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Fuzzy games with a countable space of actions and applications to systems of generalized quasi-variational inequalities

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Abstract

We introduce an abstract fuzzy economy (generalized fuzzy game) model with a countable space of actions, and we study the existence of fuzzy equilibrium. As application, we prove the existence of solutions for the systems of generalized quasi-variational inequalities with random fuzzy mappings, defined in this paper. Our results bring novelty to the current literature by considering random fuzzy mappings whose values are fuzzy sets over complete countable metric spaces.

Keywords: abstract fuzzy economy; fuzzy equilibrium; incomplete information; random fixed point; random quasi-variational inequalities; random fuzzy mapping

1 Introduction

The classical abstract economy (or generalized game) model as formalized by Borglin and Keiding [1] or Shafer and Sonnenschein [2] consists of a finite set of agents, each characterized by certain constraints and preferences, described by correspondences. This model has played a central role in the study of the economies considered by Arrow and Debreu [3], and it has been generalized in many directions during the last decades. Kim and Lee [4] proved that the theory of fuzzy sets, initiated by Zadeh [5], has become a good framework for studying the equilibrium existence in the case of generalized games when the constraints or preferences are imprecise due to the agents' behavior. These authors proved the first theorems related to this topic. For newer results, the reader should refer, for instance, to [6–8].

The purpose of this paper is twofold. Firstly, it extends the study of the abstract economy model with private information and a countable set of actions defined by the author (see [9]) in the fuzzy setting, by taking into account those situations in which the agents may have partial control over the actions they choose. Special attention is paid to the existence of fuzzy equilibrium. The uncertainties derived by the individual character of the agents in election situations are described with the help of fuzzy random mappings, as defined by Huang [10]. Secondly, the theorem regarding the existence of fuzzy equilibrium for the fuzzy games with countable action sets is applied to prove the existence of solutions for the systems of generalized quasi-equilibrium inequalities with random fuzzy mappings, which we introduce. The present work is motivated and inspired by the research going on in this field. We mention that Chang and Zhu [11] are the ones who first introduced the

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concept of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities in the deterministic and fuzzy setting have been studied. For example, we quote Noor [12], Park and Jeong [13] and Ding and Park [14]. The random variational inequalities have been considered and studied recently (see, for instance, [15–19]). The results of this paper, including the ones derived in the non-fuzzy setting, are brand new and have not been reported in literature previously.

The rest of the paper is organized as follows. Some notational and terminological conventions are given in the following section, then Section 3.1 introduces the model of an abstract fuzzy economy with private information and a countable space of actions. The main result concerns the existence of fuzzy equilibrium, stated in Section 3.2. Section 4.1 defines new types of systems of random quasi-variational inequalities with random fuzzy mappings. The existence of their solutions is proved in Section 4.2. Section 4.3 contains the results in a non-fuzzy setting, shown as consequences of the previously obtained theorems. Finally, Section 5 presents the conclusions of our research.

2 Notation and definitions

The fuzzy set theory, introduced by Zadeh [5], is a framework very much used in mathematical economics. The fuzzy sets are characterized by the fact that the degree of membership of any of its elements can be any number in the unit interval $[0,1]$, as opposed to the fact that only the binary pair $\{0,1\}$ can give candidates for the degree of membership of the elements of crisp sets.

We present below some notions related to this theory.

If Y is a topological space, then a function A from Y into $[0,1]$ is called a fuzzy set on Y (see Chang [20]). The family of all fuzzy sets on Y is denoted by $\mathcal{F}(Y)$. If X and Y are topological spaces, then a mapping $P: X \to \mathcal{F}(Y)$ is called a *fuzzy mapping*. If P is a fuzzy mapping, then for each $x \in X$, $P(x)$ is a fuzzy set on Y and $P(x)(y) \in [0,1]$, $y \in Y$ is called the *degree of membership of* y in $P(x)$. Let $A \in \mathcal{F}(Y)$, $a \in [0,1]$, then the set $(A)_a = \{y \in Y : A(y) > a\}$ is called a *strong a-cut set* of the fuzzy set A.

For the reader's convenience, we review a few basic definitions and results concerning the measurability of correspondences.

Let $(T, \mathcal{T})$ be a measurable space, Y be a topological space and $F: T \to 2^Y$ be a correspondence. F is *weakly measurable* if $F^l(A) \in \mathcal{T}$ for each open subset A of Y (the lower inverse F^l, also called the *weak inverse*, of a subset A of Y is defined by $F^l(A) = \{x \in A : F(x) \cap A \neq \emptyset\}$). F is *measurable* if $F^l(A) \in \mathcal{T}$ for each closed subset A of Y. If $(T, \mathcal{T})$ is a measurable space, Y is a countable set and $F: T \to 2^Y$ is a correspondence, then F is measurable if for each $y \in Y$, $F^{-1}(y) = \{t \in T : y \in F(t)\}$ is $\mathcal{T}$-measurable. For a correspondence $F: T \to 2^Y$ from a measurable space into a metrizable space, we have that if F is measurable, then it is also weakly measurable and if F is compact valued and weakly measurable, it is measurable. Different aspects on measurability of correspondences can be found, for instance, in [21–24].

Random fuzzy mappings have been defined in order to model random mechanisms generating imprecisely-valued data which can be properly described by using fuzzy sets.

Let Y be a topological space, $\mathcal{F}(Y)$ be a collection of all fuzzy sets over Y and $(\Omega, \mathcal{F})$ be a measurable space. A fuzzy mapping $P: \Omega \to \mathcal{F}(Y)$ is said to be *measurable* (see [25]) if for any given $a \in [0,1]$, $(P(\cdot))_a : \Omega \to 2^Y$ is a measurable set-valued mapping. A fuzzy mapping $P: \Omega \to \mathcal{F}(Y)$ is said *to have a measurable graph* if for any given $a \in [0,1]$, the set-valued mapping $(P(\cdot))_a : \Omega \to 2^Y$ has a measurable graph. A fuzzy mapping $P : \Omega \times X \to \mathcal{F}(Y)$ is called a *random fuzzy mapping* if, for any given $x \in X$, $P(\cdot, x) : \Omega \to \mathcal{F}(Y)$ is a measurable fuzzy mapping.

3 Fuzzy equilibrium existence for abstract fuzzy economies with private information and a countable set of actions

A new direction in mathematical economics and in game theory concerns the fact that the uncertainties which characterize the individual feature of agents' decisions, involved in different economic activities, must be included in the mathematical models. The uncertainties can be described using random fuzzy mappings. Thus the framework of fuzziness has become part of the language of applied mathematics.

This section is devoted to defining a new model of abstract fuzzy economy with private information and a countable set of actions and to proving the existence of fuzzy equilibrium. The model is an extension of the classical deterministic models of abstract economy due to Borglin and Keiding [1] or Shafer and Sonnenschein [2]. They considered only a finite set of agents. This model generalizes the one defined in [9] by considering the fuzzy setting. It is different from other models of abstract fuzzy economy with private information existent in the literature (see [26] or [27]), because the values of random fuzzy constraint mappings and of random fuzzy preference mappings are fuzzy sets over countable complete metric spaces and the theory of the distributions of correspondences is used. Other models introduced by the author and based on measurability requirements are those from [7, 28, 29].

3.1 The model of an abstract fuzzy economy with private information

The abstract economy is an intermediate model which makes the connection between the Nash-type general ones, which describe the general competitive situations, without being specific (see [30]) and the one of Arrow-Debreu, which belongs to the economic field and therefore is a specialized model (see [3]).

We define a new model of an abstract fuzzy economy with private information and a countable set of actions.

Definition 1 An abstract fuzzy economy (or a generalized fuzzy game) with private information and a countable space of actions is defined as follows:

$$\Gamma = \bigl(\bigl((\Omega_i, \mathcal{Z}_i)_{i\in I}, \mu\bigr), \bigl(S_i, X_i, (A_i, a_i), (P_i, p_i)\bigr)_{i\in I}\bigr),$$

where I is a non-empty finite set (the set of agents) and:

(a) $X_i : \Omega_i \to \mathcal{F}(S_i)$ is the action (strategy) fuzzy mapping of agent i;
(b) $A_i : \Omega_i \times \mathcal{D}_{X,z} \to \mathcal{F}(S_i)$ is a random fuzzy mapping (the constraint mapping of agent i);
(c) $P_i : \Omega_i \times \mathcal{D}_{X,z} \to \mathcal{F}(S_i)$ is a random fuzzy mapping (the preference mapping of agent i);
(d) $a_i : \mathcal{D}_{X,z} \to (0,1]$ is a random fuzzy constraint function and $p_i : \mathcal{D}_{X,z} \to (0,1]$ is a random fuzzy preference function;
(e) $z_i \in (0,1]$ is such that for all $(\omega_i, h_g) \in \Omega_i \times \mathcal{D}_{X,z}$, $(A_i(\omega_i, h_g))_{a_i(h_g)} \subset (X_i(\omega_i))_{z_i}$ and $(P_i(\omega, h_g))_{p_i(h_g)} \subset (X_i(\omega))_{z_i}$.

Now, we will explain the elements of the model and we will also give an interpretation.

I is a non-empty finite set (the set of agents). For each $i \in I$, the space of actions S_i is a countable complete metric space and $(\Omega_i, \mathcal{Z}_i)$ is a measurable space. $(\Omega, \mathcal{F})$ is the product measurable space $(\prod_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{Z}_i)$, and μ is a probability measure on $(\Omega, \mathcal{F})$. For a point $\omega = (\omega_1, \dots, \omega_n) \in \Omega$, we define the coordinate projections $\tau_i(\omega) = \omega_i$. The random mapping $\tau_i(\omega)$ is interpreted as player i's private information related to his action.

We also denote, for each $i \in I$, $\mathrm{Meas}(\Omega_i, S_i)$ the set of measurable mappings from $(\Omega_i, \mathcal{Z}_i)$ to S_i. An element g_i of $\mathrm{Meas}(\Omega_i, S_i)$ is called a *pure strategy* for player i. A *pure strategy profile* g is an n-vector function $(g_1, g_2, \dots, g_n)$ that specifies a pure strategy for each player.

We suppose that there exists a fuzzy mapping $X_i : \Omega_i \to \mathcal{F}(S_i)$ such that each agent i can choose an action from $(X_i(\omega_i))_{z_i} \subset S_i$ for each $\omega_i \in \Omega_i$. The function $g_i : \Omega_i \to S_i$ is said to be a selection of $(X_i(\cdot))_{z_i}$ if $g_i(\omega_i) \in (X_i(\omega_i))_{z_i}$ for every $\omega_i \in \Omega_i$.

$D_{(X_i(\cdot))_{z_i}}$ is the set $\{(\mu\tau_i^{-1})g_i^{-1} : g_i \text{ is a measurable selection of } (X_i(\cdot))_{z_i}\}$ and $\mathcal{D}_{X,z} := \prod_{i\in I} \mathcal{D}_{(X_i(\cdot))_{z_i}}$.

For each $i \in I$, we denote $h_{g_i} = (\mu\tau_i^{-1})g_i^{-1}$, where g_i is a measurable selection of $(X_i(\cdot))_{z_i}$ and $h_g = (h_{g_1}, h_{g_2}, \dots, h_{g_n})$.

For each agent i, the constraints and the preferences are described by using the random fuzzy mappings A_i respectively P_i. In the state of the world $\omega \in \Omega = \prod_{i\in I} \Omega_i$, the number $P_i(\omega_i, h_g)(y) \in [0,1]$ associated to (h_g, y) can be interpreted as the degree of intensity with which y is preferred to $g_i(\omega_i)$ or the degree of truth with which y is preferred to $g_i(\omega_i)$. We also can see the value $A_i(\omega_i, h_g)(y) \in [0,1]$, associated to (h_g, y), as the belief of the player i that in the state ω_i he can choose $y \in Y$. The element z_i is the action level in each state of the world, $a_i(h_g)$ expresses the perceived degree of feasibility of the strategy g and $p_i(h_g)$ represents the preference level of the strategy g.

The notion of fuzzy equilibrium for this model is introduced below. It generalizes the deterministic definition of equilibrium we owe to Shafer and Sonnenschein [2] and the stochastic one proposed by the author in [9].

Definition 2 A fuzzy equilibrium for Γ is defined as a strategy profile $g^* = (g_1^*, g_2^*, \dots, g_n^*) \in \prod_{i\in I} \mathrm{Meas}(\Omega_i, S_i)$ such that for each $i \in I$:

(1) $g_i^*(\omega_i) \in (A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})}$ for each $\omega_i \in \Omega_i$;

(2) $(A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})} \cap (P_i(\omega_i, h_{g^*}))_{p_i(h_{g^*})} = \phi$ for each $\omega_i \in \Omega_i$.

3.2 Existence of fuzzy equilibrium for abstract fuzzy economies with a countable set of actions

This subsection is designed to establish the existence of fuzzy equilibrium of the abstract fuzzy economies. The assumptions concern the measurability and the upper semicontinuity of correspondences which define the model.

The following definitions and properties are essential tools used to prove the existence of fuzzy equilibrium.

Let X, Y be topological spaces and $F : X \to 2^Y$ be a correspondence. F is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $F(x) \subset V$, there exists an open neighborhood U of x in X such that $F(y) \subset V$ for each $y \in U$.

Lemma 1 [19] *Let X and Y be two topological spaces and let A be an open subset of X. Suppose that $F_1 : X \to 2^Y$, $F_2 : X \to 2^Y$ are upper semicontinuous such that $F_2(x) \subset F_1(x)$*

for all $x \in A$. *Then the correspondence* $F : X \to 2^Y$ *defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A \end{cases}$$

is also upper semicontinuous.

We follow Yu and Zhang [31]. Let Y be a countable complete metric space, $(T, \mathcal{T}, \lambda)$ be an atomless probability space and $F : T \to 2^Y$ be a measurable correspondence. The function $f : T \to Y$ is said to be *a selection of* F if $f(t) \in F(t)$ for λ-almost $t \in T$. Let us denote $\mathcal{D}_F = \{\lambda f^{-1} : f \text{ is a measurable selection of } F\}$.

We will present some regular properties of $\mathcal{D}_F$, also obtained by Yu and Zhang [31]. The next lemmata state the convexity and the compactness of $\mathcal{D}_F$ for any correspondence F.

Lemma 2 [31] *Let* Y *be a countable complete metric space*, $(T, \mathcal{T}, \lambda)$ *be an atomless probability space and* $F : T \to 2^Y$ *be a measurable correspondence. Then* $\mathcal{D}_F$ *is non-empty and convex in the space* $\mathcal{M}(Y)$ *- the space of probability measures on* Y, *endowed with the topology of weak convergence*.

Lemma 3 [31] *Let* Y *be a countable complete metric space*, $(T, \mathcal{T}, \lambda)$ *be an atomless probability space and* $F : T \to 2^Y$ *be a measurable correspondence. If* F *is compact valued, then* $\mathcal{D}_F$ *is compact in* $\mathcal{M}(Y)$.

Lemma 4 [31] *Let* X *be a metric space*, $(T, \mathcal{T}, \lambda)$ *be an atomless probability space*, Y *be a countable complete metric space and* $F : T \times X \to 2^Y$ *be a correspondence. Let us assume that for any fixed* x *in* X, $F(\cdot, x)$ (*also denoted by* F_x) *is a compact-valued measurable correspondence, and for each fixed* $t \in T$, $F(t, \cdot)$ *is upper semicontinuous on* X. *Also, let us assume that there exists a compact-valued correspondence* $H : T \times X \to 2^Y$ *such that* $F(t, x) \subset H(t)$ *for all* t *and* x. *Then* $\mathcal{D}_{F_x}$ *is upper semicontinuous on* X.

We will also need the Kuratowski-Ryll-Nardzewski selection theorem in order to prove our main result.

Theorem 1 (Kuratowski-Ryll-Nardzewski selection theorem) [32] *A weakly measurable correspondence with non-empty closed values from a measurable space into a Polish space admits a measurable selector.*

Let us denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{D}_{X,z}$.

Theorem 2 is our main result in this paper. It generalizes Theorem 3 in [9] by considering the fuzzy framework. We emphasize that our arguments to the existence of fuzzy equilibrium are different from the approaches used in literature, since the modeling of the private information and the countable space of actions assumes a new setting. The proof relies in particular on the Kuratowski-Ryll-Nardzewski selection theorem and on the Fan fixed point theorem. The possible applications of this result may concern the aspects of the economies or markets in which countable aspects matter.

Theorem 2 *Let* $\Gamma = (((\Omega_i, \mathcal{Z}_i)_{i\in I}, \mu), (S_i, X_i, (A_i, a_i), (P_i, p_i))_{i\in I})$ *be an abstract fuzzy economy, with private information and countable spaces of actions, where* μ *is atomless and for each* $i \in I$:

(a) *the correspondence* $(X_i(\cdot))_{z_i} : \Omega_i \to 2^{S_i}$ *is compact valued*;

(b) *for each* $\lambda \in \mathcal{D}_{X,z}$, *the correspondence* $(A_i(\cdot, \lambda))_{a_i(\lambda)} : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, *the correspondence* $(A_i(\omega_i, \cdot))_{a_i(\cdot)} : \mathcal{D}_{X,z} \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values*;

(c) *for each* $\lambda \in \mathcal{D}_{X,z}$, *the correspondence* $(P_i(\cdot, \lambda))_{p_i(\lambda)} : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, *the correspondence* $(P_i(\omega_i, \cdot))_{p_i(\cdot)} : \mathcal{D}_{X,z} \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values*;

(d) *for each* $\omega_i \in \Omega_i$ *and each* $g \in \prod_{i\in I} \text{Meas}(\Omega_i, S_i)$, $g_i(\omega_i) \notin (P_i(\omega_i, h_g))_{p_i(h_g)}$;

(e) *the set* $U_i^{\omega_i} := \{\lambda \in \mathcal{D}_{X,z} : (A_i(\omega_i, \lambda))_{a_i(\lambda)} \cap P_i((\omega_i, \lambda))_{p_i(\lambda)} = \emptyset\}$ *is open in* $\mathcal{D}_{X,z}$ *for each* $\omega_i \in \Omega_i$.

Then there exists $g^* \in \prod_{i\in I} \text{Meas}(\Omega_i, S_i)$ *an equilibrium for* Γ.

Proof The fixed point approach can be effectively exploited. To do this, we will construct several correspondences.

Let $i \in I$ be fixed.

Let us denote

$$U_i := \{(\omega_i, \lambda) \in \Omega_i \times \mathcal{D}_{X,z} : (A_i(\omega_i, \lambda))_{a_i(\lambda)} \cap P_i((\omega_i, \lambda))_{p_i(\lambda)} = \emptyset\} \quad \text{and}$$
$$U_i^{\omega_i} := \{\lambda \in \mathcal{D}_{X,z} : (A_i(\omega_i, \lambda))_{a_i(\lambda)} \cap P_i((\omega_i, \lambda))_{p_i(\lambda)} = \emptyset\}.$$

Let us define $F_i : \Omega_i \times \mathcal{D}_{X,z} \to 2^{S_i}$ by

$$F_i(\omega_i, \lambda) = \begin{cases} (A_i(\omega_i, \lambda))_{a_i(\lambda)} \cap (P_i(\omega_i, \lambda))_{p_i(\lambda)} & \text{if } (\omega_i, \lambda) \notin U_i, \\ (A_i(\omega_i, \lambda))_{a_i(\lambda)} & \text{if } (\omega_i, \lambda) \in U_i \end{cases} \quad \text{and}$$
$$\Phi : \mathcal{D}_{X,z} \to 2^{\mathcal{D}_{X,z}}, \quad \Phi(\lambda) = \prod_{i\in I} \mathcal{D}_{F_i}(\lambda)$$

for each $\lambda \in \mathcal{D}_{X,z}$, where $\mathcal{D}_{F_i}(\lambda) = \{h_{g_i} = (\mu\tau_i^{-1})g_i^{-1} : g_i \text{ is a measurable selection of } F_i(\cdot, \lambda)\}$.

We will apply the Ky Fan fixed point theorem to the correspondence Φ and we will obtain the existence of a fixed point, which will be the equilibrium point for the abstract economy Γ. For this purpose, we check the properties of the involved sets and the correspondences F_i and Φ.

Firstly, we note that $D_{(X_i(\cdot))_{z_i}}$ is non-empty and convex according to Lemma 2 and it is compact according to Lemma 3. Consequently, the set $\mathcal{D}_{X,z}$ is also non-empty, compact and convex.

According to assumptions (b) and (c), the correspondence F_i has non-empty and compact values and it is measurable with respect to Ω_i. Assumption (e) implies that the set $U_i^{\omega_i}$ is open in $\mathcal{D}_{X,z}$ and assumptions (b) and (c) imply that for all $\omega_i \in \Omega_i$, $(A_i(\omega_i, \cdot))_{a_i(\cdot)}, (P_i(\omega_i, \cdot))_{p_i(\cdot)} : D_{(X(\cdot))_z} \to 2^{S_i}$ are upper semicontinuous; therefore, we can apply Lemma 1 to assert that F_i is upper semicontinuous with respect to $\lambda \in \mathcal{D}_{X,z}$.

Furthermore, for each $\lambda \in \mathcal{D}_{X,z}$, $\mathcal{D}_{F_i}(\lambda)$ is non-empty, convex and compact. The non-emptiness of each $\mathcal{D}_{F_i}(\lambda)$ is implied by the existence of a measurable selection from the correspondence F_i, according to the Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2 and Lemma 3 also assure the convexity and compactness of the set $\mathcal{D}_{F_i}(\lambda)$, where $\lambda \in \mathcal{D}_{X,z}$.

According to Lemma 4, the correspondence $\mathcal{D}_{F_i}$ is upper semicontinuous. Then the correspondence Φ is upper semicontinuous and has non-empty, compact and convex values. We have also proved that it is defined on a non-empty, convex and compact set. We can apply the Ky Fan fixed point theorem [33] to Φ, and we obtain that there exists a fixed point $\lambda^* \in \Phi(\lambda^*)$. In particular, for each player i, $\lambda_i^* \in \mathcal{D}_{F_i}(\lambda^*)$. From the definition of $\mathcal{D}_{F_i}(\lambda^*)$, we conclude that for each player i, there exists $g_i^* \in \operatorname{Meas}(\Omega_i, S_i)$ such that g_i^* is a selection of $F_i(\cdot, \lambda^*)$ and $h_{g_i^*} = (\mu\tau_i^{-1})(g_i^*)^{-1} = \lambda_i^*$. Let us denote $h_{g^*} = (h_{g_1^*}, \dots, h_{g_n^*})$.

We prove that g^* is an equilibrium for Γ. For each $i \in I$, because g_i^* is a selection of $F_i(\cdot, h_{g_1^*}, \dots, h_{g_n^*})$, it follows that $g_i^*(\omega_i) \in (A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})} \cap (P_i(\omega_i, h_{g^*}))_{p_i(h_{g^*})}$ if $(\omega_i, h_{g^*}) \notin U_i$ or $g_i^*(\omega_i) \in (A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})}$ if $(\omega_i, h_{g^*}) \in U_i$.

According to assumption (d), it follows that $g_i^*(\omega_i) \notin (P_i(\omega_i, h_{g^*}))_{p_i(h_{g^*})}$ for each $\omega_i \in \Omega_i$. Then $g_i^*(\omega_i) \in (A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})}$ and $(\omega_i, h_{g^*}) \in U_i$. This is equivalent to the fact that $g_i^*(\omega_i) \in (A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})}$ and $(A_i(\omega_i, h_{g^*}))_{a_i(h_{g^*})} \cap P_i(\omega_i, h_{g^*})_{p_i(h_{g^*})} = \emptyset$ for each $\omega_i \in \Omega_i$. Consequently, $g^* = (g_1^*, g_2^*, \dots, g_n^*)$ is an equilibrium for Γ. □

4 Random quasi-variational inequalities with random fuzzy mappings

The techniques to prove the equilibrium existence for the abstract economies are often used to solve other problems related to this field, especially the variational inequalities, mini-max theorems, some classes of quasi-equilibrium problems and the existence of equilibrium for the exchange economies. A number of applications of the results from Section 3 can be further obtained. We focus here on the definition of a new type of systems of quasi-variational inequalities with random fuzzy mappings whose values are fuzzy sets over complete countable metric spaces and on the proof of the existence of their solutions.

The theory of variational inequality considers a large spectrum of interesting and important tools used in various fields of mathematics, physics, economics and engineering sciences.

4.1 New types of systems of generalized quasi-variational inequalities

Motivated by the large literature developed since Chang and Zhu [11] introduced the concept of variational inequalities for fuzzy mappings and since Noor and Elsanousi [17] defined the notion of a random variational inequality, we also propose new types of systems of generalized quasi-variational inequalities, which seem to be very welcome in this context.

We will use the same setting as in Section 3.

For each $i \in I$, let S_i be a countable complete metric space. Let $A_i : \Omega_i \times \mathcal{D}_X \to \mathcal{F}(S_i)$ be a fuzzy mapping, and let $a_i : \mathcal{D}_X \to (0,1]$ be a fuzzy function. Let $\psi_i : \Omega_i \times \mathcal{D}_X \times S_i \to \mathbb{R} \cup \{-\infty, +\infty\}$.

Now, we introduce a new type of systems of generalized quasi-variational inequalities as follows.

(1) Find $\lambda^* \in \mathcal{D}_X$ such that for every $i \in I$ and for all $\omega_i \in \Omega_i$:

(i) $\lambda_i^*(\omega_i) \in (A_i(\omega_i, \lambda^*))_{a_i(\lambda^*)}$;

(ii) $\sup_{y_i \in (A_i((\omega_i, \lambda^*)))_{a_i(\lambda^*)}} \psi_i(\omega_i, \lambda^*, y_i) \leq 0$,

where $(A_{i_{x^*}})_{a_i(x^*)} = \{z \in Y_i : A_{i_{x^*}}(z) \geq a_i(x^*)\}$.

We note that (1) extends a special deterministic case of quasi-variational inequalities considered by several authors (*e.g.* see Yuan [19] and the references therein) and the random quasi-variational inequalities studied by Tan and Yuan in [18] or Yuan (see [34] and the references therein).

If $A_i : \Omega_i \times \mathcal{D}_X \to 2^{S_i}$ is a classical correspondence and $\psi_i : \Omega_i \times \mathcal{D}_X \times S_i \to \mathbb{R} \cup \{-\infty, +\infty\}$, then we get the following variational inequality as a special case of (1).

(2) Find $\lambda^* \in \mathcal{D}_X$ such that for every $i \in I$ and for all $\omega_i \in \Omega_i$:

(i) $\lambda_i^*(\omega_i) \in A_i(\omega_i, \lambda^*)$;

(ii) $\sup_{y_i \in A_i(\omega_i,\lambda^*)} \psi_i(\omega_i, \lambda^*, y_i) \leq 0$.

Finally, we define the following system of generalized quasi-variational inequalities, in case that for each $i \in I$, S_i is a countable completely metrizable topological vector space, S_i' is the dual space of S_i, $A_i : \Omega_i \times \mathcal{D}_X \to \mathcal{F}(S_i)$, $G_i : \Omega_i \times S_i \to \mathcal{F}(S_i')$ are fuzzy mappings and $a_i : \mathcal{D}_X \to (0,1]$, $g_i : \mathcal{S}_i \to (0,1]$ are fuzzy functions.

(3) Find $\lambda^* \in \mathcal{D}_X$ such that for every $i \in I$ and for all $\omega_i \in \Omega_i$:

(i) $\lambda_i^*(\omega_i) \in (A_i(\omega_i, \lambda^*))_{a_i(\lambda^*)}$;

(ii) $\sup_{y_i \in (A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}} \sup_{\nu \in (G_i(\omega_i,y_i))_{g_i(y_i)}} \operatorname{Re}\langle \nu, \lambda_i^*(\omega_i) - y_i \rangle \leq 0$,

where the real part of pairing between S_i' and S_i is denoted by $\operatorname{Re}\langle \nu, x\rangle$ for each $\nu \in S_i'$ and $x \in S_i$.

If $A_i : \Omega_i \times \mathcal{D}_X \to 2^{S_i}$ and $G_i : \Omega_i \times S_i \to 2^{S_i'}$ are classical correspondences, then we get the following variational inequality.

(4) Find $\lambda^* \in \mathcal{D}_X$ such that for every $i \in I$ and for all $\omega_i \in \Omega_i$:

(i) $\lambda^*(\omega_i) \in A_i(\omega_i, \lambda^*)$;

(ii) $\sup_{y_i \in A_i(\omega_i,\lambda^*)} \sup_{\nu \in G_i(\omega_i,y_i)} \operatorname{Re}\langle \nu, \lambda_i^*(\omega_i) - y_i \rangle \leq 0$.

System (4) extends a special deterministic case of quasi-variational inequalities considered by several authors (*e.g.* see Yuan [35] and the references therein) and the random quasi-variational inequalities studied by Yuan (see [19] and the references therein).

Our systems of quasi-variational inequalities are different from the existent ones in literature in the following aspects: random fuzzy mappings are considered and their values are fuzzy sets over complete countable metric spaces.

4.2 Existence of the solutions of systems of generalized quasi-variational inequalities with random fuzzy mappings

In this subsection, we establish new results concerning the existence of the systems of generalized random quasi-variational inequalities with random fuzzy mappings. The proofs are based mainly on the theorem of fuzzy equilibrium existence for the abstract fuzzy economy.

In order to obtain the theorems in the next subsections, we set here the following common conditions.

Let I be a non-empty and finite set. For each $i \in I$, S_i is a countable complete metric space and $(\Omega_i, \mathcal{Z}_i)$ is a measurable space. Let $(\Omega, \mathcal{Z})$ be a product measurable space $(\prod_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{Z}_i)$ and let μ be an atomless probability measure on $(\Omega, \mathcal{Z})$. For each $i \in I$, the correspondence $X_i : \Omega_i \to \mathcal{F}(S_i)$ is measurable.

This is our first theorem.

Theorem 3 *Suppose that the following conditions are satisfied for each* $i \in I$:

(a) *the correspondence* $(X_i(\cdot))_{z_i} : \Omega_i \to 2^{S_i}$ *has compact values*;

(b) *for each* $\lambda \in \mathcal{D}_{X,z}$, *the correspondence* $(A_i(\cdot,\lambda))_{a_i(\lambda)} : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, *the correspondence* $(A_i(\omega_i,\cdot))_{a_i(\cdot)} : \mathcal{D}_{X,z} \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values.*

Let us assume that the mapping $\psi_i : \Omega_i \times \mathcal{D}_{X,z} \times S_i \to \mathbb{R} \cup \{-\infty,+\infty\}$ *is such that*:

(c) $\lambda \to \{y \in S_i : \psi_i(\omega,\lambda,y) > 0\} : \mathcal{D}_{X,z} \to 2^{S_i}$ *is upper semicontinuous with compact values on* $\mathcal{D}_{X,z}$ *for each fixed* $\omega_i \in \Omega_i$;

(d) $\lambda_i(\omega_i) \notin \{y \in S_i : \psi_i(\omega_i,\lambda,y) > 0\}$ *for each fixed* $(\omega_i,\lambda) \in \Omega_i \times \mathcal{D}_{X,z}$;

(e) *for each* $\omega_i \in \Omega_i$, $\{\lambda \in \mathcal{D}_{X,z} : \alpha_i(\omega_i,\lambda) > 0\}$ *is weakly open in* $\mathcal{D}_{X,z}$, *where* $\alpha_i : \Omega_i \times \mathcal{D}_{X,z} \to \mathbb{R}$ *is defined by*

$$\alpha_i(\omega_i,\lambda) = \sup_{y\in(A_i(\omega_i,\lambda))_{a_i(\lambda)}} \psi_i(\omega_i,\lambda,y) \quad \textit{for each } (\omega_i,\lambda) \in \Omega_i \times \mathcal{D}_{X,z};$$

(f) $\{\omega_i : \alpha_i(\omega_i,\lambda) > 0\} \in \mathcal{Z}_i$ *for each* $\lambda \in \mathcal{D}_{X,z}$.

Then there exists $\lambda^* \in \mathcal{D}_{X,z}$ *such that for every* $i \in I$ *and for every* $\omega_i \in \Omega_i$:

(i) $\lambda_i^*(\omega_i) \in (A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}$;

(ii) $\sup_{y\in(A_i(\omega,\lambda^*))_{a_i(\lambda^*)}} \psi_i(\omega_i,\lambda^*,y) \le 0$.

Proof For every $i \in I$, let $P_i : \Omega_i \times \mathcal{D}_{X,z} \to \mathcal{F}(S_i)$ and $p_i : \mathcal{D}_{X,z} \to (0,1]$ such that $(P_i(\omega,\lambda))_{p_i(\lambda)} = \{y \in S_i : \psi_i(\omega_i,\lambda,y) > 0\}$ for each $(\omega_i,\lambda) \in \Omega_i \times \mathcal{D}_{X,z}$.

We shall show that the abstract economy $\Gamma = (((\Omega_i,\mathcal{Z}_i)_{i\in I},\mu),(S_i,X_i,(A_i,a_i),(P_i,p_i))_{i\in I})$ satisfies all the hypotheses of Theorem 2.

For this purpose, let us consider $\omega_i \in \Omega_i$.

According to (c), we have that $\lambda \to (P_i(\omega_i,\lambda))_{p_i(\lambda)} : \mathcal{D}_{X,z} \to 2^{S_i}$ is upper semicontinuous with non-empty values and according to (d), $\lambda_i(\omega_i) \notin (P_i(\omega_i,\lambda))_{p_i(\lambda)}$ for each $\lambda \in \mathcal{D}_{X,z}$.

According to the definition of α_i, we note that, for each $\omega_i \in \Omega_i$, $\{\lambda \in \mathcal{D}_{X,z} : (A_i(\omega_i,\lambda))_{a_i(\lambda)} \cap (P_i(\omega_i,\lambda))_{p_i(\lambda)} \neq \emptyset\} = \{\lambda \in \mathcal{D}_{X,z} : \alpha_i(\omega,\lambda) > 0\}$ so that $\{\lambda \in \mathcal{D}_{X,z} : (A_i(\omega_i,\lambda))_{a_i(\lambda)} \cap (P_i(\omega_i,\lambda))_{p_i(\lambda)} \neq \emptyset\}$ is weakly open in $\mathcal{D}_{X,z}$ by (e).

According to (b) and (f), it follows that for each $\lambda \in \mathcal{D}_{X,z}$, the correspondences $(A_i(\cdot,\lambda))_{a_i(\lambda)} : \Omega_i \to 2^{S_i}$ and $(P_i(\omega_i,\lambda))_{p_i(\lambda)} : \Omega_i \to 2^{S_i}$ are measurable.

Thus, the abstract fuzzy economy $\Gamma = (((\Omega_i,\mathcal{Z}_i)_{i\in I},\mu),(S_i,X_i,(A_i,a_i),(P_i,p_i))_{i\in I})$ satisfies all the hypotheses of Theorem 2. Therefore, there exists $\lambda^* \in \mathcal{D}_{X,z}$ such that for every $i \in I$ and for every $\omega_i \in \Omega_i$:

$$\lambda_i^*(\omega_i) \in \bigl(A_i\bigl(\omega_i,\lambda^*\bigr)\bigr)_{a_i(\lambda^*)} \quad \text{and}$$

$$\bigl(A_i\bigl(\omega_i,\lambda^*\bigr)\bigr)_{a_i(\lambda^*)} \cap \bigl(P_i\bigl(\omega_i,\lambda^*\bigr)\bigr)_{p_i(\lambda^*)} = \phi;$$

that is, there exists $\lambda^* \in \mathcal{D}_{X,z}$ such that for every $i \in I$ and for every $\omega_i \in \Omega_i$:

(i) $\lambda_i^*(\omega) \in (A_i(\omega,\lambda^*))_{a_i(\lambda^*)}$;

(ii) $\sup_{y\in(A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}} \psi_i(\omega_i,\lambda^*,y) \le 0$. □

Remark 1 The above theorem can be compared with Theorem 5 in [34] which states an existence result for the solutions of random quasi-variational inequalities in case that $A_i(\omega,\cdot) : \prod_{i\in I} X_i \to 2^{X_i}$ is upper semicontinuous for each $\omega \in \Omega$ and $i \in I$.

If $|I| = 1$, we obtain the following corollary.

Corollary 1 *Let S be a countable complete metric space, $(\Omega, \mathcal{Z}, \mu)$ be a measure space, where μ is atomless. Suppose that the following conditions are satisfied*:

(a) *the correspondence $(X(\cdot))_z : \Omega_i \to 2^S$ has compact values*;

(b) *for each $\lambda \in \mathcal{D}_{X,z}$, the correspondence $(A(\cdot,\lambda))_{a(\lambda)} : \Omega \to 2^S$ is measurable and, for all $\omega \in \Omega$, the correspondence $(A(\omega,\cdot))_{a(\cdot)} : \mathcal{D}_{X,z} \to 2^S$ is upper semicontinuous with non-empty compact values.*

The mapping $\psi : \Omega \times \mathcal{D}_{X,z} \times S \to \mathbb{R} \cup \{-\infty, +\infty\}$ is such that:

(c) *$\lambda \to \{y \in Y : \psi(\omega,\lambda,y) > 0\} : \mathcal{D}_{X,z} \to 2^S$ is upper semicontinuous with compact values on $\mathcal{D}_{X,z}$ for each fixed $\omega \in \Omega$*;

(d) *$\lambda(\omega) \notin \{y \in S : \psi(\omega,\lambda,y) > 0\}$ for each fixed $(\omega,\lambda) \in \Omega \times \mathcal{D}_{X,z}$*;

(e) *for each $\omega \in \Omega$, $\{\lambda \in \mathcal{D}_{X,z} : \alpha(\omega,\lambda) > 0\}$ is weakly open in $\mathcal{D}_{X,z}$, where $\alpha : Z \times \mathcal{D}_{X,z} \to \mathbb{R}$ is defined by*

$$\alpha(\omega,\lambda) = \sup_{y \in (A(\omega,\lambda))_{a(\lambda)}} \psi(\omega,\lambda,y) \quad \textit{for each } (\omega,\lambda) \in Z \times \mathcal{D}_{X,z};$$

(f) *$\{\omega : \alpha(\omega,\lambda) > 0\} \in \mathcal{Z}$ for each $\lambda \in \mathcal{D}_{X,z}$.*

Then there exists $\lambda^* \in \mathcal{D}_{X,z}$ such that and for every $\omega \in \Omega$:

(i) *$\lambda^*(\omega) \in (A(\omega,\lambda^*))_{a_i(\lambda^*)}$*;

(ii) *$\sup_{y \in (A(\omega,\lambda^*))_{a(\lambda^*)}} \psi(\omega,\lambda^*,y) \le 0$.*

As a consequence of Theorem 3, we prove the following Tan and Yuan's type [18] system of generalized random quasi-variational inequalities with random fuzzy mappings.

Theorem 4 *Suppose that the following conditions are satisfied for each $i \in I$*:

(a) *the correspondence $(X_i(\cdot))_{z_i} : \Omega_i \to 2^{S_i}$ has compact values*;

(b) *for each $\lambda \in \mathcal{D}_{X,z}$, the correspondence $(A_i(\cdot,\lambda))_{a_i(\lambda)} : \Omega_i \to 2^{S_i}$ is measurable and, for all $\omega_i \in \Omega_i$, the correspondence $(A_i(\omega_i,\cdot))_{a_i(\cdot)} : \mathcal{D}_{X,z} \to 2^{S_i}$ is upper semicontinuous with non-empty compact values.*

$G_i : \Omega_i \times S_i \to \mathcal{F}(S_i')$ and $g_i : S_i \to (0,1]$ are such that:

(c) *for each fixed $(\omega_i, y) \in \Omega_i \times S_i$, $\lambda \to \{y \in S_i : \sup_{u \in (G_i(\omega_i,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i(\omega_i) - y\rangle > 0\} : \mathcal{D}_{X,z} \to 2^{S_i}$ is upper semicontinuous with compact values*;

(d) *for each fixed $\omega_i \in \Omega_i$, the set $\{\lambda \in \mathcal{D}_{X,z} : \sup_{y \in (A_i(\omega_i,\lambda))_{a_i(\lambda)}} \sup_{u \in (G_i(\omega_i,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i(\omega_i) - y\rangle > 0\}$ is weakly open in $\mathcal{D}_{X,z}$*;

(e) *$\{\omega_i \in \Omega_i : \sup_{u \in (G_i(\omega,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i(\omega_i) - y\rangle > 0\} \in \mathcal{Z}_i$ for each $\lambda \in \mathcal{D}_{X,z}$.*

Then there exists $\lambda^* \in \mathcal{D}_{X,z}$ such that for every $i \in I$ and for every $\omega_i \in \Omega_i$:

(i) *$\lambda_i^*(\omega_i) \in (A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}$*;

(ii) *$\sup_{u \in (G_i(\omega_i,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i^*(\omega_i) - y\rangle \le 0$ for all $y \in (A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}$.*

Proof Let us define $\psi_i : \Omega_i \times \mathcal{D}_{X,z} \times S_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\psi_i(\omega_i,\lambda,y) = \sup_{u \in (G_i(\omega_i,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i(\omega_i) - y\rangle \quad \text{for each } (\omega_i,\lambda,y) \in \Omega_i \times \mathcal{D}_{X,z} \times S_i.$$

We have that $\lambda_i(\omega_i) \notin \{y \in S_i : \psi_i(\omega_i,\lambda,y) > 0\}$ for each fixed $(\omega_i,\lambda) \in \Omega_i \times \mathcal{D}_{X,z}$.

All the hypotheses of Theorem 3 are satisfied. According to Theorem 3, there exists $\lambda^* \in \mathcal{D}_{X,z}$ such that for every $i \in I$ and for every $\omega_i \in \Omega_i$:

$$\lambda_i^*(\omega_i) \in \bigl(A_i(\omega_i, \lambda^*)\bigr)_{a_i(\lambda^*)} \quad \text{and}$$

$$\sup_{y\in(A_i(\omega_i,\lambda^*))_{a_i(\lambda^*)}} \sup_{u\in(G_i(\omega_i,y))_{g_i(y)}} \operatorname{Re}\langle u, \lambda_i^*(\omega_i) - y\rangle \leq 0.$$

Similar to Corollary 1, a result can be reached in case that $|I| = 1$. □

The following theorem is obtained as a particular case of Theorem 3.

Theorem 5 *Suppose that the following conditions are satisfied for each* $i \in I$:

(a) *the correspondence* $(X_i(\cdot))_{z_i} : \Omega_i \to 2^{S_i}$ *has compact values*;

(b) *for each* $\lambda \in \mathcal{D}_{X,z}$, *the correspondence* $(A_i(\cdot, \lambda))_{a_i(\lambda)} : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, *the correspondence* $(A_i(\omega_i, \cdot))_{a_i(\cdot)} : \mathcal{D}_{X,z} \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values.*

Then there exists $\lambda^* \in \mathcal{D}_{X,z}$ *such that for every* $i \in I$ *and for every* $\omega_i \in \Omega_i$, $\lambda_i^*(\omega_i) \in (A_i(\omega_i, \lambda^*))_{a_i(\lambda^*)}$.

If $|I| = 1$, we obtain the following result.

Theorem 6 *Let S be a countable complete metric space and let* $(\Omega, \mathcal{Z}, \mu)$ *be a measure space, where* μ *is atomless. Suppose that the following conditions are satisfied*:

(a) *the correspondence* $(X(\cdot))_z : \Omega \to 2^S$ *has compact values*;

(b) *for each* $\lambda \in \mathcal{D}_{X,z}$, *the correspondence* $(A(\cdot, \lambda))_{a(\lambda)} : \Omega \to 2^S$ *is measurable and, for all* $\omega \in \Omega$, *the correspondence* $(A(\omega, \cdot))_{a(\cdot)} : \mathcal{D}_{X,z} \to 2^S$ *is upper semicontinuous with non-empty compact values.*

Then there exists $\lambda^* \in \mathcal{D}_{X,z}$ *such that, and for every* $\omega \in \Omega$, $\lambda^*(\omega) \in (A(\omega, \lambda^*))_{a(\lambda^*)}$.

4.3 Classical systems of generalized quasi-variational inequalities

Several results concerning systems of generalized random quasi-variational inequalities in a non-fuzzy setting are obtained as consequences of the theorems stated in the last subsection. They are new and have not been reported in literature. We work under the same conditions which we have set at the beginning of Section 4.2. We present Theorem 7 as a consequence of Theorem 3.

Theorem 7 *Suppose that the following conditions are satisfied for each* $i \in I$:

(a) *the correspondence* $X_i : \Omega_i \to 2^{S_i}$ *is measurable and has compact values*;

(b) *the correspondence* A_i *is such that for each* $\lambda \in \mathcal{D}_X$, $A_i(\cdot, \lambda) : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, $A_i(\omega_i, \cdot) : \mathcal{D}_X \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values.*

Let us assume that the mapping $\psi_i : \Omega_i \times \mathcal{D}_X \times S_i \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is such that*:

(c) $\lambda \to \{y \in Y : \psi_i(\omega, \lambda, y) > 0\} : \mathcal{D}_X \to 2^{S_i}$ *is upper semicontinuous with compact values on* $\mathcal{D}_X$ *for each fixed* $\omega_i \in \Omega_i$;

(d) $\lambda_i(\omega_i) \notin \{y \in S_i : \psi_i(\omega_i, \lambda, y) > 0\}$ *for each fixed* $(\omega_i, \lambda) \in \Omega_i \times \mathcal{D}_X$;

(e) *for each* $\omega_i \in \Omega_i$, $\{\lambda \in \mathcal{D}_X : \alpha_i(\omega_i, \lambda) > 0\}$ *is weakly open in* $\mathcal{D}_X$, *where* $\alpha_i : \Omega_i \times \mathcal{D}_X \to \mathbb{R}$ *is defined by* $\alpha_i(\omega_i, \lambda) = \sup_{y\in A_i(\omega_i,\lambda)} \psi_i(\omega_i, \lambda, y)$ *for each* $(\omega_i, \lambda) \in \Omega_i \times \mathcal{D}_X$;

(f) $\{\omega_i : \alpha_i(\omega_i, \lambda) > 0\} \in \mathcal{Z}_i$ *for each* $\lambda \in \mathcal{D}_X$.

Then there exists $\lambda^* \in \mathcal{D}_X$ *such that for every* $i \in I$ *and for every* $\omega_i \in \Omega_i$,

(i) $\lambda_i^*(\omega_i) \in A_i(\omega_i, \lambda^*)$;

(ii) $\sup_{y \in A_i(\omega, \lambda^*)} \psi_i(\omega_i, \lambda^*, y) \le 0$.

The next theorem concerns the correspondences with values in complete countable metric spaces.

Theorem 8 *Suppose that the following conditions are satisfied for each* $i \in I$:

(a) *the correspondence* $X_i : \Omega_i \to 2^{S_i}$ *is measurable with compact values*;

(b) *for each* $\lambda \in \mathcal{D}_X$, $A_i(\cdot, \lambda) : \Omega_i \to 2^{S_i}$ *is measurable and, for all* $\omega_i \in \Omega_i$, $A_i(\omega_i, \cdot) : \mathcal{D}_X \to 2^{S_i}$ *is upper semicontinuous with non-empty compact values*.

Then there exists $\lambda^* \in \mathcal{D}_X$ *such that for every* $i \in I$ *and for every* $\omega_i \in \Omega_i$, $\lambda_i^*(\omega_i) \in A_i(\omega_i, \lambda^*)$.

5 Concluding remarks

The main object of this paper was to define a new model of an abstract fuzzy economy with private information and a countable set of actions and to prove the existence of fuzzy equilibrium under the upper semicontinuity assumptions. The uncertainties derived from the individual character of the agents in election situations have been described by using fuzzy random mappings. The techniques for proving the existence of equilibrium have been used to solve a new type of systems of quasi-variational inequalities with random fuzzy mappings which has been introduced in a special section.

Further attention is needed for the study of applications of the established results in the fuzzy economic field and fuzzy game theory. The methods and the setting in this paper can be adapted to solve other classes of generalized variational inequalities or quasi-equilibrium problems and they can especially be used in the general equilibrium theory.

Competing interests
The author declares that they have no competing interests.

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