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Strong convergence of a parallel iterative algorithm in a reflexive Banach space

Yuan Qing[1] and Songtao Lv[2*]

*Correspondence: sqlvst@yeah.net
[2]School of Mathematics and Information Science, Shangqiu Normal University, Shangqiu, Henan, China
Full list of author information is available at the end of the article

Abstract
In this paper, a parallel iterative algorithm is investigated for common zeros of a family of m-accretive operators. Strong convergence theorems are established in a reflexive Banach space.
MSC: 47H06; 47H09

Keywords: accretive operator; resolvent; nonexpansive mapping; proximal point algorithm; zero

1 Introduction

In this paper, we are concerned with the problem of finding common zero points of a finite family of accretive operators in a reflexive Banach space. Many nonlinear problems arising in applied areas such as image recovery and signal processing are mathematically modeled as fixed or zero point problems. Interest in accretive operators stems mainly from their firm connection with equations of evolution is an important class of nonlinear operators. It is well known that many physically significant problems can be modeled by initial value problems (IVP) of the following form:

$$x'(t) + Ax(t) = 0, \qquad x(0) = x_0, \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. If $x(t)$ is dependent on t, then (1.1) is reduced to $Au = 0$ whose solutions correspond to the equilibrium points of (1.1). An early fundamental result in the theory of accretive operators, due to Browder [1], states that IVP (1.1) is solvable if A is locally Lipschitz and accretive on E. One of the most popular techniques for solving zero points of accretive operators is the proximal point algorithms, which have been studied by many authors; see [2–27] and the references therein.

In this paper, we propose a viscosity proximal point algorithm for treating common zeros of a finite family of accretive operators. Strong convergence of the algorithm is obtained in the framework of reflexive Banach spaces.

Let E be a Banach space with the dual E^*. Let R^+ be the positive real number set. Let $\varphi : [0,\infty] := R^+ \to R^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a gauge function. The duality mapping

$J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the case that $\varphi(t) = t$, we write J for J_φ and call J the normalized duality mapping.

Following Browder [28], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to J_φ). It is well known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis.

A Banach space E is said to be strictly convex if and only if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$.

E is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex.

Let $U_E = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to be have a Gâteaux differentiable norm if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. E is said to be uniformly smooth or said to have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U_E$.

It is well known that Fréchet differentiability of the norm of E implies Gâteaux differentiability of the norm of E. It is well known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. For an accretive operator A, we can define a single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be α-contractive iff there exists a constant $\alpha \in [0,1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$, $\forall x, y \in C$. T is said to be nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. It is well known that many nonlinear problems can be reduced to the search for fixed points of nonexpansive mappings; see [29–35] and the references therein. Iterative methods are often used for finding and approximating such fixed points.

Let x be a fixed element in C and let T be a nonexpansive mapping with a nonempty fixed point set. For each $t \in (0,1)$, let x_t be the unique solution of the equation $y = tx + (1 - t)Ty$. In the framework of reflexive Banach spaces, Qin *et al.* [15] recently proved that $\{x_t\}$ converges strongly to a fixed point of T as $t \to 0$; see [15] and the references therein.

In this paper, we propose a parallel iterative algorithm for treating common zeros of a family of m-accretive operators. Strong convergence theorems are established in a reflexive Banach space.

Lemma 1.1 [36] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}\bigl(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.2 [37] *Let C be a closed convex subset of a strictly convex Banach space E. Let $N \geq 1$ be some positive integer and let $T_m : C \to C$ be a nonexpansive mapping. Let $\{\delta_m\}$ be a real number sequence in $(0,1)$ such that $\sum_{m=1}^N \delta_m = 1$. Suppose that $\bigcap_{m=1}^N F(T_m)$ is nonempty. Then the mapping $\sum_{m=1}^N \delta_m T_m$ is nonexpansive with $F(\sum_{m=1}^N \delta_m T_m) = \bigcap_{m=1}^N F(T_m)$.*

The following lemma can be obtained from [38] immediately.

Lemma 1.3 *Let E be a reflexive Banach space and has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be nonempty closed convex subset of E. Let $f : C \to C$ be an α-contractive mapping and let $T : C \to C$ be a nonexpansive mapping. Let $x_t \in C$ be the unique fixed point of the mapping $tf + (1 - t)T$, where $t \in (0,1)$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point $\bar{x}$ of T, where $\bar{x}$ is the unique solution to the following variational inequality: $\langle f(\bar{x}) - \bar{x}, J_\varphi(p - \bar{x})\rangle \leq 0$, $\forall p \in \bigcap_{m=1}^N A_m^{-1}(0)$.*

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [39].

Lemma 1.4 *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds*:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds*:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall x, y \in E.$$

Lemma 1.5 [40] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying $b_{n+1} \leq (1-a_n)b_n + a_n c_n$, $\forall n \geq n_0$, where n_0 is some positive integer, $\{a_n\}$ is a number sequence in $(0,1)$ such that $\sum_{n=n_0}^{\infty} a_n = \infty$, $\{c_n\}$ is a number sequence such that $\limsup_{n\to\infty} c_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

2 Main results

Theorem 2.1 *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let $N \geq 1$ be some positive integer and let A_i be an m-accretive operator in E for each $i \in \{1, 2, \ldots, N\}$. Assume that $\bigcap_{i=1}^N \overline{D(A_i)}$ is convex and $\bigcap_{i=1}^N A_i^{-1}(0)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_{n,i}\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in $\bigcap_{i=1}^N \overline{D(A_i)}$ generated in the following iterative process: $x_1 \in \bigcap_{i=1}^N \overline{D(A_i)}$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N \delta_{n,i} J_{r_i} x_n, \quad \forall n \geq 1,$$

where f is an α-contraction on $\bigcap_{i=1}^N \overline{D(A_i)}$, $\{r_i\}$ be a positive real numbers sequence and $J_{r_i} = (I + r_i A_i)^{-1}$. Assume that the following conditions are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\sum_{i=1}^N \delta_{n,i} = 1$, $\lim_{n\to\infty} \delta_{n,i} = \delta_i$.

Then $\{x_n\}$ converges strongly to $\bar{x}$, which is the unique solution to the following variational inequality: $\langle f(\bar{x}) - \bar{x}, J_\varphi(p - \bar{x})\rangle \leq 0$, $\forall p \in \bigcap_{i=1}^N A_i^{-1}(0)$.

Proof First, we show that $\{x_n\}$ is bounded. By fixing $p \in \bigcap_{i=1}^N A_i^{-1}(0)$, we get

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \left\| \sum_{i=1}^N \delta_{n,i} J_{r_m} x_n - p \right\| \\
&\leq \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \sum_{i=1}^N \delta_{n,i} \|J_{r_i} x_n - p\| \\
&\leq (1 - \alpha_n(1-\alpha))\|x_n - p\| + \alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha}.
\end{aligned}$$

This implies that

$$\|x_{n+1}-p\| \le \max\left\{\frac{\|f(p)-p\|}{1-\alpha}, \|x_1-p\|\right\}.$$

We find that $\{x_n\}$ is bounded. Putting $y_n = \sum_{i=1}^N \delta_{n,i}J_{r_i}x_n$, we see that

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \left\|\sum_{i=1}^N \delta_{n,i}J_{r_i}x_n - \sum_{i=1}^N \delta_{n,i}J_{r_n}x_{n-1}\right\| \\ &\quad + \left\|\sum_{i=1}^N \delta_{n,i}J_{r_i}x_{n-1} - \sum_{i=1}^N \delta_{n-1,i}J_{r_i}x_{n-1}\right\| \\ &\le \|x_n - x_{n-1}\| + \sum_{i=1}^N |\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|.\end{aligned}$$

Define $z_n := \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. This gives

$$\begin{aligned}z_n - z_{n-1} &= \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} - \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}} \\ &= \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n} - \frac{\alpha_{n-1}f(x_{n-1}) + \gamma_{n-1}y_{n-1}}{1-\beta_{n-1}} \\ &= \frac{\alpha_n}{1-\beta_n}\bigl(f(x_n)-y_n\bigr) - \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl(f(x_{n-1})-y_{n-1}\bigr) + y_n - y_{n-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\|z_n - z_{n-1}\| &\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n)-y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1})-y_{n-1}\bigr\| + \|y_n - y_{n-1}\| \\ &\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n)-y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1})-y_{n-1}\bigr\| + \|x_n - x_{n-1}\| \\ &\quad + \sum_{i=1}^N |\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\| &\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n)-y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1})-y_{n-1}\bigr\| \\ &\quad + \sum_{i=1}^N |\delta_{n,i}-\delta_{n-1,i}|\|J_{r_i}x_{n-1}\|.\end{aligned}$$

From the conditions (b), (c), and (d), we get

$$\limsup_{n\to\infty}\bigl(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|\bigr) \le 0.$$

In light of Lemma 1.1, we find that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Since $x_{n+1}-x_n = (1-\beta_n)(z_n - x_n)$, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Setting $T = \sum_{i=1}^N \delta_i J_{r_i}$, we from Lemma 1.2 see that T is nonexpansive with $F(T) = \bigcap_{i=1}^N F(J_{r_i}) = \bigcap_{i=1}^N A_i^{-1}(0)$. Note that

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\| + \beta_n \|x_n - Tx_n\| + \gamma_n \left\| \sum_{i=1}^N \delta_{n,i} J_{r_i} x_n - Tx_n \right\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\| + \beta_n \|x_n - Tx_n\| + \gamma_n \sum_{i=1}^N |\delta_{n,i} - \delta_i| \|J_{r_i} x_n\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1 - \beta_n)\|x_n - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n \|f(x_n) - Tx_n\| + \gamma_n \sum_{i=1}^N |\delta_{n,i} - \delta_i| \|J_{r_i} x_n\|.
\end{aligned}$$

It follows from the conditions (b), (c), and (d) that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.1}$$

Next, we show that $\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, J_\varphi(x_n - \bar{x})\rangle \le 0$. Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, J_\varphi(x_n - \bar{x})\rangle = \lim_{j\to\infty} \langle f(\bar{x}) - \bar{x}, J_\varphi(x_{n_j} - \bar{x})\rangle. \tag{2.2}$$

Since E is reflexive, we may further assume that $x_{n_j} \rightharpoonup \hat{x}$ for some $\hat{x} \in \bigcap_{i=1}^N \overline{D(A_i)}$. Since J_φ is weakly continuous, we find from Lemma 1.4 that

$$\limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - \hat{x}\|) + \Phi(\|x - \hat{x}\|), \quad \forall x \in E.$$

Putting $f(x) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|)$, $\forall x \in E$, we have

$$f(x) = f(\hat{x}) + \Phi(\|x - \hat{x}\|), \quad \forall x \in E. \tag{2.3}$$

It follows from (2.1) that

$$\begin{aligned}
f(T\hat{x}) &= \limsup_{j\to\infty} \Phi(\|x_{n_j} - T\hat{x}\|) \\
&\le \limsup_{j\to\infty} \Phi(\|Tx_{n_j} - T\hat{x}\|) \\
&\le \limsup_{j\to\infty} \Phi(\|x_{n_j} - \hat{x}\|) = f(\bar{x}).
\end{aligned} \tag{2.4}$$

On the other hand, we find from (2.3) that

$$f(T\hat{x}) = f(\hat{x}) + \Phi(\|T\hat{x} - \hat{x}\|). \tag{2.5}$$

In view of (2.4) and (2.5), we find that $\Phi(\|T\hat{x}-\hat{x}\|)\leq 0$. This implies that $T\hat{x}=\hat{x}$; that is, $\hat{x}\in F(T)=\bigcap_{i=1}^{N}A_i^{-1}(0)$. In light of (2.2), we find that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},J_\varphi(x_n-\bar{x})\rangle\leq 0. \tag{2.6}$$

Now, we are in a position to prove $x_n\to\bar{x}$ as $n\to\infty$. Using Lemma 1.1, we find that

$$\begin{aligned}\Phi(\|x_{n+1}-\bar{x}\|) &= \Phi\Bigg(\Bigg\|\alpha_n\big(f(x_n)-f(\bar{x})\big)+\alpha_n\big(f(\bar{x})-\bar{x}\big)+\beta_n(x_n-\bar{x})\\ &\qquad+\gamma_n\Bigg(\sum_{i=1}^{N}\beta_{n,i}J_{r_i}x_n-\bar{x}\Bigg)\Bigg\|\Bigg)\\ &\leq\Phi\Bigg(\Bigg\|\alpha_n\big(f(x_n)-f(\bar{x})\big)+\beta_n(x_n-\bar{x})+\gamma_n\Bigg(\sum_{i=1}^{N}\beta_{n,i}J_{r_i}x_n-\bar{x}\Bigg)\Bigg\|\Bigg)\\ &\qquad+\alpha_n\langle f(\bar{x})-\bar{x},J_\varphi(x_{n+1}-\bar{x})\rangle\\ &\leq\big(1-\alpha_n(1-\alpha)\big)\Phi(\|x_n-\bar{x}\|)+\alpha_n\langle f(\bar{x})-\bar{x},J_\varphi(x_{n+1}-\bar{x})\rangle.\end{aligned}$$

It follows from Lemma 1.5 that $\Phi(\|x_n-\bar{x}\|)\to 0$. This implies that $\lim_{n\to\infty}\|x_n-\bar{x}\|=0$. This completes the proof. □

If $A_1=A_2=\cdots=A_N$, the restriction of strict convexness imposed on the framework of the space can be removed. Indeed, we have the following result.

Corollary 2.2 *Let E be a reflexive Banach space which has a weakly continuous duality map J_φ. Let A be an m-accretive operator in E such that $\overline{D(A)}$ is convex and $A^{-1}(0)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in $\overline{D(A)}$ generated in the following iterative process: $x_1\in\overline{D(A)}$ and*

$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n J_r x_n,\quad\forall n\geq 1,$$

where f is an α-contraction on $\overline{D(A_i)}$, r be a positive real number and $J_r=(I+rA)^{-1}$. Assume that the following conditions are satisfied:
(a) $\alpha_n+\beta_n+\gamma_n=1$;
(b) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(c) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$.
Then $\{x_n\}$ converges strongly to $\bar{x}$, which is the unique solution to the following variational inequality: $\langle f(\bar{x})-\bar{x},J_\varphi(p-\bar{x})\rangle\leq 0$, $\forall p\in A^{-1}(0)$.

In the framework of Hilbert spaces, we find from Theorem 2.1 the following result.

Corollary 2.3 *Let E be a Hilbert space. Let $N\geq 1$ be some positive integer and let A_i be a maximal monotone operator in E for each $i\in\{1,2,\dots,N\}$. Assume that $\bigcap_{i=1}^{N}\overline{D(A_i)}$ is convex and $\bigcap_{i=1}^{N}A_i^{-1}(0)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_{n,i}\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in $\bigcap_{i=1}^{N}\overline{D(A_i)}$ generated in the following iterative process:*

$x_1 \in \bigcap_{i=1}^N \overline{D(A_i)}$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N \delta_{n,i} J_{r_i} x_n, \quad \forall n \geq 1,$$

where f is an α-contraction on $\bigcap_{i=1}^N \overline{D(A_i)}$, $\{r_i\}$ be a positive real numbers sequence and $J_{r_i} = (I + r_i A_i)^{-1}$. Assume that the following conditions are satisfied:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(d) $\sum_{i=1}^N \delta_{n,i} = 1$, $\lim_{n\to\infty} \delta_{n,i} = \delta_i$.

Then $\{x_n\}$ converges strongly to $\bar{x}$, which is the unique solution to the following variational inequality: $\langle f(\bar{x}) - \bar{x}, p - \bar{x}\rangle \leq 0$, $\forall p \in \bigcap_{i=1}^N A_i^{-1}(0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper.

Author details
[1]Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang, China. [2]School of Mathematics and Information Science, Shangqiu Normal University, Shangqiu, Henan, China.

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