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Fixed points in uniform spaces

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Abstract

We improve Angelov's fixed point theorems of Φ-contractions and j-nonexpansive maps in uniform spaces and investigate their fixed point sets using the concept of virtual stability. Some interesting examples and an application to the solution of a certain integral equation in locally convex spaces are also given.

Keywords: fixed points; Φ-contractions; uniform spaces

1 Introduction

In 1987 [1], Angelov introduced the notion of Φ-contractions on Hausdorff uniform spaces, which simultaneously generalizes the well-known Banach contractions on metric spaces as well as γ-contractions [2] on locally convex spaces, and he proved the existence of their fixed points under various conditions. Later in 1991 [3], he also extended the notion of Φ-contractions to j-nonexpansive maps and gave some conditions to guarantee the existence of their fixed points. However, there is a minor flaw in his proof of Theorem 1 [3] where the surjectivity of the map j is implicitly used without any prior assumption. Additionally, we observe that such a map j can be naturally replaced by a multi-valued map J to obtain a more general, yet interesting, notion of J-nonexpansiveness. Therefore, in this work, we aim to correct and simplify the proof of Theorem 1 [3] as well as extend the notion of j-nonexpansive maps to J-nonexpansive maps and investigate the existence of their fixed points. Then we introduce J-contractions, a special kind of J-nonexpansive maps, that play the similar role as Banach contractions in yielding the uniqueness of fixed points. With the notion of J-contractions, we are able to recover results on Φ-contractions proved in [1] as well as present some new fixed point theorems in which one of them naturally leads to a new existence theorem for the solution of a certain integral equation in locally convex spaces. Finally, we prove that, under a mild condition, J-nonexpansive maps are always virtually stable in the sense of [4] and hence their fixed point sets are retracts of their convergence sets. An example of a virtually stable J-nonexpansive map whose fixed point set is not convex is also given.

2 Fixed point theorems

For any set S, we will use $\mathcal{P}^f(S)$ and $|S|$ to denote the set of all nonempty finite subsets of S and the cardinality of S, respectively. Let $(E,\mathcal{A})$ be a Hausdorff uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{A} = \{d_\alpha : \alpha \in A\}$ indexed by A, $\emptyset \neq X \subseteq E$, and $J : A \to \mathcal{P}^f(A)$. Interested readers should consult [5] for general topological concepts of uniform spaces, and [6] for the complete development of fixed point theory in uniform spaces that motivates this work. We first give the definition of a J-nonexpansive map as follows:

Definition 2.1 A self-map $T: X \to X$ is said to be J-nonexpansive if for each $\alpha \in A$,

$$d_\alpha(Tx, Ty) \leq \sum_{\beta \in J(\alpha)} d_\beta(x, y),$$

for any $x, y \in X$.

Example 2.2 Let $1 < p < \infty$, $E = \ell_p$ be equipped with the weak topology, and $T: \ell_p \to \ell_p$ be defined by

$$T(x_1, x_2, \dots) = \left(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, \dots\right),$$

for any $(x_1, x_2, \dots) \in \ell_p$. Then $\mathcal{A} = \{|f| : f \in \ell_p^*\}$, where $|f|(x) = |f(x)|$ for each $x \in \ell_p$.

By Theorem 4.6 in [7], we have

$$\begin{aligned} \left|f(Tx - Ty)\right| \leq{}& \left|\frac{\|f\|}{3}(x_1 - y_1 + x_3 - y_3)\right| + \left|\frac{\|f\|}{3}(x_2 - y_2 + x_4 - y_4)\right| \\ &+ \left|\|f\|(x_1 - y_1)\right| + \left|\|f\|(x_2 - y_2)\right| + \left|f(x - y)\right|, \end{aligned}$$

for each $f \in \ell_p^*$, $x = (x_1, x_2, \dots) \in \ell_p$ and $y = (y_1, y_2, \dots) \in \ell_p$. Here, $\|f\| = \sup\{|f(x)| : x \in X, \|x\| \leq 1\}$.

By letting $J: \ell_p^* \to \mathcal{P}^f(\ell_p^*)$ be defined by $J(f) = \{|f|, |g_1|, |g_2|, |g_3|, |g_4|\}$, for each $f \in \ell_p^*$, where

$$g_1(x) = \frac{\|f\|}{3}(x_1 + x_3), \qquad g_2(x) = \frac{\|f\|}{3}(x_2 + x_4), \qquad g_3(x) = \|f\|x_1, \qquad g_4(x) = \|f\|x_2,$$

for each $x = (x_1, x_2, \dots) \in \ell_p$, it follows that T is J-nonexpansive.

The above definition of a J-nonexpansive map clearly extends the definition of a j-nonexpansive map in [3]. Before giving general existence criteria for fixed points of J-nonexpansive maps, we need the following notations. For each $\alpha \in A$ and $n \in \mathbf{N}$, we let

$$A_n(\alpha) = \left\{(\alpha_1, \dots, \alpha_n) : \alpha_1 \in J(\alpha) \text{ and } \alpha_k \in J(\alpha_{k-1}) \text{ for } 1 < k \leq n\right\}$$

and

$$A(\alpha) = \left\{(\alpha_1, \alpha_2, \dots) : \alpha_1 \in J(\alpha) \text{ and } \alpha_k \in J(\alpha_{k-1}) \text{ for } k > 1\right\}.$$

When there is no ambiguity, we will denote an element of both $A_n(\alpha)$ and $A(\alpha)$ simply by (α_k). Notice that for each $\alpha \in A$ and $n \in \mathbf{N}$, the sets $A_n(\alpha)$ and $\pi_n(A(\alpha))$ are finite, where π_n denotes the nth coordinate projection $(\alpha_k) \mapsto \alpha_n$.

Lemma 2.3 *Every J-nonexpansive map is continuous.*

Proof Suppose $T : X \to X$ is J-nonexpansive. Let $x \in X$ and (x_γ) be a net in X converging to x. Then for each $\alpha \in A$, we have

$$d_\alpha(Tx_\gamma, Tx) \leq \sum_{\beta \in J(\alpha)} d_\beta(x_\gamma, x).$$

Since (x_γ) converges to x, $(d_\beta(x_\gamma, x))$ converges to 0 for any $\beta \in A$, and this proves the continuity of T. □

Theorem 2.4 *Let $T : X \to X$ be J-nonexpansive whose $A(\alpha)$ is finite for any $\alpha \in A$. Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that*

(i) *the sequence $(T^n x_0)$ has a convergence subsequence, and*
(ii) *for each $\alpha \in A$ and $(\alpha_k) \in A(\alpha)$, $\lim_{n\to\infty} d_{\alpha_n}(x_0, Tx_0) = 0$.*

Proof ($\Rightarrow$): It is obvious by letting x_0 be a fixed point of T.

($\Leftarrow$): Suppose that $(T^{n_i} x_0)$ converges to some $z \in X$. Let $\alpha \in A$ and $(\alpha_k) \in A(\alpha)$. Then $\lim_{i\to\infty} d_\alpha(z, T^{n_i} x_0) = 0$ and $\lim_{n\to\infty} d_{\alpha_n}(x_0, Tx_0) = 0$. We can choose $N \in \mathbf{N}$ sufficiently large so that $d_\alpha(z, T^{n_i} x_0) < \epsilon$ and $d_{\alpha_{n_i}}(x_0, Tx_0) < \epsilon$, for all $i \geq N$. It follows that

$$\begin{aligned} d_\alpha\big(z, T^{n_i+1} x_0\big) &\leq d_\alpha\big(z, T^{n_i} x_0\big) + d_\alpha\big(T^{n_i} x_0, T^{n_i}(Tx_0)\big) \\ &\leq d_\alpha\big(z, T^{n_i} x_0\big) + \sum_{(\alpha_k) \in A_{n_i}(\alpha)} d_{\alpha_{n_i}}(x_0, Tx_0) \\ &\leq \big(1 + \big|A(\alpha)\big|\big)\epsilon. \end{aligned}$$

Since α is arbitrary, $(T^{n_i+1} x_0)$ converges to z. By the continuity of T, we have $z = Tz$ and hence z is a fixed point of T. □

As a corollary of the previous theorem, we immediately obtain Theorem 1 [3], with a corrected and simplified proof, as follows:

Corollary 2.5 *Let $T : X \to X$ be a j-nonexpansive map. If there exists $x_0 \in X$ such that*

(i) *the sequence $(T^n x_0)$ has a convergence subsequence, and*
(ii) *for every $\alpha \in A$, $\lim_{n\to\infty} d_{j^n(\alpha)}(x_0, Tx_0) = 0$,*

then T has a fixed point.

Proof The proof follows directly from the previous theorem by considering the map $J : \alpha \mapsto \{j(\alpha)\}$. Notice that $A(\alpha) = \{(j^n(\alpha))\}$ which is finite. □

We will now consider a special kind of J-nonexpansive maps that resemble Banach contractions in yielding the uniqueness of fixed points. Let Φ denote the family of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Φ1) ϕ is non-decreasing and continuous from the right, and
(Φ2) $\phi(t) < t$ for any $t > 0$.

Notice that $\phi(0) = 0$, and we will call $\phi \in \Phi$ subadditive if $\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 \geq 0$. Also, for a subfamily $\{\phi_\alpha\}_{\alpha \in A}$ of Φ, $\alpha \in A$, $(\alpha_k) \in A_n(\alpha)$ and $i \leq n$, we let

$$\phi^i_{(\alpha_k)} = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_i}.$$

Definition 2.6 A self-map $T : X \to X$ is said to be a J-contraction if for each $\alpha \in A$, there exists $\phi_\alpha \in \Phi$ such that

$$d_\alpha(Tx, Ty) \le \sum_{\beta \in J(\alpha)} \phi_\alpha\big(d_\beta(x,y)\big),$$

for any $x, y \in X$, and ϕ_α is subadditive whenever $|J(\alpha)| > 1$.

Clearly, a Φ-contraction as defined in [1] is a J-contraction and a J-contraction is always J-nonexpansive. A natural example of a J-contraction can be obtained by adding (finitely many) appropriate Φ-contractions as shown in the following example.

Example 2.7 Given two Φ-contractions $T_1 : X \to X$ and $T_2 : X \to X$ as defined [1]. Then there exist $j_1, j_2 : A \to A$, and for each $\alpha \in A$, there exist $\phi_{1,\alpha}, \phi_{2,\alpha} \in \Phi$ such that

$$d_\alpha(T_1x, T_1y) \le \phi_{1,\alpha}\big(d_{j_1(\alpha)}(x,y)\big) \quad \text{and} \quad d_\alpha(T_2x, T_2y) \le \phi_{2,\alpha}\big(d_{j_2(\alpha)}(x,y)\big),$$

for any $\alpha \in A$ and $x, y \in X$. If for each $\alpha \in A$, $j_1(\alpha) \neq j_2(\alpha)$ and there is a subadditive $\phi_{3,\alpha} \in \Phi$ so that $\phi_{1,\alpha}(t) \le \phi_{3,\alpha}(t)$ and $\phi_{2,\alpha}(t) \le \phi_{3,\alpha}(t)$ for any $t \ge 0$, then the map $H = T_1 + T_2$ is clearly a J-contraction with respect to $J(\alpha) = \{j_1(\alpha), j_2(\alpha)\}$ and $\phi_{H,\alpha} = \phi_{3,\alpha}$ for any $\alpha \in A$.

Lemma 2.8 *If $T : X \to X$ is a J-contraction. Then we have*

$$d_\alpha\big(T^n x, T^n y\big) \le \sum_{(\alpha_k) \in A_n(\alpha)} \phi_\alpha \circ \phi^{n-1}_{(\alpha_k)}\big(d_{\alpha_n}(x,y)\big),$$

for any $\alpha \in A$, $n \ge 2$ and $x, y \in X$.

Proof Recall that ϕ_α is assumed to be subadditive whenever $|J(\alpha)| > 1$. Then, for any $\alpha \in A$, $n \ge 2$ and $x, y \in X$, we clearly have

$$\begin{aligned}
d_\alpha\big(T^n x, T^n y\big) &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\big(d_{\alpha_1}\big(T^{n-1}x, T^{n-1}y\big)\big) \\
&\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\Big(\sum_{\alpha_2 \in J(\alpha_1)} \phi_{\alpha_1}\big(d_{\alpha_2}\big(T^{n-2}x, T^{n-2}y\big)\big)\Big) \\
&\le \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \phi_\alpha \circ \phi_{\alpha_1}\big(d_{\alpha_2}\big(T^{n-2}x, T^{n-2}y\big)\big) \\
&\;\;\vdots \\
&\le \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \cdots \sum_{\alpha_n \in J(\alpha_{n-1})} \phi_\alpha \circ \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_{n-1}}\big(d_{\alpha_n}(x,y)\big) \\
&= \sum_{(\alpha_k) \in A_n(\alpha)} \phi_\alpha \circ \phi^{n-1}_{(\alpha_k)}\big(d_{\alpha_n}(x,y)\big).
\end{aligned}$$

□

We now obtain some general criteria for the existence of fixed points of J-contractions.

Theorem 2.9 *Suppose X is sequentially complete and $T : X \to X$ is a J-contraction whose $A(\alpha)$ is finite for any $\alpha \in A$. If T satisfies the following conditions*:

(i) *for each* $\alpha \in A$, *there exists* $c_\alpha \in \Phi$ *such that*

$$\phi_{\alpha_i}(t) \le c_\alpha(t),$$

for any $(\alpha_k) \in A(\alpha)$, $i \in \mathbf{N}$, $t \ge 0$, *and*

(ii) *there exists* $x_0 \in X$ *such that for each* $\alpha \in A$, $(\alpha_k) \in A(\alpha)$, $i \in \mathbf{N}$ *and* $n, m \in \mathbf{N}$, *we have*

$$d_{\alpha_i}\big(T^n x_0, T^m x_0\big) \le M_\alpha(x_0),$$

for some $M_\alpha(x_0) \in \mathbf{R}$,

then T *has a fixed point. Moreover, if for each* $\alpha \in A$ *and* $x, y \in X$, *there exists* $F_\alpha(x,y) \in \mathbf{R}_0^+$ *such that*

$$d_{\alpha_i}(x,y) \le F_\alpha(x,y),$$

for all $(\alpha_k) \in A(\alpha)$ *and* $i \in \mathbf{N}$, *then the fixed point of* T *is unique.*

Proof For each $\alpha \in A$ and $n, m, N \in \mathbf{N}$, since ϕ_α is non-decreasing, we have

$$\begin{aligned} d_\alpha\big(T^n x_0, T^m x_0\big) &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\big(d_{\alpha_1}\big(T^{n-1}x_0, T^{m-1}x_0\big)\big) \\ &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\big(\sup\big\{d_{\alpha_1}\big(T^{n-1}x_0, T^{m-1}x_0\big) : n, m \ge N\big\}\big), \end{aligned}$$

and by letting $h_N^\alpha := \sup\{d_\alpha(T^n x_0, T^m x_0) : n, m \ge N\}$, it follows that

$$\begin{aligned} h_N^\alpha &\le \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\big(\sup\big\{d_{\alpha_1}\big(T^{n-1}x_0, T^{m-1}x_0\big) : n, m \ge N\big\}\big) \\ &= \sum_{\alpha_1 \in J(\alpha)} \phi_\alpha\big(h_{N-1}^{\alpha_1}\big) \\ &\le \sum_{\alpha_1 \in J(\alpha)} \sum_{\alpha_2 \in J(\alpha_1)} \phi_\alpha\big(\phi_{\alpha_1}\big(h_{N-2}^{\alpha_2}\big)\big) \\ &\ \ \vdots \\ &\le \sum_{(\alpha_k) \in A_{N-1}(\alpha)} \phi_\alpha \circ \phi_{(\alpha_k)}^{N-1}\big(h_1^{\alpha_{N-1}}\big) \\ &\le \sum_{(\alpha_k) \in A_{N-1}(\alpha)} c_\alpha^N\big(M_\alpha(x_0)\big) \\ &\le \big|A(\alpha)\big| c_\alpha^N\big(M_\alpha(x_0)\big). \end{aligned} \tag{1}$$

Also, for a given $t \ge 0$, since $0 \le c_\alpha^N(t) = c_\alpha(c_\alpha^{N-1}(t)) < c_\alpha^{N-1}(t)$, we have $\lim_{N\to\infty} c_\alpha^N(t) = r_\alpha$ for some $r_\alpha \ge 0$. Since c_α is right continuous, we have $\lim_{N\to\infty} c_\alpha(c_\alpha^{N-1}(t)) = c_\alpha(r_\alpha)$, and hence $c_\alpha(r_\alpha) = r_\alpha$. Therefore, $r_\alpha = 0$. By (1), it follows that $\lim_{N\to\infty} h_N^\alpha = 0$. Since α is arbitrary, $(T^k x_0)$ is a Cauchy sequence and, by sequential completeness, converges to some $z \in X$. Notice also that z must be a fixed point of T by continuity.

Now suppose that for each $x, y \in X$ and $\alpha \in A$, there exists $F_\alpha(x,y) \in \mathbf{R}_0^+$ such that $d_{\alpha_i}(x,y) \leq F_\alpha(x,y)$ for all $(\alpha_k) \in A(\alpha)$ and $i \in \mathbf{N}$. If x, y are fixed points of T, then by Lemma 2.8, we have for each $\alpha \in A$ and $n \in \mathbf{N}$,

$$\begin{aligned} d_\alpha(x,y) &= d_\alpha\big(T^n x, T^n y\big) \\ &\leq \sum_{(\alpha_k)\in A_n(\alpha)} \phi_\alpha \circ \phi_{(\alpha_k)}^{n-1}\big(d_{\alpha_n}(x,y)\big) \\ &\leq \sum_{(\alpha_k)\in A_n(\alpha)} c_\alpha^n\big(d_{\alpha_n}(x,y)\big) \\ &\leq \big|A(\alpha)\big| c_\alpha^n\big(F_\alpha(x,y)\big). \end{aligned}$$

Since $\lim_{n\to\infty} c_\alpha^n(F_\alpha(x,y)) = 0$, we must have $x = y$. □

As a corollary of the previous theorem, we immediately obtain Theorem 1 in [1] as follows.

Corollary 2.10 *Suppose X is a bounded and sequentially complete subset of E and $T : X \to X$ is Φ-contraction. If*

(i) *for each $\alpha \in A$, there exists $c_\alpha \in \Phi$ such that $\phi_{j^n(\alpha)}(t) \leq c_\alpha(t)$ for all $n \in \mathbf{N}$ and $t \geq 0$,*

(ii) *for each $n \in \mathbf{N}$, $\sup\{d_{j^n(\alpha)}(x,y) : x,y \in X\} \leq p(\alpha) := \sup\{d_\alpha(x,y) : x,y \in X\}$,*

then there exists a unique fixed point $x \in X$ of T.

Proof For each $x_0, x, y \in X$, $\alpha \in A$, $(\alpha_k) \in A(\alpha)$ and $i, m, n \in \mathbf{N}$, by letting $J(\alpha) = \{j(\alpha)\}$ and $M_\alpha(x_0) = p(\alpha) = F_\alpha(x,y)$, we have $A(\alpha) = \{(\alpha, j(\alpha), j^2(\alpha), \ldots, j^k(\alpha), \ldots)\}$, $d_{\alpha_i}(T^m x_0, T^n x_0) = d_{j^i(\alpha)}(T^m x_0, T^n x_0) \leq M_\alpha(x_0)$ and $d_{\alpha_i}(x,y) \leq F_\alpha(x,y)$. Hence, by Theorem 2.9, T has a unique fixed point. □

Theorem 2.11 *Suppose X is sequentially complete and $T : X \to X$ is a self-map satisfying: for each $\alpha \in A$ and $k \in \mathbf{N}$, there exist $\phi_{\alpha,k} \in \Phi$, a finite set $D_{\alpha,k}$ and a map $P_{\alpha,k} : D_{\alpha,k} \to A$ such that*

$$d_\alpha\big(T^k x, T^k y\big) \leq \sum_{\gamma\in D_{\alpha,k}} \phi_{\alpha,k}\big(d_{P_{\alpha,k}(\gamma)}(x,y)\big),$$

for any $x, y \in X$.

1. *If there exists $x_0 \in X$ such that for each $\alpha \in A$ there exists $M_\alpha(x_0) \in \mathbf{R}_0^+$ so that $\sum_{k\in\mathbf{N}} |D_{\alpha,k}| \phi_{\alpha,k}(M_\alpha(x_0)) < \infty$ and*

$$d_{P_{\alpha,k}(\gamma)}(x_0, Tx_0) \leq M_\alpha(x_0),$$

for all $k \in \mathbf{N}$ and $\gamma \in D_{\alpha,k}$, then T has a fixed point in X.

2. *If for each $\alpha \in A$ and $x, y \in X$, there exists $F_\alpha(x,y) \in \mathbf{R}_0^+$ such that $\sum_{k\in\mathbf{N}} |D_{\alpha,k}| \phi_{\alpha,k}(F_\alpha(x,y)) < \infty$ and*

$$d_{P_{\alpha,k}(\gamma)}(x,y) \leq F_\alpha(x,y),$$

for all $k \in \mathbf{N}$ and $\gamma \in D_{\alpha,k}$, then T has a unique fixed point in X and, for any $x \in X$, the sequence $(T^n x)$ converges to the fixed point of T.

Proof First notice that T is clearly a J-contraction.

1. For any $\alpha \in A$ and $m > n \in \mathbf{N}$, we have

$$\begin{aligned} d_\alpha\left(T^n x_0, T^m x_0\right) &\le \sum_{n \le i < m} d_\alpha\left(T^i x_0, T^{i+1} x_0\right) \\ &\le \sum_{n \le i < m} \sum_{\gamma \in D_{\alpha,i}} \phi_{\alpha,i}\left(d_{P_{\alpha,i}(\gamma)}(x_0, Tx_0)\right) \\ &\le \sum_{n \le i < m} |D_{\alpha,i}| \phi_{\alpha,i}\left(M_\alpha(x_0)\right). \end{aligned}$$

Also, since $\sum_{k \in \mathbf{N}} |D_{\alpha,k}| \phi_{\alpha,k}(M_\alpha(x_0)) < \infty$, $(T^k x_0)$ is a Cauchy sequence and converges to a fixed point of T by the sequential completeness of X and the continuity of T.

2. For any $x \in X$, $\alpha \in A$ and $m > n \in \mathbf{N}$, we have

$$\begin{aligned} d_\alpha\left(T^n x, T^m x\right) &\le \sum_{n \le i < m} d_\alpha\left(T^i x, T^{i+1} x\right) \\ &\le \sum_{n \le i < m} \sum_{\gamma \in D_{\alpha,i}} \phi_{\alpha,i}\left(d_{P_{\alpha,i}(\gamma)}(x, Tx)\right) \\ &\le \sum_{n \le i < m} |D_{\alpha,i}| \phi_{\alpha,i}\left(F_\alpha(x, Tx)\right). \end{aligned}$$

Also, since $\sum_{k \in \mathbf{N}} |D_{\alpha,k}| \phi_{\alpha,k}(F_\alpha(x, Tx)) < \infty$, $(T^k x)$ is a Cauchy sequence and converges to a fixed point of T by the sequential completeness of X and the continuity of T.

Now, since for each $\alpha \in A$, $k \in \mathbf{N}$ and $x, y \in F(T)$,

$$\begin{aligned} d_\alpha(x, y) &= d_\alpha\left(T^k x, T^k y\right) \\ &\le \sum_{\gamma \in D_{\alpha,k}} \phi_{\alpha,k}\left(d_{P_{\alpha,k}(\gamma)}(x, y)\right) \\ &\le \sum_{\gamma \in D_{\alpha,k}} \phi_{\alpha,k}\left(F_\alpha(x, y)\right) \\ &= |D_{\alpha,k}| \phi_{\alpha,k}\left(F_\alpha(x, y)\right), \end{aligned}$$

and $\lim_{k \to \infty} |D_{\alpha,k}| \phi_{\alpha,k}(F_\alpha(x, y)) = 0$, we have the uniqueness. □

Corollary 2.12 (Theorem 5 in [1]) *Let us suppose*

(i) *for each $\alpha \in A$ and $n > 0$, there exist $\phi_{\alpha,n} \in \Phi$ and $j(\alpha, n) \in A$ such that*

$$d_\alpha\left(T^n x, T^n y\right) \le \phi_{\alpha,n}\left(d_{j(\alpha,n)}(x, y)\right),$$

for any $x, y \in X$,

(ii) *there exists $x_0 \in X$ such that $d_{j(\alpha,n)}(x_0, Tx_0) \le p(\alpha) < \infty$ $(n = 1, 2, \dots)$, $\sum_n \phi_{\alpha,n}(p(\alpha)) < \infty$ and $j : A \times \mathbf{N} \to A$.*

Then T has at least one fixed point in X.

Proof By letting $D_{\alpha,k} = \{j(\alpha,k)\}$ for any $\alpha \in A$ and $k \in \mathbf{N}$ and $P_{\alpha,k} = \pi_k|_{D_{\alpha,k}}$. Then for each $i \in \mathbf{N}$, we have $|D_{\alpha,i}| = 1$ and $M_\alpha(x_0) = p(\alpha)$. By Theorem 2.11(2), T has a fixed point. □

Theorem 2.13 *Suppose X is sequentially complete and $T : X \to X$ is a J-contraction whose $A(\alpha)$ is finite for each $\alpha \in A$. If, for each $\alpha \in A$, there exists $c_\alpha \in \Phi$ satisfying:*

(i) *$c_\alpha(t)/t$ is non-decreasing in t,*

(ii) *$\phi_{\alpha_n}(t) \le c_\alpha(t)$ for any $(\alpha_k) \in A(\alpha)$, $n \in \mathbf{N}$ and $t \in [0,\infty)$, and*

(iii) *there exist $x_0 \in X$ and $M_\alpha(x_0) \in \mathbf{R}^+$ such that $d_{\alpha_n}(x_0, Tx_0) \le M_\alpha(x_0)$ for any $(\alpha_k) \in A(\alpha)$ and $n \in \mathbf{N}$,*

then T has a fixed point in X.

Proof Let $D_{\alpha,i} = A_i(\alpha)$, $P_{\alpha,i}((\alpha_k)) = \alpha_i$, and $\phi_{\alpha,i}(t) = c_\alpha^i(t)$ for any $i \in \mathbf{N}$, $\alpha \in A$, $(\alpha_k) \in A_i(\alpha)$, and $t \in [0,\infty)$. Then for any $\alpha \in A$ and $x, y \in X$, we have, by Lemma 2.8,

$$\begin{aligned} d_\alpha\bigl(T^i x, T^i y\bigr) &\le \sum_{(\alpha_k)\in A_i(\alpha)} \phi_\alpha \circ \phi_{(\alpha_k)}^{i-1}\bigl(d_{\alpha_i}(x,y)\bigr) \\ &\le \sum_{(\alpha_k)\in A_i(\alpha)} c_\alpha^i\bigl(d_{\alpha_i}(x,y)\bigr) \\ &= \sum_{(\alpha_k)\in D_{\alpha,i}} \phi_{\alpha,i}\bigl(d_{P_{\alpha,i}((\alpha_k))}(x,y)\bigr). \end{aligned}$$

Since

$$\frac{|D_{\alpha,i+1}|\phi_{\alpha,i+1}(M_\alpha(x_0))}{|D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0))} = \frac{|A_{i+1}(\alpha)|c_\alpha^{i+1}(M_\alpha(x_0))}{|A_i(\alpha)|c_\alpha^i(M_\alpha(x_0))} \le \frac{c_\alpha(c_\alpha^i(M_\alpha(x_0)))}{c_\alpha^i(M_\alpha(x_0))} \le \frac{c_\alpha(M_\alpha(x_0))}{M_\alpha(x_0)} < 1,$$

for any $i \in \mathbf{N}$, we have $\sum_{i\in\mathbf{N}} |D_{\alpha,i}|\phi_{\alpha,i}(M_\alpha(x_0)) < \infty$. Then by Theorem 2.11(1), T has a fixed point. □

Corollary 2.14 (Theorem 2 in [1]) *Let us suppose*

(i) *the operator $T : X \to X$ is a Φ-contraction,*

(ii) *for each $\alpha \in A$ there exists a Φ-function c_α such that $\phi_{j^n(\alpha)}(t) \le c_\alpha(t)$ for all $n \in \mathbf{N}$ and $c_\alpha(t)/t$ is non-decreasing,*

(iii) *there exists an element $x_0 \in X$ such that $d_{j^n(\alpha)}(x_0, Tx_0) \le p(\alpha) < \infty$ $(n = 1, 2, \ldots)$.*

Then T has at least one fixed point in X.

Proof By letting $J(\alpha) = \{j(\alpha)\}$ for any $\alpha \in A$ and $M_\alpha(x_0) = p(\alpha)$. Then $|A(\alpha)| = 1$, and, by Theorem 2.13, T has a fixed point. □

Example 2.15 Given a sequentially complete locally convex space X, and two Φ-contractions $T_1, T_2 : X \to X$; *i.e.*, there exist $j_1, j_2 : A \to A$, and for each $\alpha \in A$, there exist $\phi_{1,\alpha}, \phi_{2,\alpha} \in \Phi$ such that

$$d_\alpha(T_1 x, T_1 y) \le \phi_{1,\alpha}\bigl(d_{j_1(\alpha)}(x,y)\bigr) \quad \text{and} \quad d_\alpha(T_2 x, T_2 y) \le \phi_{2,\alpha}\bigl(d_{j_2(\alpha)}(x,y)\bigr),$$

for any $\alpha \in A$ and $x, y \in X$. Suppose further that

(i) $j_1^{n+1} = j_2^n \circ j_1$ and $j_1^n \circ j_2 = j_2^{n+1}$ for any $n \in \mathbf{N}$,

(ii) for each $\alpha \in A$, $\phi_{1,\alpha}(t) = c_1(\alpha)t$ and $\phi_{2,\alpha}(t) = c_2(\alpha)t$ for some $c_1(\alpha) + c_2(\alpha) \in (0,1)$, and

(iii) there exists $x_0 \in X$ such that $d_{j_1^n(\alpha)}(x_0, T_1x_0) \le p_1(x_0,\alpha) < \infty$ and $d_{j_2^n(\alpha)}(x_0, T_2x_0) \le p_2(x_0,\alpha) < \infty$ for any $\alpha \in A$ and $n = 1,2,\ldots$.

Then $H = \frac{T_1+T_2}{2}$ is a J-contraction with $J(\alpha) = \{j_1(\alpha), j_2(\alpha)\}$ and $\phi_{H,\alpha}(t) = (c_1(\alpha) + c_2(\alpha))t$. Also, by (i) and (iii), we have $|A(\alpha)| = 2 < \infty$ and

$$d_{\alpha_n}(x_0, Hx_0) \le \frac{d_{\alpha_n}(x_0, T_1x_0) + d_{\alpha_n}(x_0, T_2x_0)}{2} \le \frac{p_1(x_0,\alpha) + p_2(x_0,\alpha)}{2}.$$

Hence, H satisfies all conditions in Theorem 2.13, and it has a fixed point in X. Notice that H may not be a Φ-contraction, by choosing j_1 and j_2 so that $d_{j_1(\alpha)} + d_{j_2(\alpha)} \notin \mathcal{A}$ for some $\alpha \in A$, and hence Theorem 2 in [1] cannot be applied.

We now end this section by giving an application to the solution of a certain integral equation in locally convex spaces.

Example 2.16 Following terminologies in [8], let X be an $\mathcal{S}$-space topologized by the family of seminorms $\{|\cdot|_\alpha : \alpha \in A\}$ and $C([0,T];X)$ the space of all continuous functions from $[0,T]$ into X topologized by the family of seminorms $\{\|\cdot\|_\alpha : \alpha \in A\}$, where $\|x\|_\alpha := \sup_{t\in[0,T]} |x(t)|_\alpha$ for any $x \in C([0,T];X)$. Let $L(X)$ denote the set of all continuous linear operators on X,

$$L_0(X) = \{l \in L(X) : \forall\alpha \in A, \exists M(\alpha) > 0, \forall x \in X, |lx|_\alpha \le M(\alpha)|x|_\alpha\},$$

and let $\{S(t)\}_{t\ge 0}$ be a C_0-semigroup on X such that $S : [0,\infty) \to L_0(X)$ is locally bounded.

Now, we replace H3 and H5 in [8] by conditions (N1), (N2) and (N3) as follows:

(N1) $B : C([0,T];X) \to C([0,T];X)$ is an operator such that there exists $J_B : A \to \mathcal{P}^f(A)$ so that for any $\alpha \in A$, there is $k_{\alpha,B} \in L^1_{\mathrm{loc}}([0,T];[0,\infty))$ such that

$$|Bx(t) - By(t)|_\alpha \le k_{\alpha,B}(t) \sum_{\beta\in J_B(\alpha)} |x(t) - y(t)|_\beta,$$

for any $x, y \in C([0,T];X)$.

(N2) $f : [0,T] \times X \times X \to X$ is continuous and there exist $J_f : A \to \mathcal{P}^f(A)$ and $K_f \in L^1_{\mathrm{loc}}([0,T];[0,\infty))$ such that for each $\alpha \in A$,

$$|f(t,u_1,v_1) - f(t,u_2,v_2)|_\alpha \le K_f(t)\left(\sum_{\beta\in J_f(\alpha)} |u_1 - u_2|_\beta + |v_1 - v_2|_\alpha\right),$$

for any $t \in [0,T]$ and $u_1, u_2, v_1, v_2 \in X$,

(N3) $K_f \cdot k_{\alpha,B} \in L^1_{\mathrm{loc}}([0,T];[0,\infty))$.

Consider the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s), Bx(s))\,ds; \quad t \in [0,T] \tag{2}$$

whose solution is closely related to the mild solution of the differential equation

$$\frac{dx}{dt} = ax + f\big(t, x(t), Bx(t)\big),$$

where a denotes the infinitesimal generator of $\{S(t)\}_{t\geq 0}$.

We now define an operator G on $C_{x_0}([0,T];X) = \{x \in C([0,T];X) : x(0) = x_0\}$ by

$$(Gx)(t) = S(t)x_0 + \int_0^t S(t-s)f\big(s, x(s), Bx(s)\big)\,ds,$$

for any $x \in C_{x_0}([0,T];X)$. Following the proof of Theorem 3 in [8] and for each $t > 0$, $S(t) \in L_0(X)$, then we can show that, for any $\alpha \in A$, there exists $M(\alpha) > 0$ such that

$$\|Gx - Gy\|_\alpha \leq H_\alpha\left(\sum_{\beta\in J_f(\alpha)} \|x-y\|_\beta + \sum_{\beta\in J_B(\alpha)} \|x-y\|_\beta\right),$$

where $H_\alpha = \max\{M(\alpha)\int_0^T K_f(s)\,ds, M(\alpha)\int_0^T K_f(s)k_{\alpha,B}(s)\,ds\}$. It is easy to see that if for each $\alpha \in A$, $H_\alpha \in (0,1)$ and $J_f(\alpha) \cap J_B(\alpha) = \emptyset$, then G is a J-contraction with $J_G(\alpha) = J_f(\alpha) \cup J_B(\alpha)$.

In particular, if we assume further that for each $\alpha \in A$, $J_f(\alpha) = \{\alpha\}$, $|J_B(\alpha)| = 1$ such that $J_B \circ J_B = J_B$ and $H_\alpha = H_{J_B(\alpha)} < \frac{1}{2}$. Then for any $k \in \mathbf{N}$ and $x, y \in C_{x_0}([0,T];X)$, we have

$$\begin{aligned}\big\|G^k x - G^k y\big\|_\alpha &\leq H_\alpha^k \|x-y\|_\alpha + \left(\sum_{i=1}^k (2H_{J_B(\alpha)})^{k-i} H_\alpha^i\right)\|x-y\|_{J_B(\alpha)} \\ &= H_\alpha^k \|x-y\|_\alpha + \left(\sum_{i=1}^k 2^{k-i} H_\alpha^k\right)\|x-y\|_{J_B(\alpha)} \\ &\leq 2^{k-1} H_\alpha^k \left(\|x-y\|_\alpha + \sum_{i=1}^k \|x-y\|_{J_B(\alpha)}\right).\end{aligned}$$

Now, by letting $\phi_{\alpha,k}(t) = 2^{k-1}H_\alpha^k t$, $D_{\alpha,k} = \{(1,\alpha), (1, J_B(\alpha))(2, J_B(\alpha)), \ldots, (k, J_B(\alpha))\}$, $P_{\alpha,k}(\gamma) = \pi_2(\gamma)$, and $F_\alpha(x,y) = \max\{\|x-y\|_\alpha, \|x-y\|_{J_B(\alpha)}\}$, we have

(i) $\|x-y\|_{P_{\alpha,k}(\gamma)} \leq F_\alpha(x,y)$ for any $x, y \in C_{x_0}([0,T];X)$, $k \in \mathbf{N}$, $\alpha \in A$, and $\gamma \in D_{\alpha,k}$,

(ii) $\sum_{k\in\mathbf{N}} |D_{\alpha,k}|\phi_{\alpha,k}(F_\alpha(x,y)) = \sum_{k\in\mathbf{N}} \frac{k+1}{2}(2H_\alpha)^k F_\alpha(x,y) < \infty$ for any $x, y \in C_{x_0}([0,T];X)$ and $\alpha \in A$.

Therefore, by Theorem 2.11(2), G has a unique fixed point, so the integral equation (2) has a unique solution.

3 Fixed point sets

In this section, we will show that, under a mild condition, a J-nonexpansive map is always virtually stable. This immediately gives a connection between the fixed point set and the convergence set of a J-nonexpansive map. Recall that a continuous self-map $T : X \to X$, whose fixed point set $F(T)$ is nonempty, on a Hausdorff space X is said to be virtually stable [4] if for each $x \in F(T)$ and each neighborhood U of x, there exist a neighborhood V of x and an increasing sequence (k_n) of positive integers such that $T^{k_n}(V) \subseteq U$ for all $n \in \mathbf{N}$. When the sequence (k_n) is independent of the point x and the neighborhood U, we simply call T a uniformly virtually stable map with respect to (k_n). For example, a (quasi-)

nonexpansive self-map, whose fixed point set is nonempty, on a metric space is always uniformly virtually stable with respect to the sequence (n) of all natural numbers. An important feature of a virtually stable map is the connection between its fixed point set and its convergence set as given in the following theorem.

Theorem 3.1 ([4], Theorem 2.6) *Suppose X is a regular space. If $T: X \to X$ is virtually stable, then $F(T)$ is a retract of $C(T)$, where $C(T)$ is the (Picard) convergence set of T defined as follows*:

$$C(T) = \{x \in X : \text{the sequence } (T^n x) \text{ converges}\}.$$

As in the previous section, let $(E, \mathcal{A})$ be a Hausdorff uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{A} = \{d_\alpha : \alpha \in A\}$ indexed by A and $\emptyset \neq X \subseteq E$. The following theorem gives a general criterion for a self-map on X to be virtually stable.

Theorem 3.2 *Let $T: X \to X$ be a self-map whose fixed point set $F(T)$ is nonempty, and which satisfies the following conditions*:

(i) *for each $\alpha \in A$ and $k \in \mathbf{N}$, there exist a finite set $D_{\alpha,k}$ and a map $P_{\alpha,k} : D_{\alpha,k} \to A$ such that*

$$d_\alpha(T^k x, T^k y) \leq \sum_{\gamma \in D_{\alpha,k}} d_{P_{\alpha,k}(\gamma)}(x, y),$$

for any $x, y \in X$,

(ii) *there exists $N \in \mathbf{N}$ such that $|D_{\alpha,n}| \leq |D_{\alpha,N}|$ and $P_{\alpha,n}(D_{\alpha,n}) \subseteq P_{\alpha,N}(D_{\alpha,N})$ for any $n \geq N$ and $\alpha \in A$.*

Then T is uniformly virtually stable with respect to the sequence of all natural numbers.

Proof Let $z \in F(T)$ and let U be a neighborhood of z. We may assume that $U = \bigcap_{i=1}^m \{w \in X : d_{\alpha_i}(w, z) < \epsilon\}$ for some $\epsilon > 0$ and $\alpha_1, \ldots, \alpha_m \in A$. For each $n \in \mathbf{N}$, let

$$V_n = \bigcap_{i=1}^m \bigcap_{\gamma \in D_{\alpha_i,n}} \left\{w \in X : d_{P_{\alpha_i,n}(\gamma)}(w, z) < \frac{\epsilon}{|D_{\alpha_i,n}|}\right\}.$$

By (ii), there exists $N \in \mathbf{N}$ such that $|D_{\alpha_i,n}| \leq |D_{\alpha_i,N}|$ and $P_{\alpha_i,n}(D_{\alpha_i,n}) \subseteq P_{\alpha_i,N}(D_{\alpha_i,N})$ for any $n \geq N$ and $i = 1, \ldots, m$. Let $V = V_1 \cap V_2 \cap \cdots \cap V_N$ which is clearly a nonempty open subset of X, $y \in V$, $l \in \mathbf{N}$ and $i \in \{1, \ldots, m\}$. It follows that

$$d_{\alpha_i}(T^l y, z) = d_{\alpha_i}(T^l y, T^l z) \leq \sum_{\gamma \in D_{\alpha_i,l}} d_{P_{\alpha_i,l}(\gamma)}(y, z).$$

If $l < N$, then

$$d_{\alpha_i}(T^l y, z) < \sum_{\gamma \in D_{\alpha_i,l}} \frac{\epsilon}{|D_{\alpha_i,l}|} = \epsilon.$$

If $l \geq N$, since $P_{\alpha_i,l}(\gamma) \in P_{\alpha_i,l}(D_{\alpha_i,l}) \subseteq P_{\alpha_i,N}(D_{\alpha_i,N})$, we have $d_{P_{\alpha_i,l}(\gamma)}(y,z) < \frac{\epsilon}{|D_{\alpha_i,N}|}$ for each $\gamma \in D_{\alpha_i,l}$, and hence

$$d_{\alpha_i}\bigl(T^l y, z\bigr) < \sum_{\gamma \in D_{\alpha_i,l}} \frac{\epsilon}{|D_{\alpha_i,N}|} = \frac{\epsilon|D_{\alpha_i,l}|}{|D_{\alpha_i,N}|} \leq \epsilon.$$

Hence, T is uniformly virtually stable with respect to the sequence of all natural numbers. □

Corollary 3.3 *Suppose that T is J-nonexpansive with $F(T) \neq \emptyset$. If there exists $N \in \mathbf{N}$ such that $|A_n(\alpha)| \leq |A_N(\alpha)|$ and $\pi_n(A_n(\alpha)) \subseteq \pi_N(A_N(\alpha))$ for any $n \geq N$ and $\alpha \in A$, then T is uniformly virtually stable with respect to the sequence of all natural numbers.*

Proof By letting $D_{\alpha,n} = A_n(\alpha)$ and $P_{\alpha,n} = \pi_n|_{A_n(\alpha)}$ for any $n \in \mathbf{N}$ and $\alpha \in A$, we have

$$d_\alpha\bigl(T^l x, T^l y\bigr) \leq \sum_{\gamma \in D_{\alpha,l}} d_{P_{\alpha,l}(\gamma)}(x,y),$$

for any $x, y \in X$. The result then follows from Theorem 3.2. □

Example 3.4 Let $E = \ell_2$ equipped with the weak topology and $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \dots) = \left(\frac{|x_1 + x_3|}{3}, \frac{|x_2 + x_4|}{3}, x_3, x_4, \dots\right),$$

for any $(x_1, x_2, \dots) \in \ell_2$. Then $\mathcal{A} = \{|f| : f \in \ell_2\}$, and by Lemma 4.5 and Theorem 4.6 in [7], we have

$$\begin{aligned}
&\bigl|f\bigl(T^n x - T^n y\bigr)\bigr| \\
&\quad\leq 2\|f\|\biggl[\frac{\sqrt{2}}{9}\bigl(|x_1 - y_1 + x_3 - y_3| + |x_2 - y_2 + x_4 - y_4|\bigr) \\
&\qquad+ \frac{\sqrt{2}(|x_1 - y_1| + |x_2 - y_2| + |x_1 - y_1 + x_3 - y_3| + |x_2 - y_2 + x_4 - y_4|)}{9 - 6\sqrt{2}}\biggr] \\
&\qquad+ \|f\|\biggl(\frac{1}{3}|x_1 - y_1| + |x_1 - y_1 + x_3 - y_3| + \frac{1}{3}|x_2 - y_2| + |x_2 - y_2 + x_4 - y_4|\biggr) \\
&\qquad+ \|f\||x_1 - y_1| + \|f\||x_2 - y_2| + \bigl|f(x - y)\bigr|,
\end{aligned}$$

for each $f \in \ell_2$, $n \in \mathbf{N}$, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots) \in \ell_2$.

By letting $J : \ell_2 \to \mathcal{P}(\ell_2)$ be defined by $J(f) = \{|f|, |g_1|, |g_2|, |g_3|, |g_4|\}$ for each $f \in \ell_2$, where

$$\begin{aligned}
g_1(x) &= \|f\|\left(\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9 - 6\sqrt{2}} + 1\right)(x_1 + x_3), \\
g_2(x) &= \|f\|\left(\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9 - 6\sqrt{2}} + 1\right)(x_2 + x_4), \\
g_3(x) &= \|f\|\left(\frac{2\sqrt{2}}{9 - 6\sqrt{2}} + \frac{4}{3}\right)x_1, \qquad g_4(x) = \|f\|\left(\frac{2\sqrt{2}}{9 - 6\sqrt{2}} + \frac{4}{3}\right)x_2,
\end{aligned}$$

for each $x = (x_1, x_2, \dots) \in \ell_2$, it follows that T is J-nonexpansive.

Notice that $(0, 0, \dots)$ is a fixed point of T, and for each $f \in \ell_2$ and $n, m \in \mathbf{N}$, $\pi_n(A(|f|)) = \pi_m(A(|f|))$. Then, by Theorem 3.2, T is virtually stable and hence the fixed point set of T is a retract of the convergence set of T. Moreover, the fixed point set is not convex because $x = (1, 1, 2, 2, 0, \dots)$ and $y = (1, 1, -4, -4, 0, \dots)$ are fixed points of T, while the convex combination $\frac{1}{2}x + \frac{1}{2}y = (1, 1, -1, -1, 0, \dots)$ is not.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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