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Contraction conditions using comparison functions on b-metric spaces

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Abstract

In this paper, we consider the setting of b-metric spaces to establish results regarding the common fixed points of two mappings, using a contraction condition defined by means of a comparison function. An example is presented to support our results comparing with existing ones.

MSC: 49H09; 47H10

Keywords: b-metric space; common fixed point; contraction condition; comparison function

1 Introduction

The contraction principle of Banach [1], proved in 1922, was followed by diverse works about fixed points theory regarding different classes of contractive conditions on some spaces such as: quasi-metric spaces [2, 3], cone metric spaces [4, 5], partially ordered metric spaces [6–8], G-metric spaces [9], partial metric spaces [10–13], Menger spaces [14], metric-type spaces [15], and fuzzy metric spaces [16–18]. Also, there have been developed studies on approximate fixed point or on qualitative aspects of numerical procedures for approximating fixed points see, for example [19, 20].

The concept of b-metric spaces was introduced by Bakhtin [21] in 1989, who used it to prove a generalization of the Banach principle in spaces endowed with such kind of metrics. Since then, this notion has been used by many authors to obtain various fixed point theorems. Aydi *et al.* in [22] proved common fixed point results for single-valued and multi-valued mappings satisfying a weak ϕ-contraction in b-metric spaces. Roshan *et al.* in [23] used the notion of almost generalized contractive mappings in ordered complete b-metric spaces and established some fixed and common fixed point results. Starting from the results of Berinde [24], Păcurar [25] proved the existence and uniqueness of fixed points of ϕ-contractions on b-metric spaces. Hussain and Shah in [26] introduced the notion of a cone b-metric space, generalizing both notions of b-metric spaces and cone metric spaces. In this paper they also considered topological properties of cone b-metric spaces and results on KKM mappings in the setting of cone b-metric spaces. Fixed point theorems of contractive mappings in cone b-metric spaces without the assumption of the normality of a corresponding cone are proved by Huang and Xu in [27]. The setting of partially ordered b-metric spaces was used by Hussain *et al.* in [28] to study tripled coincidence points of mappings which satisfy nonlinear contractive conditions, extending those results of Berinde and Borcut [29] for metric spaces to b-metric spaces. Using the concept of a g-monotone mapping, Shah and Hussain in [30] proved common fixed point theorems involving g-non-decreasing mappings in b-metric spaces, generalizing several results of Agarwal *et al.* [31] and Ćirić *et al.* [32]. Some results of Suzuki [33] are extended to the case of metric-type spaces and cone metric-type spaces.

The aim of this paper is to consider and establish results on the setting of b-metric spaces, regarding common fixed points of two mappings, using a contraction condition defined by means of a comparison function. An example is given to support our results.

2 Preliminaries

Definition 1 Let X be a nonempty set and $d: X \times X \to [0, +\infty)$. A function d is called a b-metric with constant (base) $s \geq 1$ if:

(1) $d(x, y) = 0$ iff $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is obvious that a b-metric space with base $s = 1$ is a metric space. There are examples of b-metric spaces which are not metric spaces (see, *e.g.*, Singh and Prasad [34]).

The notions of a Cauchy sequence and a convergent sequence in b-metric spaces are defined by Boriceanu [35].

Definition 2 Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(1) A sequence $\{x_n\}$ is called convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ when $n \to +\infty$.
(2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, when $n, m \to +\infty$.

As usual, a b-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Regarding the properties of a b-metric space, we recall that if the limit of a convergent sequence exists, then it is unique. Also, each convergent sequence is a Cauchy sequence. But note that a b-metric, in the general case, is not continuous (see Roshan *et al.* [23]).

The continuity of a mapping with respect to a b-metric is defined as follows.

Definition 3 Let (X, d) and (X', d') be two b-metric spaces with constant s and s', respectively. A mapping $T: X \to X'$ is called continuous if for each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d, then Tx_n converges to Tx with respect to d'.

Definition 4 Let $s \geq 1$ be a constant. A mapping $\varphi: [0, +\infty) \to [0, +\infty)$ is called comparison function with base $s \geq 1$, if the following two axioms are fulfilled:

(a) φ is non-decreasing,
(b) $\lim_{n \to +\infty} \varphi^n(t) = 0$ for all $t > 0$.

Clearly, if φ is a comparison function, then $\varphi(t) < t$ for each $t > 0$.

For different properties and applications of comparison functions on partial metric spaces, we refer the reader to [36].

3 Main results

Now we are ready to prove our main results.

Theorem 1 *Let (X,d) be a complete b-metric space with a constant s and $T,S\colon X\to X$ two mappings on X. Suppose that there is a constant $L<\frac{1}{1+s}$ and a comparison function φ such that the inequality*

$$sd(Tx,Sy)\le\varphi\bigl(\max\bigl\{sd(x,Tx),sd(y,Sy),L\bigl[d(x,Sy)+d(Tx,y)\bigr]\bigr\}\bigr) \tag{3.1}$$

holds for each $x,y\in X$. Suppose that one of the mappings T or S is continuous. Then T and S have a unique common fixed point.

Proof Let $x_0\in X$ be arbitrary. We define a sequence $\{x_n\}$ as follows:

$$x_{2n+1}=Tx_{2n},\qquad x_{2n+2}=Sx_{2n+1},\quad n\in\mathbb{N}. \tag{3.2}$$

Suppose that there is some $n\in\mathbb{N}$ such that $x_n=x_{n+1}$. If $n=2k$, then $x_{2k}=x_{2k+1}$ and from the contraction condition (3.1) with $x=x_{2k}$ and $y=x_{2k+1}$ we have

$$\begin{aligned} sd(x_{2k+1},x_{2k+2}) &= sd(Tx_{2k},Sx_{2k+1}) \\ &\le\varphi\bigl(\max\bigl\{sd(x_{2k},Tx_{2k}),sd(x_{2k+1},Sx_{2k+1}), \\ &\quad L\bigl[d(x_{2k},Sx_{2k+1})+d(Tx_{2k},x_{2k+1})\bigr]\bigr\}\bigr) \\ &=\varphi\bigl(\max\bigl\{sd(x_{2k},x_{2k+1}),sd(x_{2k+1},x_{2k+2}),Ld(x_{2k},x_{2k+2})\bigr\}\bigr). \end{aligned}$$

Hence, as we supposed that $x_{2k}=x_{2k+1}$ and as a comparison function φ is non-decreasing,

$$\begin{aligned} sd(x_{2k+1},x_{2k+2}) &\le\varphi\bigl(\max\bigl\{sd(x_{2k+1},x_{2k+2}),L\bigl[s\bigl(d(x_{2k},x_{2k+1})+d(x_{2k+1},x_{2k+2})\bigr)\bigr]\bigr\}\bigr) \\ &=\varphi\bigl(\max\bigl\{sd(x_{2k+1},x_{2k+2}),Lsd(x_{2k+1},x_{2k+2})\bigr\}\bigr) \\ &=\varphi\bigl(sd(x_{2k+1},x_{2k+2})\bigr). \end{aligned}$$

If we assume that $d(x_{2k+1},x_{2k+2})>0$, then we have, as $\varphi(t)<t$ for $t>0$,

$$sd(x_{2k+1},x_{2k+2})\le\varphi\bigl(sd(x_{2k+1},x_{2k+2})\bigr)<sd(x_{2k+1},x_{2k+2}),$$

a contradiction. Therefore, $d(x_{2k+1},x_{2k+2})=0$. Hence $x_{2k+1}=x_{2k+2}$. Thus we have $x_{2k}=x_{2k+1}=x_{2k+2}$. By (3.2), it means $x_{2k}=Tx_{2k}=Sx_{2k}$, that is, x_{2k} is a common fixed point of T and S.

If $n=2k+1$, then using the same arguments as in the case $x_{2k}=x_{2k+1}$, it can be shown that x_{2k+1} is a common fixed point of T and S.

From now on, we suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$.

Now we shall prove that

$$sd(x_n,x_{n+1})\le\varphi\bigl(sd(x_{n-1},x_n)\bigr)\quad\text{for each } n\in\mathbb{N}. \tag{3.3}$$

There are two cases which we have to consider.

Case I. $n = 2k$, $k \in \mathbb{N}$.

From the contraction condition (3.1) with $x = x_{2k}$ and $y = x_{2k-1}$ we get

$$\begin{aligned} sd(x_{2k+1}, x_{2k}) &= sd(Tx_{2k}, Sx_{2k-1}) \\ &\le \varphi\big(\max\big\{sd(x_{2k}, Tx_{2k}), sd(x_{2k-1}, Sx_{2k-1}), \\ &\quad L\big[d(x_{2k}, Sx_{2k-1}) + d(Tx_{2k}, x_{2k-1})\big]\big\}\big) \\ &= \varphi\big(\max\big\{sd(x_{2k}, x_{2k+1}), sd(x_{2k-1}, x_{2k}), Ld(x_{2k+1}, x_{2k-1})\big\}\big). \end{aligned}$$

Since $L < 1/2$, we get

$$\begin{aligned} sd(x_{2k+1}, x_{2k}) &\le \varphi\bigg(\max\bigg\{sd(x_{2k-1}, x_{2k}), sd(x_{2k}, x_{2k+1}), \\ &\quad \frac{s}{2}\big(d(x_{2k-1}, x_{2k}) + d(x_{2k}, x_{2k+1})\big)\bigg\}\bigg) \\ &= \varphi\big(\max\big\{sd(x_{2k-1}, x_{2k}), sd(x_{2k}, x_{2k+1})\big\}\big). \end{aligned}$$

Now, if we suppose that $\max\{sd(x_{2k}, x_{2k-1}), sd(x_{2k}, x_{2k+1})\} = sd(x_{2k}, x_{2k+1})$, then by the property (a) of φ in Definition 4 we get

$$sd(x_{2k}, x_{2k+1}) \le \varphi\big(sd(x_{2k}, x_{2k+1})\big) < sd(x_{2k}, x_{2k+1}),$$

a contradiction. Therefore, from the above inequality we have

$$sd(x_{2k}, x_{2k+1}) \le \varphi\big(sd(x_{2k-1}, x_{2k})\big). \tag{3.4}$$

Thus we proved that (3.3) holds for $n = 2k$.

Case II. $n = 2k + 1$, $k \in \mathbb{N}$.

Using the same argument as in the Case I, it can be proved that (3.3) holds for $n = 2k + 1$, that is,

$$sd(x_{2k+1}, x_{2k+2}) \le \varphi\big(sd(x_{2k}, x_{2k+1})\big). \tag{3.5}$$

From (3.4) and (3.5) we conclude that the inequality (3.3) holds for all $n \in \mathbb{N}$.

From (3.3), by the induction it is easy to prove that

$$sd(x_n, x_{n+1}) \le \varphi^n\big(sd(x_0, x_1)\big) \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Since $\lim_{n\to+\infty} \varphi^n(t) = 0$ for all $t > 0$, from (3.6) it follows that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.7}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$. Since $L < \frac{1}{1+s}$ implies $s - 2L > 0$ and $1 - L(1 + s) > 0$, from (3.7) we conclude that there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n-1}) < \frac{1 - L - Ls}{2s}\epsilon \tag{3.8}$$

for all $n \ge n_0$.

Let $m, n \in \mathbb{N}$ with $m > n$. By induction on m, we shall prove that

$$d(x_n, x_m) < \epsilon \quad \text{for all } m > n \geq n_0. \tag{3.9}$$

Let $n \geq n_0$ and $m = n + 1$. Then from (3.3) and (3.8) we get

$$d(x_n, x_m) = d(x_n, x_{n+1}) \leq d(x_n, x_{n-1}) < \frac{1 - L - Ls}{2s}\epsilon < \epsilon.$$

Thus (3.9) holds for $m = n + 1$.

Assume now that (3.9) holds for some $m \geq n + 1$. We have to prove that (3.9) holds for $m + 1$.

We have to consider four cases.

Case I. n is odd, $m + 1$ is even.

From the contraction condition (3.1) we get

$$\begin{aligned} sd(x_n, x_{m+1}) &= sd(Tx_{n-1}, Sx_m) \\ &\leq \varphi\bigl(\max\bigl\{sd(x_{n-1}, x_n), sd(x_m, x_{m+1}), \\ &\quad L\bigl[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)\bigr]\bigr\}\bigr). \end{aligned}$$

Hence we get, as $d(x_m, x_{m+1}) < d(x_{n-1}, x_n)$ and $\varphi(t) < t$ for all $t > 0$,

$$sd(x_n, x_{m+1}) < \max\bigl\{sd(x_{n-1}, x_n), L\bigl[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)\bigr]\bigr\}. \tag{3.10}$$

If from (3.10) we have $sd(x_n, x_{m+1}) < sd(x_{n-1}, x_n)$, then by (3.8),

$$d(x_n, x_{m+1}) < d(x_{n-1}, x_n) < \frac{1 - L - Ls}{2s}\epsilon < \epsilon.$$

If (3.10) implies $sd(x_n, x_{m+1}) < L[d(x_{n-1}, x_{m+1}) + d(x_n, x_m)]$, then by the (general) triangle inequality,

$$sd(x_n, x_{m+1}) < Lsd(x_{n-1}, x_n) + Lsd(x_n, x_{m+1}) + Ld(x_n, x_m).$$

Hence we get, as $L < 1/(1 + s)$ implies $L/(1 - L) < 2L < 1 \leq s$,

$$\begin{aligned} d(x_n, x_{m+1}) &< \frac{L}{1 - L}\left[d(x_{n-1}, x_n) + \frac{1}{s}d(x_n, x_m)\right] \\ &< 2L\left[d(x_{n-1}, x_n) + \frac{1}{s}d(x_n, x_m)\right]. \end{aligned}$$

Now, by (3.8) and the induction hypothesis (3.9),

$$\begin{aligned} d(x_n, x_{m+1}) &< 2L\frac{1 - L - Ls}{2s}\epsilon + \frac{2L}{s}\epsilon < \frac{1 - 2L - L(s - 1)}{s}\epsilon + \frac{2L}{s}\epsilon \\ &\leq \frac{1 - 2L}{s}\epsilon + \frac{2L}{s}\epsilon = \frac{1}{s}\epsilon \leq \epsilon. \end{aligned}$$

Thus we proved that in this case (3.9) holds for $m+1$. Therefore, by induction, we conclude that in Case I the inequality (3.9) holds for all $m > n$.

Case II. n is even, $m+1$ is odd. The proof of (3.9) in this case is similar to one given in Case I.

Case III. n is even, $m+1$ is even.

Using the (general) triangle inequality and the contraction condition (3.1), we obtain

$$\begin{aligned} d(x_n, x_{m+1}) &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{m+1}) \\ &= sd(x_n, x_{n+1}) + sd(Tx_n, Sx_m) \\ &\leq sd(x_n, x_{n+1}) + \varphi\bigl(\max\bigl\{sd(x_n, x_{n+1}), sd(x_m, x_{m+1}), \\ &\quad L\bigl[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\bigr]\bigr\}\bigr) \\ &= sd(x_n, x_{n+1}) + \varphi\bigl(\max\bigl\{sd(x_n, x_{n+1}), \\ &\quad L\bigl[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\bigr]\bigr\}\bigr). \end{aligned}$$

Hence we get, as $d(x_m, x_{m+1}) < d(x_{n-1}, x_n)$ and $\varphi(t) < t$ for all $t > 0$,

$$d(x_n, x_{m+1}) < sd(x_n, x_{n+1}) + \max\bigl\{sd(x_n, x_{n+1}), L\bigl[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\bigr]\bigr\}. \tag{3.11}$$

If the inequality (3.11) implies $d(x_n, x_{m+1}) < sd(x_n, x_{n+1}) + sd(x_n, x_{n+1})$, then from (3.8) we get

$$d(x_n, x_{m+1}) < 2s\frac{1-L-Ls}{2s}\epsilon = \epsilon.$$

If (3.11) implies

$$d(x_n, x_{m+1}) < sd(x_n, x_{n+1}) + L\bigl[d(x_n, x_{m+1}) + d(x_{n+1}, x_m)\bigr],$$

then by the (general) triangle inequality we have

$$\begin{aligned} d(x_n, x_{m+1}) &< sd(x_n, x_{n+1}) + Ld(x_n, x_{m+1}) + Lsd(x_{n+1}, x_n) + Lsd(x_n, x_m) \\ &= (1+L)sd(x_n, x_{n+1}) + Ld(x_n, x_{m+1}) + Lsd(x_n, x_m). \end{aligned}$$

Hence we get

$$(1-L)d(x_n, x_{m+1}) \leq (1+L)sd(x_n, x_{n+1}) + Lsd(x_n, x_m).$$

Now, by (3.8) and the induction hypothesis (3.3), we have

$$(1-L)d(x_n, x_{m+1}) < \frac{(1+L)s[(1-L)-Ls]}{2s}\epsilon + Ls\epsilon < \bigl[(1-L)-Ls\bigr]\epsilon + Ls\epsilon = (1-L)\epsilon.$$

Hence

$$d(x_n, x_{m+1}) < \epsilon.$$

Thus we proved that (3.9) holds for $m+1$. Therefore, by induction, we conclude that in Case III the inequality (3.9) holds for all $m > n$.

Case IV. n is odd, $m+1$ is odd. The proof of (3.9) in this case is similar to one given in Case III.

Therefore, we proved that in all of four cases the inequality (3.9) holds.

From (3.9) it follows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete b-metric space, then $\{x_n\}$ converges to some $u \in X$ as $n \to +\infty$.

Now we shall prove that if one of the mappings T or S is continuous, then $Tu = Su = u$. Without loss of generality, we can suppose that S is continuous. Clearly, as $x_n \to u$, then by (3.2) we have $Sx_{2n+1} = x_{2n+2} \to u$ as $n \to +\infty$. Since $x_{2n+1} \to u$ and S is continuous, then $Sx_{2n+1} \to Su$. Thus, by the uniqueness of the limit in a b-metric space, we have $Su = u$. Now, from the contraction condition (3.1),

$$\begin{aligned} sd(Tu, u) &= sd(Tu, Su) \\ &\leq \varphi\bigl(\max\bigl\{sd(u, Tu), sd(u, Su), L\bigl[d(Tu, u) + d(u, Su)\bigr]\bigr\}\bigr) \\ &= \varphi\bigl(sd(u, Tu)\bigr). \end{aligned}$$

If we suppose that $d(u, Tu) > 0$, then we have

$$sd(u, Tu) \leq \varphi\bigl(sd(u, Tu)\bigr) < sd(u, Tu),$$

a contradiction. Therefore, $d(u, Tu) = 0$. Hence $Tu = u$. Thus we proved that u is a common fixed point of T and S.

Suppose now that u and v are different common fixed points of T and S, that is, $d(u, v) > 0$. Then

$$\begin{aligned} sd(u, v) &= sd(Tu, Sv) \\ &\leq \varphi\bigl(\max\bigl\{sd(u, Tu), sd(v, Sv), L\bigl(d(u, Sv) + d(v, Tu)\bigr)\bigr\}\bigr) \\ &= \varphi\bigl(2Ld(u, v)\bigr). \end{aligned}$$

Since $2L < 1 \leq s$, then we get $sd(u, v) \leq \varphi(sd(u, v)) < sd(u, v)$, a contradiction. Thus we proved that S and T have a unique common fixed point in X. □

If $S = T$ in Theorem 1, then we have the following result.

Corollary 1 *Let (X, d) be a complete b-metric space with a constant s and $T : X \to X$ two mappings on X. Suppose that there is a constant $L < \frac{1}{2}$ and a comparison function φ such that the inequality*

$$sd(Tx, Ty) \leq \varphi\bigl(\max\bigl\{sd(x, Tx), sd(y, Ty), L\bigl[d(x, Ty) + d(Tx, y)\bigr]\bigr\}\bigr) \tag{3.12}$$

holds for each $x, y \in X$. Suppose that a mapping T is continuous. Then T has a unique fixed point.

Omitting the continuity assumption of mapping T or S in Theorem 1, modifying the contraction condition (3.1) and imposing on a comparison function φ a corresponding condition, then we can prove the following theorem.

Theorem 2 *Let (X,d) be a complete b-metric space with a constant s and $T,S\colon X\to X$ two mappings on X. Suppose that there is a constant $L<\frac{1}{1+s}$ and a comparison function φ such that the inequality*

$$sd(Tx,Ty)\le\varphi\bigl(\max\bigl\{sd(x,Tx),d(y,Ty),L\bigl(d(x,Ty)+d(Tx,y)\bigr)\bigr\}\bigr) \tag{3.13}$$

holds for all $x,y\in X$. If in addition a comparison function φ satisfies the following condition:

$$\limsup_{\beta\to\alpha}\varphi(\beta)<\alpha,\quad \alpha>0, \tag{3.14}$$

then T and S have a unique common fixed point.

Proof Since the contraction condition (3.13) implies the contraction condition (3.1) in Theorem 1, then from the proof of Theorem 1 it follows that a sequence $\{x_n\}$, defined as in (3.3), converges to some $u\in X$, that is,

$$Tx_{2n}=x_{2n+1}\to u\quad\text{and}\quad Sx_{2n+1}=x_{2n+2}\to u\quad\text{as } n\to+\infty. \tag{3.15}$$

Now we prove that $Su=u$. From the contraction condition (3.13) and by the monotonicity of φ we obtain

$$\begin{aligned} sd(x_{2n+1},Su) &= sd(Tx_{2n},Su)\\ &\le\varphi\bigl(\max\bigl\{sd(x_{2n},x_{2n+1}),d(u,Su),L\bigl(d(x_{2n+1},u)+d(x_{2n},Su)\bigr)\bigr\}\bigr)\\ &\le\varphi\bigl(\max\bigl\{sd(x_{2n},x_{2n+1}),sd(u,x_{2n+1})+sd(x_{2n+1},Su),\\ &\qquad L\bigl(d(x_{2n+1},u)+sd(x_{2n},x_{2n+1})+sd(x_{2n+1},Su)\bigr)\bigr\}\bigr). \end{aligned} \tag{3.16}$$

Since φ is non-decreasing and $L<1$, from (3.16) we get

$$sd(x_{2n+1},Su)\le\varphi\bigl(sd(x_{2n},x_{2n+1})+sd(u,x_{2n+1})+sd(x_{2n+1},Su)\bigr). \tag{3.17}$$

Set

$$t_n=sd(x_{2n},x_{2n+1})+sd(u,x_{2n+1})+sd(x_{2n+1},Su).$$

Then, in virtue of (3.15),

$$\limsup_{n\to\infty}t_n=\limsup_{n\to\infty}sd(x_{2n+1},Su)=r, \tag{3.18}$$

where $r\ge0$. Let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ such that $t_{n_k}\to r$ as $k\to\infty$. For simplicity, denote $\{t_{n_k}\}$ again by $\{t_n\}$. Then from (3.18),

$$\lim_{n\to\infty}t_n=\lim_{n\to\infty}sd(x_{2n+1},Su)=r. \tag{3.19}$$

Suppose that $r > 0$. Then from (3.19), (3.17), and the assumption (3.14) of φ, we have

$$r = \lim_{n\to\infty} t_n = \lim_{n\to\infty} sd(x_{2n+1}, Su) \leq \lim_{t_n\to r} \varphi(t_n) < r,$$

a contradiction. Therefore,

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} sd(x_{2n+1}, Su) = 0.$$

Hence we have $x_{2n+1} \to Su$ as $n \to \infty$. Since by (3.15), $x_{2n+1} \to u$, and as the limit in a b-metric space is unique, it follows that $Su = u$. Now, by (3.13),

$$\begin{aligned} sd(Tu, u) &= sd(Tu, Su) \\ &\leq \varphi\bigl(\max\bigl\{sd(u, Tu), d(u, Su), L\bigl(d(Tu, u) + d(u, Su)\bigr)\bigr\}\bigr) \\ &= \varphi\bigl(sd(u, Tu)\bigr). \end{aligned}$$

If we suppose that $d(u, Tu) > 0$, then we have $sd(Tu, u) \leq \varphi(sd(u, Tu)) < sd(u, Tu)$, a contradiction. Therefore, $d(Tu, u) = 0$, that is, $Tu = u$. Thus we proved that $Tu = Su = u$. □

If $S = T$ in Theorem 2, then we get the following result.

Corollary 2 *Let (X, d) be a complete b-metric space with a constant s and $T: X \to X$ a mapping on X. Suppose that there is a constant $L < \frac{1}{1+s}$ and a comparison function φ such that the inequality*

$$sd(Tx, Ty) \leq \varphi\bigl(\max\bigl\{sd(x, Tx), d(y, Ty), L\bigl[d(x, Ty) + d(Tx, y)\bigr]\bigr\}\bigr)$$

holds for all $x, y \in X$. If in addition a comparison function φ satisfies the inequality (3.14), *then T has a unique fixed point.*

Now we give an example to support our results.

Example 1 Let $X = [0, 1]$ endowed with the b-metric

$$d: X \times X \to [0, +\infty), \quad d(x, y) = (x - y)^2,$$

with constant $s = 2$. Consider mappings $T, S: X \to X$, $Tx = \frac{1}{4}x$, $Sx = \frac{1}{8}x$, and the comparison function $\varphi: [0, +\infty) \to [0, +\infty)$, $\varphi(t) = \frac{t}{t+1}$. Clearly, (X, d) is a complete metric space, and S is continuous with respect to d, so we have to verify the contraction condition (3.1). There are three cases to be considered.

Case I. $y = 2x$. Hence $Tx = Sy$, $d(Tx, Sy) = 0$, and, therefore, the inequality (3.1) holds.

Case II. $y > 2x$. Then $\frac{1}{8}y > \frac{1}{4}x$, and

$$\begin{aligned} 2d(Tx, Sy) &= 2\left(\frac{1}{8}y - \frac{1}{4}x\right)^2 \leq \frac{1}{32}y^2 \leq \frac{49}{64 + 49y^2}y^2 = \varphi\left(\frac{49}{32}y^2\right) = \varphi\bigl(2d(y, Sy)\bigr) \\ &= \varphi\bigl(\max\bigl\{2d(x, Tx), 2d(y, Sy), \bigl(d(x, Sy) + d(Tx, y)\bigr)\bigr\}\bigr). \end{aligned}$$

Thus in this case the contraction condition (3.1) holds.

Case III. $y < 2x$. Then

$$\begin{aligned} 2d(Tx,Sy) &= 2\left(\frac{1}{4}x - \frac{1}{8}y\right)^2 \\ &\leq \frac{1}{8}x^2 \leq \frac{9}{8}x^2 = \varphi\bigl(2d(x,Tx)\bigr) \\ &\leq \varphi\bigl(\max\bigl\{2d(x,Tx), 2d(y,Sy), \bigl(d(x,Sy) + d(Tx,y)\bigr)\bigr\}\bigr). \end{aligned}$$

Therefore, we showed that the contraction condition (3.1) is satisfied in all cases. Thus we can apply our Theorem 1, and T and S have a unique common fixed point $u = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
2. Caristi, J: Fixed point theorems for mapping satisfying inwardness conditions. Trans. Am. Math. Soc. **215**, 241-251 (1976)
3. Hicks, TL: Fixed point theorems for quasi-metric spaces. Math. Jpn. **33**(2), 231-236 (1988)
4. Altun, I, Durmaz, G: Some fixed point results in cone metric spaces. Rend. Circ. Mat. Palermo **58**, 319-325 (2009)
5. Choudhury, BS, Metiya, N: Coincidence point and fixed point theorems in ordered cone metric spaces. J. Adv. Math. Stud. **5**(2), 20-31 (2012)
6. Aydi, H, Shatanawi, W, Postolache, M, Mustafa, Z, Tahat, N: Theorems for Boyd-Wong type contractions in ordered metric spaces. Abstr. Appl. Anal. **2012**, Article ID 359054 (2012)
7. Chandok, S, Postolache, M: Fixed point theorem for weakly Chatterjea-type cyclic contractions. Fixed Point Theory Appl. **2013**, Article ID 28 (2013)
8. Shatanawi, W, Postolache, M: Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces. Fixed Point Theory Appl. **2013**, Article ID 271 (2013)
9. Shatanawi, W, Pitea, A: Fixed and coupled fixed point theorems of omega-distance for nonlinear contraction. Fixed Point Theory Appl. **2013**, Article ID 275 (2013)
10. Altun, I, Simsek, H: Some fixed point theorems on dualistic partial metric spaces. J. Adv. Math. Stud. **1**(1-2), 1-8 (2008)
11. Aydi, H: Fixed point results for weakly contractive mappings in ordered partial metric spaces. J. Adv. Math. Stud. **4**(2), 1-12 (2011)
12. Khan, AR, Abbas, M, Nazir, T, Ionescu, C: Fixed points of multivalued contractive mappings in partial metric spaces. Abstr. Appl. Anal. **2014**, Article ID 230708 (2014)
13. Shatanawi, W, Postolache, M: Coincidence and fixed point results for generalized weak contractions in the sense of Berinde. Fixed Point Theory Appl. **2013**, Article ID 54 (2013)
14. Menger, K: Statistical metrics. Proc. Natl. Acad. Sci. USA **28**, 535-537 (1942)
15. Cosentino, M, Salimi, P, Vetro, P: Fixed point on metric-type spaces. Acta Math. Sci. **34**(4), 1-17 (2014)
16. Grabiec, M: Fixed points in fuzzy metric spaces. Fuzzy Sets Syst. **27**, 385-389 (1988)
17. Gregori, V, Sapena, A: On fixed point theorems in fuzzy metric spaces. Fuzzy Sets Syst. **125**, 245-252 (2002)
18. Ionescu, C, Rezapour, S, Samei, M: Fixed points of some new contractions on intuitionistic fuzzy metric spaces. Fixed Point Theory Appl. **2013**, Article ID 168 (2013)
19. Haghi, RH, Postolache, M, Rezapour, S: On T-stability of the Picard iteration for generalized ϕ-contraction mappings. Abstr. Appl. Anal. **2012**, Article ID 658971 (2012)
20. Miandaragh, MA, Postolache, M, Rezapour, S: Some approximate fixed point results for generalized alpha-contractive mappings. Sci. Bull. 'Politeh.' Univ. Buchar., Ser. A, Appl. Math. Phys. **75**(2), 3-10 (2013)
21. Bakhtin, IA: The contraction mapping principle in almost metric spaces. In: Functional Analysis, vol. 30, pp. 26-37. Ul'yanovsk Gos. Ped. Inst., Ul'yanovsk (1989)

22. Aydi, H, Bota, MF, Karapinar, E, Moradi, S: A common fixed point for weak ϕ-contractions on *b*-metric spaces. Fixed Point Theory **13**(2), 337-346 (2012)
23. Roshan, JR, Parvaneh, V, Sedghi, S, Shobkolaei, N, Shatanawi, W: Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered *b*-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 130 (2013)
24. Berinde, V: Generalized contractions in quasimetric spaces. In: Seminar on Fixed Point Theory. Preprint, vol. 3, pp. 3-9. "Babeş-Bolyai" University, Cluj-Napoca (1993)
25. Păcurar, M: A fixed point result for ϕ-contractions and fixed points on *b*-metric spaces without the boundness assumption. Fasc. Math. **43**(1), 127-136 (2010)
26. Hussain, N, Shah, MH: KKM mappings in cone *b*-metric spaces. Comput. Math. Appl. **61**(4), 1677-1684 (2011)
27. Huang, H, Xu, S: Fixed point theorems of contractive mappings in cone *b*-metric spaces and applications. Fixed Point Theory Appl. **2013**, Article ID 112 (2013)
28. Hussain, N, Dorić, N, Kadelburg, Z, Radenović, S: Suzuki-type fixed point results in metric type spaces. Fixed Point Theory Appl. **2012**, Article ID 126 (2012)
29. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**(15), 4889-4897 (2011)
30. Shah, MH, Hussain, N: Nonlinear contractions in partially ordered quasi *b*-metric spaces. Commun. Korean Math. Soc. **27**, 117-128 (2012)
31. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1-8 (2008)
32. Ćirić, L, Cakić, N, Rojović, M, Ume, JS: Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2008**, Article ID 131294 (2008)
33. Suzuki, T: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. **136**(5), 1861-1869 (2008)
34. Singh, SL, Prasad, B: Some coincidence theorems and stability of iterative procedures. Comput. Math. Appl. **55**, 2512-2520 (2008)
35. Boriceanu, M: Strict fixed point theorems for multivalued operators in *b*-metric spaces. Int. J. Mod. Math. **4**(3), 285-301 (2009)
36. Hussain, N, Kadelburg, Z, Radenović, S, Al-Solami, F: Comparison functions and fixed point results in partial metric spaces. Abstr. Appl. Anal. **2012**, Article ID 605781 (2012)

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