

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Abstract

Recently, some (common) coupled fixed theorems in various abstract spaces have appeared as a generalization of existing (usual) fixed point results. Unexpectedly, we noticed that most of such (common) coupled fixed theorems are either weaker or equivalent to existing fixed point results in the literature. In particular, we prove that the very recent paper of Turkoglu and Sangurlu 'Coupled fixed point theorems for mixed *g*-monotone mappings in partially ordered metric spaces [Fixed Point Theory and Applications 2013, 2013:348]' can be considered as a consequence of the existing fixed point theorems on the topic in the literature. Furthermore, we give an example to illustrate that the main results of Turkoglu and Sangurlu (Fixed Point Theory Appl. 2013:348, 2013) has limited applicability compared to the mentioned existing fixed point result.

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1 Introduction and preliminaries

In many recent publications in fixed point theory auxiliary functions are used to generalize the contractive conditions on the maps defined on various spaces. On the other hand, as appears in some studies, not all of these generalizations are meaningful. Moreover, some of the results are equivalent to, or even weaker than the existing theorems.

In this paper we discuss the insufficiency of one of these recent generalizations given by Turkoglu and Sangurlu in [1]. Our discussion can also be applied to revise some other existing results.

We first recall the definition of the following auxiliary functions and some of their basic properties.

Let Φ denote all functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy

(1) φ is continuous and nondecreasing,

(2) $\varphi(t) = 0$ if and only if $t = 0$,

(3) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$, $\forall t, s \in [0,\infty)$,

Let Ψ denote all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

For consistency, we use the following definitions of coupled fixed point, coupled common fixed point and coupled coincidence point.

Definition 1.1 Let $F : X \times X \to X$ and $g : X \to X$ be given mappings.

(1) A point $(x, y) \in X \times X$ is called a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.
(2) A point $(x, y) \in X \times X$ is called a coupled coincidence point of F and g if $gx = F(x, y)$ and $gy = F(y, x)$.
(3) A point $(x, y) \in X \times X$ is called a coupled common fixed point of F if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.2 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. The mapping F is said to have mixed monotone property on X if it is monotone nondecreasing in x and monotone nonincreasing in y, that is,

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y), \\ &y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_1) \geq F(x, y_2). \end{aligned} \tag{1}$$

We next recollect the main results of Turkoglu and Sangurlu [1] by removing the typos and emphasizing the definitions of auxiliary functions which are missing in the original paper. Notice also that in the original paper of Turkoglu and Sangurlu [1], Theorem 4 is superfluous, since it is a consequence of Theorem 5.

Theorem 1.1 [1] *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ and F, g satisfy*

$$\varphi\big(d\big(F(x,y), F(u,v)\big)\big) \leq \frac{1}{2}\varphi\big(d(gx,gu) + d(gy,gv)\big) - \psi\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) \tag{2}$$

for all $x, y, u, v \in X$ with $gx \leq gu$ and $gy \geq gv$, $F(X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous.

Suppose that either

(1) *F is continuous or*
(2) *X has the following property*:
    (a) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*
    (b) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

Then there exist $x, y \in X$ such that $x = gx = F(x, y)$ and $y = gy = F(y, x)$, that is, F and g have a coupled common fixed point in $X \times X$.

2 Main results

In the following example, we shall emphasize the insufficiency of the main results of Turkoglu and Sangurlu [1].

Example 2.1 Let $X = [0, \infty)$ be endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the maps $F : X \times X \to X$ and $g : X \to X$ by $F(x, y) = \frac{3}{5}x - \frac{1}{5}y$ and $g(x) = x$ for all $x, y \in X$. Then for all $x, y, u, v \in X$ with $y = v$, we have

$$d\big(F(x,y), F(u,v)\big) = \frac{3}{5}|x - u| \quad \text{and} \quad d(gx, gy) + d(gy, gv) = |x - u|. \tag{3}$$

Thus,

$$d\bigl(F(x,y),F(u,v)\bigr) > \frac{1}{2}\bigl[d(gx,gy) + d(gy,gv)\bigr].$$

Regarding the nondecreasing character of the functions in Φ, we deduce that

$$\varphi\bigl(d\bigl(F(x,y),F(u,v)\bigr)\bigr) > \frac{1}{2}\varphi\bigl(d(gx,gy) + d(gy,gv)\bigr).$$

Since the functions in the class Ψ are nonnegative, it is impossible to satisfy the inequality (2) for any function $\psi \in \Psi$. Hence, Theorem 1.1 cannot provide the existence of a coupled common fixed point of F and g. On the other hand, it is easy to see that $(0,0)$ is a coupled common fixed point of F and g.

The weakness of Theorem 1.1 can also be observed with the following theorem.

Theorem 2.1 *Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$. Suppose that there exist $\varphi \in \Phi$, $\psi \in \Psi$ for which F and g satisfy*

$$\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \leq \varphi\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) - \psi\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) \tag{4}$$

for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$, where $F(X \times X) \subseteq g(X)$, $g(X)$ is complete and g is continuous.

Suppose that either

(1) *F is continuous or*

(2) *X has the following property:*

  (a) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*

  (b) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

Then there exist $x,y \in X$ such that $x = gx = F(x,y)$ and $y = gy = F(y,x)$, that is, F and g have a coupled common fixed point in $X \times X$.

Proof The proof of this theorem is standard. Indeed, the desired result is obtained by mimicking the lines in the proof of Turkoglu and Sangurlu [1]. Since there is no difficulty in this process, we omit the details. □

Notice that if take $g(x) = x$ in Theorem 2.1, then we derive the following result, which was proved in [2].

Theorem 2.2 *Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0,y_0)$*

and $y_0 \geq F(y_0, x_0)$. *Suppose that there exist* $\varphi \in \Phi$, $\psi \in \Psi$ *and that* F *satisfies*

$$\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \leq \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right) \tag{5}$$

for all $x, y, u, v \in X$ *with* $x \leq u$ *and* $y \geq v$. *Suppose that either*

(1) F *is continuous or*

(2) X *has the following property*:

  (a) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$,

  (b) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* $n \in \mathbb{N}$.

Then there exist $x, y \in X$ *such that* $x = F(x,y)$ *and* $y = F(y,x)$, *that is*, F *has a coupled fixed point in* $X \times X$.

Lemma 2.1 [3] *Let* X *be a nonempty set and* $T : X \to X$ *be a function. Then there exists a subset* $E \subseteq X$ *such that* $T(E) = T(X)$ *and* $T : E \to X$ *is one-to-one.*

Theorem 2.3 *Theorem* 2.1 *is a consequence of Theorem* 2.2.

Proof By Lemma 2.1, there exists $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. Define a map $G : g(E) \times g(E) \to g(E)$ by $G(gx, gy) = F(x,y)$ and $G(gy, gx) = F(y,x)$. Since g is one-to-one on $g(E)$, G is well defined. Note that

$$\begin{aligned}
&\varphi\left(\frac{d(G(gx,gy),G(gu,gv)) + d(G(gy,gx),G(gv,vu))}{2}\right) \\
&\quad = \varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \\
&\quad \leq \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right)
\end{aligned} \tag{6}$$

for all $gx, gy \in g(E)$. Since $g(E) = g(X)$ is complete, by using Theorem 2.2, there exist $x_0, y_0 \in X$ such that $G(gx_0, gy_0) = gx_0$ and $G(gy_0, gx_0) = gy_0$. Hence, F and g have a coupled coincidence point. □

The complicated contractive conditions of the aforementioned theorems can be simplified considerably by means of the following notations. Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and $F : X \times X \to X$ be a given mapping. We define a partial order $\preceq_2$ on the product set $X \times X$ as

$$(x,y),(u,v) \in X \times X, \quad (x,y) \preceq_2 (u,v) \quad \Leftrightarrow \quad x \preceq u, \quad y \succeq v. \tag{7}$$

Let $Y = X \times X$. It is easy to show that the mapping $\eta : Y \times Y \to [0, \infty)$ defined by

$$\eta\big((x,y),(u,v)\big) = d(x,u) + d(y,v) \tag{8}$$

for all $(x,y),(u,v) \in Y$ is a metric on Y.

Now, define the mapping $T_F : Y \to Y$ by

$$T_F(x,y) = \big(F(x,y), F(y,x)\big) \quad \text{for all } (x,y) \in Y. \tag{9}$$

The following properties can easily be seen.

Lemma 2.2 (*see e.g.* [4]) *If X and Y are the metric spaces defined above, then the following properties hold.*

(1) *(X,d) is complete if and only if (Y,η) is complete*;
(2) *F has the mixed monotone property if and only if T_F is monotone nondecreasing with respect to $\preceq_2$*;
(3) *$(x,y) \in X \times X$ is a coupled fixed point of F if and only if (x,y) is a fixed point of T_F.*

Theorem 2.4 *Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists $x_0 \in X$ with $x_0 \leq T(x_0)$. Suppose also that there exist $\varphi \in \Phi$, $\psi \in \Psi$ satisfying*

$$\varphi\big(d(Tx,Ty)\big) \leq \varphi\big(d(x,y)\big) - \psi\big(d(x,y)\big) \tag{10}$$

for all $x,y \in X$. Suppose that either

(1) *T is continuous or*
(2) *X has the following property*: *If a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.*

Then there exists $x \in X$ such that $x = Tx$.

We skip the proof of Theorem 2.4 since it is standard and can be found easily in the literature; see *e.g.* [5]. More precisely, it is the analog of the proof given in [1].

Theorem 2.5 *Theorem* 2.2 *follows from Theorem* 2.4.

Proof Notice that (5) is equivalent to

$$\varphi\left(\frac{\eta(T_F((x,y),(u,v)))}{2}\right) \leq \varphi\left(\frac{\eta((x,y),(u,v))}{2}\right) - \psi\left(\frac{\eta((x,y),(u,v))}{2}\right). \tag{11}$$

By Lemma 2.2, all conditions of Theorem 2.4 are satisfied. To finalize the proof, we let

$$d_2\big((x,y),(u,v)\big) = \frac{\eta((x,y),(u,v))}{2} = \frac{d(x,u)+d(y,v)}{2}.$$

Hence, (5) turns into

$$\varphi\big(d_2\big(T_F(x,y),T_F(u,v)\big)\big) \leq \varphi\big(d_2\big((x,y),(u,v)\big)\big) - \psi\big(d_2\big((x,y),(u,v)\big)\big). \tag{12}$$

□

Theorem 2.6 *Theorem* 2.1 *is a consequence of Theorem* 2.4.

Proof Since Theorem 2.1 is a consequence of Theorem 2.2, which follows from Theorem 2.4, then Theorem 2.1 is a consequence of Theorem 2.4. □

3 Conclusion

We have presented evidence, that is, Example 2.1 and Theorem 2.1, that the results of Theorem 1.1 in [1] have more limited area of application than some existing results in the literature. Moreover, the generalizations and equivalences given in this paper can be used to show that other published theorems in the literature are in fact consequences of these generalizations. In particular, Theorem 2.11 in [6] is a consequence of Theorem 2.4. As a matter of fact, this note can be seen as a continuation of the discussion given in [4].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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