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G-Prešić operators on metric spaces endowed with a graph and fixed point theorems

Satish Shukla[1] and Naseer Shahzad[2*]

*Correspondence: nshahzad@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21859, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The purpose of this paper is to introduce the Prešić type contraction in metric spaces endowed with a graph and to prove some fixed point results for the G-Prešić operators in such spaces. The results proved here generalize, extend, and unify several comparable results in the existing literature. Some examples are included which illustrate the results proved herein.

MSC: 47H10; 54H25

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1 Introduction and preliminaries

The famous Banach contraction mapping principle is one of the most simple and useful tools in the theory of fixed points. There are several generalizations of this famous principle. In 1965, Prešić [1, 2] gave a generalization of Banach contraction principle in product spaces which ensures the convergence of a particular sequence. Prešić proved the following theorem.

Theorem 1.1 *Let (X,d) be a complete metric space, k a positive integer and $T: X^k \to X$ be a mapping satisfying the following contractive type condition*:

$$d\big(T(x_1,x_2,\dots,x_k),T(x_2,x_3,\dots,x_{k+1})\big) \le \sum_{i=1}^{k} q_i d(x_i,x_{i+1}) \tag{1.1}$$

for every $x_1,x_2,\dots,x_{k+1} \in X$, where $q_1,q_2,\dots,q_k$ are nonnegative constants such that $q_1+q_2+\cdots+q_k<1$. Then there exists a unique point $x \in X$ such that $T(x,x,\dots,x)=x$. Moreover, if $x_1,x_2,\dots,x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k}=T(x_n,x_{n+1},\dots,x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n,\dots,\lim_{n\to\infty} x_n)$.

A mapping T satisfying (1.1) is called a Prešić operator. Note that the above theorem in the case $k=1$ reduces to the well-known Banach contraction mapping principle. Therefore Theorem 1.1 is a generalization of the Banach fixed point theorem.

Prešić type operators have applications in solving nonlinear difference equations and in the study of convergence of sequences; for example, see [1–4]. In the recent years, several authors generalized and extend the result of Prešić in different directions. For more on the generalizations of Prešić type operators the reader is referred to [5–19].

The existence of fixed point in metric spaces endowed with a partial order was investigated by Ran and Reurings [20] and then by Nieto and Rodríguez-Lopez [21, 22]. For the related results see, for instance, [23–27] and references therein. In [7] Malhotra *et al.* (see also [15, 19]) considered the Prešić type mappings in partially ordered sets and proved the ordered version of theorem of Prešić. This result is further generalized by Shukla *et al.* [16].

Kirk *et al.* [28] introduced the notion of cyclic operators and generalized the Banach contraction principle by proving the fixed point results for cyclic operators. Since then, many authors have made their contribution in this area; see, for example, [29–33] and the references cited therein. Very recently, Shukla and Abbas [18] generalized both the notions of cyclic operators and of Prešić operators by introducing the notion of cyclic-Prešić operators.

On the other hands, Jachymski [34] initiate the study of fixed point theorems in metric spaces endowed with graphs. Jachymski generalized the Banach contraction principle and unified the results of Ran and Reurings [20], Nieto and Rodríguez-Lopez [21, 22] and Edelstein [35]. For other related results, see, for instance, [23, 36–41].

In this paper, we generalize the result of Prešić in the metric spaces endowed with a graph. The notion of G-Prešić operators is introduced and fixed point results for such operators are proved. The results of this paper generalize and unify the results of Prešić [1, 2], Luong and Thuan [19] and Shukla and Abbas [18] for the spaces endowed with a graph, also these results generalize the results of Ran and Reurings [20], Nieto and Rodríguez-Lopez [21, 22] and Kirk *et al.* [28] in product spaces. Some examples are provided which illustrate the results proved herein.

First we recall some definitions and results which will be needed in the sequel.

Let (X, d) be a metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{1.2}$$

If x and y are vertices in a graph G, then a path in G from x to y of length l is a sequence $\{x_i\}_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x$, $x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, l$. A graph G is called connected if there is a path between any two vertices of G. G is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper we assume that X is a nonempty set, G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Now we can state our main results.

2 Main results

First we define some notions which will be useful in the sequel.

Definition 2.1 Let (X,d) be a metric space endowed with a graph G, k a positive integer and $T: X^k \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $T(x,x,\dots,x) = x$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$. Let $x_1, x_2, \dots, x_k \in X$ be arbitrary points in X. Then the sequence $\{x_n\}$ defined by $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$ for all $n \in \mathbb{N}$ is called a Prešić-Picard sequence (in short *PP*-sequence) with initial values $x_1, x_2, \dots, x_k$. The mapping T is called a Prešić-Picard operator (in short *PP*-operator) on X, if T has a unique fixed point $u \in X$ and every *PP*-sequence in X converges to u. The mapping T is called a weakly Prešić-Picard operator (in short *WPP*-operator) on X, if for every *PP*-sequence $\{x_n\}$ in X, $\lim_{n\to\infty} x_n$ exists (it may depend on the initial values $x_1, x_2, \dots, x_k$) and is a fixed point of T. Let P_T^k denotes the set of all paths $\{x_i\}_{i=1}^k$ of k vertices such that $(x_k, T(x_1, x_2, \dots, x_k)) \in E(G)$, that is,

$$P_T^k = \bigl\{\{x_i\}_{i=1}^k : (x_i, x_{i+1}), \bigl(x_k, T(x_1, x_2, \dots, x_k)\bigr) \in E(G) \text{ for } i = 1, 2, \dots, k-1\bigr\}.$$

The space (X,d) is said to have property (P) if:

$$\begin{aligned}&\text{whenever a sequence } \{x_n\} \text{ in } X, \text{converges to } x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N},\\ &\text{then there exists a subsequence } \{x_{n_k}\} \text{ such that } (x_{n_k}, x) \in E(G) \text{ for all } k \in \mathbb{N}.\end{aligned} \tag{P}$$

Now we define a G-Prešić operator on a metric space endowed with a graph.

Definition 2.2 Let (X,d) be a metric space endowed with a graph G and k be a positive integer. Let $T: X^k \to X$ be a mapping satisfying the following conditions:

(GP1) if $\{x_i\}_{i=1}^{k+1}$ be any path in G then $(T(x_1, \dots, x_k), T(x_2, \dots, x_{k+1})) \in E(G)$;

(GP2) for every path $\{x_i\}_{i=1}^{k+1}$ in G we have

$$d\bigl(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})\bigr) \le \sum_{i=1}^{k} q_i d(x_i, x_{i+1}),$$

where q_i's are nonnegative reals such that $\sum_{i=1}^{k} q_i < 1$.

Then T is called a G-Prešić operator.

Example 2.3 Any constant function $T: X^k \to X$ is a G-Prešić operator since $E(G)$ contains all loops.

Example 2.4 Let T be any Prešić operator, that is, T satisfies (1.1); then T is a G_0-Prešić operator where the graph G_0 is defined by $E(G_0) = X \times X$.

Remark 2.5 In the case $k = 1$, G-Prešić operator reduces into a G-contraction (see Jachymski [34]). Proposition 2.1 of Jachymski [34] shows that if T is a G-contraction then it is both G^{-1}-contraction and $\widetilde{G}$-contraction. But in the case $k > 1$ a G-Prešić operator need not be a G^{-1}-Prešić operator or a $\widetilde{G}$-Prešić operator, as shown in the following example.

Example 2.6 Let $X = \{1,2,3,4,5\}$ and (X,d) be the metric space with usual metric d and G be the graph defined by

$$E(G) = \Delta \cup \{(1,3),(3,5),(1,2)\}.$$

For $k = 2$, define a mapping $T: X^2 \to X$ by $T(3,5) = 2$, $T(5,3) = 3$, $T(2,2) = T(3,3) = 1$, $T(5,5) = 2$, and $T(x,y) = \min\{x,y\}$ for all other values of x and y. Then T is a G-Prešić operator with $q_1 = 0$, $q_2 = 1/2$.

On the other hand, T is not a G^{-1}-Prešić operator. Indeed,

$$E(G^{-1}) = \Delta \cup \{(3,1),(5,3),(2,1)\}$$

and so $\{x_i\}_{i=1}^3$, where $x_1 = 5$, $x_2 = 3$, $x_3 = 1$ is a path in G^{-1}. Now $d(T(5,3),T(3,1)) = d(3,1) = 2$ and $d(5,3) = 2$, $d(3,1) = 2$, so there are no nonnegative constants q_1, q_2 such that $q_1 + q_2 < 1$ and $d(T(5,3),T(3,1)) \leq q_1 d(5,3) + q_2 d(3,1)$. Because $E(\widetilde{G}) \supseteq E(G^{-1})$ we have $\{x_i\}_{i=1}^3$, where $x_1 = 5$, $x_2 = 3$, $x_3 = 1$ is also a path in $\widetilde{G}$ and so T is not a $\widetilde{G}$-Prešić operator.

Remark 2.7 Let (X,d) be a metric space endowed with a graph G, k a positive integer and $T: X^k \to X$ be a G-Prešić operator. If $(x,y) \in E(\widetilde{G})$ then $d(T(x,x,\ldots,x),T(y,y,\ldots,y)) < d(x,y)$.

Proof Suppose $(x,y) \in E(\widetilde{G}) = E(G) \cup E(G^{-1})$. If $(x,y) \in E(G)$ then since $E(G) \supseteq \Delta$ so by (GP2) we have

$$\begin{aligned} d\big(T(x,x,\ldots,x),T(y,y,\ldots,y)\big) &\leq d\big(T(x,x,\ldots,x),T(x,\ldots,x,y)\big) \\ &\quad + d\big(T(x,\ldots,x,y),T(x,\ldots,x,y,y)\big) \\ &\quad + \cdots + d\big(T(x,y,\ldots,y),T(y,y,\ldots,y)\big) \\ &\leq q_k d(x,y) + q_{k-1} d(x,y) + \cdots + q_1 d(x,y) \\ &= \sum_{i=1}^{k} q_i d(x,y) < d(x,y). \end{aligned}$$

Similarly, if $(x,y) \in E(G^{-1})$ we obtain the same result. □

Theorem 2.8 *Let (X,d) be a metric space endowed with a graph G and k be a positive integer. Suppose $T: X^k \to X$ be a G-Prešić operator and $P_T^k \neq \emptyset$. Then for every path $\{x_i\}_{i=1}^k$ in P_T^k, the PP-sequence with initial values $x_1,x_2,\ldots,x_k$ is a Cauchy sequence.*

Proof Let $\{x_i\}_{i=1}^k$ be a path in P_T^k, then by definition we have

$$(x_i,x_{i+1}),\big(x_k,T(x_1,x_2,\ldots,x_k)\big) \in E(G) \quad \text{for } i = 1,2,\ldots,k-1. \tag{2.1}$$

Let $\{x_n\}$ be the *PP*-sequence with initial values $x_1,x_2,\ldots,x_k$, that is,

$$x_{n+k} = T(x_n,x_{n+1},\ldots,x_{n+k-1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Then by (2.1), (2.2) we have $(x_k, x_{k+1}) \in E(G)$. Therefore by (2.1), (2.2) and (GP1) we have $(x_{k+1}, x_{k+2}) \in E(G)$ and $\{x_i\}_{i=1}^{k+2}$ is a path of $k+2$ vertices in G. In a similar way, we find that, for any fix $n > 1$, $\{x_i\}_{i=1}^{n}$ is a path of n vertices in G.

For notational convenience, let $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and

$$\mu = \max\left\{\frac{d_1}{\theta}, \frac{d_2}{\theta^2}, \dots, \frac{d_k}{\theta^k}\right\}, \quad \text{where } \theta = \left[\sum_{i=1}^{k} q_i\right]^{1/k}.$$

We shall show that

$$d_n \le \mu\theta^n \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

By the definition of μ, it is clear that (2.3) is true for $n = 1, 2, \dots, k$. Now let the following k inequalities:

$$d_n \le \mu\theta^n, \qquad d_{n+1} \le \mu\theta^{n+1}, \qquad \dots, \qquad d_{n+k-1} \le \mu\theta^{n+k-1}$$

be the induction hypothesis.

Since $\{x_i\}_{i=1}^{n}$ is a path for all $n \in \mathbb{N}$ we obtain from (GP2)

$$\begin{aligned}
d_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\
&= d\bigl(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \dots, x_{n+k})\bigr) \\
&\le q_1 d(x_n, x_{n+1}) + q_2 d(x_{n+1}, x_{n+2}) + \cdots + q_k d(x_{n+k-1}, x_{n+k}) \\
&= q_1 d_n + q_2 d_{n+1} + \cdots + q_k d_{n+k-1} \\
&\le q_1\mu\theta^n + q_2\mu\theta^{n+1} + \cdots + q_k\mu\theta^{n+k-1} \\
&\le q_1\mu\theta^n + q_2\mu\theta^n + \cdots + q_k\mu\theta^n \quad \left(\text{since } \theta = \left[\sum_{i=1}^{k} q_i\right]^{1/k} < 1\right) \\
&= \left[\sum_{i=1}^{k} q_i\right]\mu\theta^n \\
&= \mu\theta^{n+k}
\end{aligned}$$

and the inductive proof of (2.3) is complete. Now we shall show that the sequence $\{x_n\}$ is a Cauchy sequence. If $m, n \in \mathbb{N}$ with $m > n$ then by (2.3) we have

$$\begin{aligned}
d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&= d_n + d_{n+1} + \cdots + d_{m-1} \\
&\le \mu\theta^n + \mu\theta^{n+1} + \cdots + \mu\theta^{m-1} \\
&\le \frac{\mu\theta^n}{1-\theta}.
\end{aligned}$$

Since $\theta = [\sum_{i=1}^{k} q_i]^{1/k} < 1$, it follows from the above inequality that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, that is, the sequence $\{x_n\}$ is a Cauchy sequence. □

Note that the above theorem cannot be considered as an existence theorem for G-Prešić operator even when the space (X,d) is complete. The following example verifies this fact.

Example 2.9 Let $X = [0,1]$, then (X,d) is a complete metric space, where d is usual metric on X. Let the graph G be defined by

$$E(G) = \Delta \cup \{(x,y) \in X \times X : x,y \in (0,1] \text{ with } y \le x\}.$$

For $k = 2$, define a mapping $T: X^2 \to X$ by

$$T(x,y) = \begin{cases} \frac{1}{3}\max\{x,y\}, & \text{if } x,y \in (0,1]; \\ 1, & \text{otherwise.} \end{cases}$$

Then it is easy to see that T is a G-Prešić operator with $q_1 = q_2 = 1/3$. For any pair $x_1, x_2 \in (0,1]$ with $x_2 \le x_1$, $\frac{x_1}{3} \le x_2$ we have $(x_1,x_2), (x_2, T(x_1,x_2)) \in E(G)$, that is, $\{x_i\}_{i=1}^2 \in P_T^k$ so $P_T^k \neq \emptyset$. Thus, all the conditions of Theorem 2.8 are satisfied and (X,d) is a complete metric space, but T has no fixed point.

The following is an existence theorem for a G-Prešić operator and provides a sufficient condition for a G-Prešić operator to be a WPP-operator.

Theorem 2.10 *Let (X,d) be a complete metric space endowed with a graph G and k be a positive integer. Suppose $T: X^k \to X$ be a G-Prešić operator and $P_T^k \neq \emptyset$. Then for every path $\{x_i\}_{i=1}^k$ in P_T^k, the PP-sequence with initial values $x_1, x_2, \ldots, x_k$ is a Cauchy sequence. In addition, if (X,d) has the property* (P) *then $T|_{P_T^k}$ is a WPP-operator.*

Proof From Theorem 2.8 it follows that for every path $\{x_i\}_{i=1}^k$ in P_T^k, the PP-sequence with initial values $x_1, x_2, \ldots, x_k$ is a Cauchy sequence. Also, by following the arguments similar to those in Theorem 2.8 we have $\{x_i\}_{i=1}^n$ is a path in G for all $n \in \mathbb{N}$. Now, by completeness of X there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

We shall show that u is a fixed point of T. By the property (P) there exists a subsequence $\{x_{n_p}\}$ such that $(x_{n_p}, u) \in E(G)$ for all $p \in \mathbb{N}$. Therefore for any $p \in \mathbb{N}$ with $n_p > k$ by (GP2) we obtain

$$\begin{aligned} d\bigl(u, T(u,u,\ldots,u)\bigr) &\le d(u, x_{n_p+1}) + d\bigl(x_{n_p+1}, T(u,u,\ldots,u)\bigr) \\ &= d(u, x_{n_p+1}) + d\bigl(T(x_{n_p-k+1}, x_{n_p-k+2}, \ldots, x_{n_p}), T(u,u,\ldots,u)\bigr) \\ &\le d(u, x_{n_p+1}) + d\bigl(T(x_{n_p-k+1}, \ldots, x_{n_p}), T(x_{n_p-k+2}, \ldots, x_{n_p}, u)\bigr) \\ &\quad + d\bigl(T(x_{n_p-k+2}, \ldots, x_{n_p}, u), T(x_{n_p-k+3}, \ldots, x_{n_p}, u, u)\bigr) \\ &\quad + \cdots + d\bigl(T(x_{n_p}, u, \ldots, u), T(u,u,\ldots,u)\bigr) \\ &\le d(u, x_{n_p+1}) + \bigl\{q_1 d_{n_p-k+1} + q_2 d_{n_p-k+2} + \cdots + q_{k-1} d_{n_p-1} \\ &\quad + q_k d(x_{n_p}, u)\bigr\} + \bigl\{q_1 d_{n_p-k+2} + q_2 d_{n_p-k+3} + \cdots + q_{k-2} d_{n_p-1} \\ &\quad + q_{k-1} d(x_{n_p}, u)\bigr\} + \cdots + q_1 d(x_{n_p}, u). \end{aligned}$$

Since $\lim_{n\to\infty} x_n = u$, letting $p \to \infty$ in the above inequality we obtain $d(u, T(u, u, \dots, u)) = 0$, that is, $T(u, u, \dots, u) = u$. Thus u is a fixed point of T. □

In the above theorem the fixed point of the mapping T may not be unique; moreover, T may not be a *PP*-operator and may have infinitely many fixed points as illustrated in the following example.

Example 2.11 Let $X = \mathbb{N} = \bigcup_{k\in\mathbb{N}_0} \mathcal{N}_k$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathcal{N}_k = \{2^k(2n-1) : n \in \mathbb{N}\}$ for all $k \in \mathbb{N}_0$. Let G be defined by

$$E(G) = \Delta \bigcup_{k\in\mathbb{N}_0} (\mathcal{N}_{2k} \times \mathcal{N}_{2k}) \bigcup_{k\in\mathbb{N}_0} (\mathcal{N}_{2k+1} \times \mathcal{N}_{2k+1}).$$

Then (X, d) is a complete metric space where d is the usual metric on X. For $k = 2$, define a mapping $T : X^2 \to X$ by

$$T(x, y) = \begin{cases} 2^{2k}, & \text{if } x, y \in \mathcal{N}_{2k} \times \mathcal{N}_{2k}, k \in \mathbb{N}_0; \\ \frac{x+y}{4}, & \text{if } x, y \in \mathcal{N}_{2k+1} \times \mathcal{N}_{2k+1}, k \in \mathbb{N}_0; \\ x + y, & \text{otherwise.} \end{cases}$$

Now it is easy to see that T is a G-Prešić operator with $q_1 = q_2 = 1/4$. For all pairs $x_1, x_2 \in \mathcal{N}_{2k}$, $k \in \mathbb{N}$ we have $(x_1, x_2), (x_2, T(x_1, x_2)) \in E(G)$, that is, $\{x_i\}_{i=1}^2 \in P_T^k$ so $P_T^k \neq \emptyset$. Also the property (P) is satisfied trivially in this case. Thus, all the conditions of Theorem 2.10 are satisfied and T has infinitely many fixed points, precisely $\mathrm{Fix}(T) = \{2^{2k} : k \in \mathbb{N}_0\}$. Therefore T is not a *PP*-operator but *WPP*-operator on P_T^k.

In the next theorem a condition for the uniqueness of a fixed point of a G-Prešić operator is provided.

Theorem 2.12 *Let (X, d) be a complete metric space endowed with a graph G, k be a positive integer and $T : X^k \to X$ be a G-Prešić operator such that all the conditions of Theorem* 2.10 *are satisfied, then $T|_{P_T^k}$ is a WPP-operator. In addition, if the subgraph G_F^T is weakly connected, where $V(G_F^T) = \mathrm{Fix}(T)$ and $E(G_F^T) \subseteq E(G)$, then $T|_{P_T^k}$ is a PP-operator.*

Proof The existence of a fixed point follows from Theorem 2.8. Suppose G_F^T is weakly connected and $u, v \in \mathrm{Fix}(T)$ with $u \neq v$. Since G_F^T is weakly connected, there exists a path $\{x_i\}_{i=0}^l$ of $l + 1$ vertices with $x_0 = u$, $x_l = v$ and $(x_i, x_{i+1}) \in E(\widetilde{G}_F^T)$ for $0 \le i \le l - 1$.

Note that T is also G_F^T-Prešić operator and so by Remark 2.7 we have $u = v$. Thus, the fixed point of T is unique and $T|_{P_T^k}$ is a *PP*-operator. □

Remark 2.13 In Example 2.11 the fixed point of the operator is not unique. Note that $\mathrm{Fix}(T) = \{2^{2k} : k \in \mathbb{N}_0\}$ is not weakly connected. Indeed, $(2^{k_1}, 2^{k_2}) \notin E(G)$ for all $k_1, k_2 \in \mathbb{N}_0$ with $k_1 \neq k_2$. Therefore, Example 2.11 shows that when considering the uniqueness, the additional condition 'G_F^T is weakly connected' in Theorem 2.12 cannot be relaxed.

Now we derive some results as a consequences of the above results. For this first we state some definitions about the Prešić type mappings which can be found in [7, 18].

Definition 2.14 [7] Let a nonempty set X is equipped with a partial order '$\sqsubseteq$' such that (X,d) is a metric space, then $(X,\sqsubseteq,d)$ is called an ordered metric space. A sequence $\{x_n\}$ in X is said to be nondecreasing with respect to '$\sqsubseteq$' if $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$. Let k be a positive integer and $T : X^k \to X$ be a mapping, then T is said to be nondecreasing with respect to '$\sqsubseteq$' if for any finite nondecreasing sequence $\{x_i\}_{i=1}^{k+1}$ we have $T(x_1,x_2,\ldots,x_k) \sqsubseteq T(x_2,x_3,\ldots,x_{k+1})$. T is said to be an ordered Prešić type contraction if:

(OP1) T is nondecreasing with respect to '$\sqsubseteq$';

(OP2) there exist nonnegative real numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ such that $\sum_{i=1}^{k}\alpha_i < 1$ and

$$d\bigl(f(x_1,x_2,\ldots,x_k),f(x_2,x_3,\ldots,x_{k+1})\bigr) \le \sum_{i=1}^{k}\alpha_i d(x_i,x_{i+1}) \tag{2.4}$$

for all $x_1,x_2,\ldots,x_{k+1} \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$.

Definition 2.15 [18] Let X be any nonempty set, k a positive integer, $T\colon X^k \to X$ an operator and $A_1,A_2,\ldots,A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if:

(1) A_i, $i = 1,2,\ldots,m$ are nonempty sets;

(2) $T(A_1 \times A_2 \times \cdots \times A_k) \subset A_{k+1}$, $T(A_2 \times A_3 \times \cdots \times A_{k+1}) \subset A_{k+2}, \ldots,$
$T(A_i \times A_{i+1} \times \cdots \times A_{i+k-1}) \subset A_{i+k}, \ldots,$ where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$.

$T\colon Y^k \to Y$ is called a cyclic-Prešić operator if the following conditions are met:

(CP1) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(CP2) there exist nonnegative real numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ such that $\sum_{i=1}^{k}\alpha_i < 1$ and

$$d\bigl(T(x_1,x_2,\ldots,x_k),T(x_2,x_3,\ldots,x_{k+1})\bigr) \le \sum_{i=1}^{k}\alpha_i d(x_i,x_{i+1}) \tag{2.5}$$

for all $x_1 \in A_i, x_2 \in A_{i+1},\ldots,x_{k+1} \in A_{i+k}$ ($i = 1,2,\ldots,m$ where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$).

Now we give some consequences of our main results.

The following corollary is the fixed point result for the ordered Prešić type contractions (for details, see [7]).

Corollary 2.16 *Let $(X,\sqsubseteq,d)$ be an ordered complete metric space. Let $T\colon X^k \to X$ be a mapping such that the following conditions hold*:

(A) *T is an ordered Prešić type contraction*;

(B) *there exist $x_1,x_2,\ldots,x_k \in X$ such that $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq T(x_1,x_2,\ldots,x_k)$*;

(C) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point $u \in X$. In addition, $\mathrm{Fix}(T)$ *is well-ordered if and only if the fixed point of T is unique.*

Proof Define a graph G by $V(G) = X$ and

$$E(G) = \bigl\{(x,y) \in X \times X : x \sqsubseteq y\bigr\}.$$

Then (OP1) implies (GP1) and (OP2) implies (GP2) therefore T is a G-Prešić operator. By condition (B) it follows that $P_T^k \neq \emptyset$ and the path $\{x_i\}_{i=1}^{k} \in P_T^k$. Condition (C) insures that (X,d) has the property (P) and well-orderedness of $\mathrm{Fix}(T)$ implies that G_F^T is weakly connected. By Theorem 2.12, T has a unique fixed point. □

Corollary 2.17 *Let $A_1, A_2, \ldots, A_m$ be closed subsets of a complete metric space (X,d), k a positive integer, and $Y = \bigcup_{i=1}^m A_i$. Let $T\colon Y^k \to Y$ be a cyclic-Prešić operator, then T has a fixed point $u \in \bigcap_{i=1}^m A_i$. In addition, fixed point of T is unique if and only if* $\mathrm{Fix}(T) \subset \bigcap_{i=1}^m A_i$.

Proof Define a graph G by $V(G) = X$ and

$$E(G) = \Delta \cup \bigl\{(x,y) \in X \times X\colon x \in A_i, y \in A_{i+1}, 1 \le i \le k\bigr\},$$

where $A_{m+j} = A_j$ for all $j \in \mathbb{N}$. Note that Y is a complete subspace of X. Since $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, therefore (GP1) holds and (CP2) implies (GP2) and so T is a G-Prešić operator. As A_i, $1 \le i \le k$ are nonempty, therefore $P_T^k \neq \emptyset$. Proposition 2.1 of [18] shows that the subspace (Y,d) has the property (P). Finally, note that if $\mathrm{Fix}(T) \subset \bigcap_{i=1}^m A_i$ then G_F^T is weakly connected, therefore proof follows from Theorem 2.12. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, Shri Vaishnav Institute of Technology and Science, Gram Baroli Sanwer Road, Indore, MP 453331, India. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21859, Saudi Arabia.

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