Li et al. *Fixed Point Theory and Applications* 2014, 2014:146
http://www.fixedpointtheoryandapplications.com/content/2014/1/146

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Solving *GNOVI* frameworks involving (γ_G, λ)-weak-*GRD* set-valued mappings in positive Hilbert spaces

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Abstract

First, a new concept, positive Hilbert spaces, is introduced and some fundamental inequalities which are applied to studying the properties of the resolvent operator associated for (γ_G, λ)-weak-*GRD* set-valued mappings are introduced and discussed in positive Hilbert spaces. Next, by using the resolvent operator and fixed point theory, an existence theorem and an approximation algorithm to solve a new class of general nonlinear ordered inclusions are established and suggested. In this field, the results obtained seem to be general in nature.

MSC: 49J40; 47H06

Keywords: general nonlinear ordered inclusion frameworks; positive Hilbert spaces; inequalities; (γ_G, λ)-weak-*GRD* set-valued mapping; approximation algorithm

1 Introduction

Generalized nonlinear variational inclusion was introduced and studied by Hassouni and Moudafi [1]; it is useful and important in, for example, optimization and control, nonlinear programming, economics, mathematics, physics and engineering sciences. From 1989, Chang and Zhu [2], Chang and Huang [3], Ding and Jong [4], Ding and Luo [5], Jin [6], Li [7], Ahmad and Bazán [8], Chang [9], Cho *et al.* [10] and in recent years, Huang and Fang [11, 12], Chang and Huang [13], Fang *et al.* [14], Lan *et al.* [15] and others studied the properties of many kinds of resolvent operators (generalized m-accretive mappings, generalized monotone mappings, maximal η-monotone mappings, H-monotone operators, (H,η)-monotone operators, (A,η)-accretive mappings) and variational inequalities (inequalities, equalities, quasi-variational inclusions, quasi-complementarity) for fuzzy mappings, generalized random multivalued mappings *etc.*

On the other hand, in 1972, a number of solutions of nonlinear equations were introduced and studied by Amann [16]; and in recent years, the nonlinear mapping fixed point theory and application have been intensively studied in ordered Banach spaces [17–19]. Therefore, it is very important and natural for generalized nonlinear ordered variational inequalities (ordered equation) to be studied and discussed.

In 2008, the author introduced the generalized nonlinear ordered variational inequalities (the ordered equations) and studied an approximation algorithm and an approximation solution for a class of generalized nonlinear ordered variational inequalities and ordered equations in ordered Banach spaces [20]. In 2009, by using the B-restricted-accretive method of the mapping A with constants α_1, α_2, the author introduced and studied a new class of general nonlinear ordered variational inequalities and equations and established an existence theorem and an approximation algorithm of solutions for this kind of generalized nonlinear ordered variational inequalities (equations) in ordered Banach spaces [21]. In 2011, by using the resolvent operator associated with an *RME* set-valued mapping, the author introduced and studied a class of nonlinear inclusion problems for ordered *RME* set-valued mappings to find $x \in X$ such that $0 \in M(x)$ ($M(x)$ is a set-valued mapping), and the existence theorem of solutions and an approximation algorithm for this kind of nonlinear inclusion problems for ordered extended set-valued mappings in ordered Hilbert spaces [22]. In 2012, the author introduced and studied a class of nonlinear inclusion problems for ordered (α, λ)-*NODM* set-valued mappings and then, applying the resolvent operator associated with (α, λ)-*NODM* set-valued mappings, established the existence theorem on the solvability and a general algorithm applied to the approximation solvability of the nonlinear inclusion problem of this class of nonlinear inclusion problems, based on the existence theorem and the new (α, λ)-*NODM* model in an ordered Hilbert space [23]. In Banach spaces, the author proved sensitivity analysis of the solution for a new class of general nonlinear ordered parametric variational inequalities to find $x = x(\lambda) : \Omega \to X$ such that $A(g(x, \lambda), \lambda) + f(x, \lambda) \geq \theta$ ($A(x)$, $g(x)$ and $F(\cdot, \cdot)$ are single-valued mappings) in 2011 [24]. In this field, the obtained results seem to be general in nature.

Very recently, in 2013, the author introduced and studied characterizations of ordered (α_A, λ)-weak-*ANODD* set-valued mappings, which was applied to solving an approximate solution for a new class of general nonlinear mixed order quasi-variational inclusions involving $\oplus$ operator in ordered Banach spaces [25] and *GNM* ordered variational inequality system with ordered Lipschitz continuous mappings in ordered Banach spaces [26]. In 2014, a class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ)-*ANODM* set-valued mappings with strong comparison mapping A [27] and sensitivity analysis for GSV parametric OVI with (α, λ)-*NODSM* mappings in ordered Banach spaces [28] were introduced and studied. Now, it is excellent that we are introducing positive Hilbert spaces and studying the properties of (γ_G, λ)-weak ordered *GRD* set-valued mappings, which is applied to finding a solution for a new class of general nonlinear ordered inclusion frameworks involving a strong comparison mapping in positive Hilbert spaces. For details, we refer the reader to [1–50] and the references therein.

2 Fundamental inequalities in positive Hilbert spaces

In this paper, unless specified otherwise, X expresses a real ordered Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, a norm $\| \cdot \|$, a zero element θ, a normal cone $\mathbf{P}$ with normal constant $N > 0$ and a partially ordered relation $\leq$ defined by a normal cone $\mathbf{P}$. For $x, y \in X$, x and y are said to be comparable to each other if and only if $x \leq y$ (or $y \leq x$) holds (denoted by $x \propto y$ for $x \leq y$ and $y \leq x$) [23]. $CB(X)$ expresses the family of all nonempty closed bounded subsets of X.

Lemma 2.1 ([25]) *Let X be an ordered Hilbert space and $\leq$ be a partially ordered relation.*

(i) *If $x \propto y$, then* $\text{lub}\{x, y\}$ *and* $\text{glb}\{x, y\}$ *exist*, $x - y \propto y - x$, *and* $\theta \leq (x - y) \vee (y - x)$;

(ii) *If* $x \vee y = \text{lub}\{x, y\}$, $x \wedge y = \text{glb}\{x, y\}$, $x \oplus y = (x - y) \vee (y - x)$, $x \odot y = (x - y) \wedge (y - x)$, *then the following relations hold*:

(1) $x \oplus y = y \oplus x$, $x \oplus x = \theta$, $x \odot y = y \odot x = -(x \oplus y)$;
(2) *let* λ *be real, then* $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(3) *let* $(x + y) \vee (u + v)$ *exist, and if* $x \propto u, v$ *and* $y \propto u, v$, *then* $(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u)$;
(4) *if* $x \leq y$ *and* $u \leq v$, *then* $x + u \leq y + v$;
(5) *if* $x \propto y$, *then* $x \oplus y = \theta$ *if and only if* $x = y$;
(6) $x \vee y = x + y - (x \wedge y)$;
(7) $\alpha x \oplus \beta x = |\alpha - \beta| x$ *if* $x \propto \theta$.

Definition 2.2 An ordered Hilbert space X is said to be a positive Hilbert space with a partially ordered relation $\leq$ (denoted by X_P) if for any $x, y \in X$, $x \geq \theta$ and $y \geq \theta$, then $\langle x, y\rangle \geq 0$.

Example 2.3 Let $X = \mathbf{R}^n$ be a real n-dimensional ordered inner product space with orthogonal basis $\{\alpha_i\}_{i=1}^n$. Setting $\mathbf{P} = \{\beta = \sum_{i=1}^n k_i\alpha_i | k_i \geq 0, k_i \in \mathbf{R}^n\ (1 \leq i \leq n)\}$, it is a normal cone, then $\mathbf{R}_P^n$ is a positive Hilbert space.

Theorem 2.4 (Inequalities I) *If* X *is an ordered Hilbert space, for* $x, y, z, w \in X$, *then*
(1) $x \leq x \vee y$, $y \leq x \vee y$, $x \wedge y \leq x$, $x \wedge y \leq y$, $x \odot y \geq x \oplus y$;
(2) $x \leq y$ *if and only if* $-y \leq -x$;
(3) $x - (y \vee z) \leq (x - y) \wedge (x - z)$, $x + (y \vee z) \leq (x + y) \vee (x + z)$;
(4) $x - (y \wedge z) \geq (x - y) \vee (x - z)$, $x + (y \wedge z) \leq (x + y) \wedge (x + z)$;
(5) *if* $\theta \leq x$, $\theta \leq y$, *then* $x \oplus y \leq x \vee y$, $x \odot y \leq x \wedge y$;
(6) $(x + y) \oplus (z + w) \geq ((x \oplus z) - (y \oplus w)) \vee ((x \oplus w) - (y \oplus z))$.

Proof Obviously, (1)-(5) hold for Lemma 2.1 and Definition 2.2.

For $x, y, z, w \in X$, we have $(x + y) \oplus (z + w) + (y \oplus w) \geq (x + y - z - w) \vee (z + w - x - y) + (y - w) \vee (w - y) \geq (x + y - z - w) + (w - y) \geq x - z$; in the same way, $(x + y) \oplus (z + w) + (y \oplus w) \geq z - x$. Therefore,

$$(x + y) \oplus (z + w) \geq (x \oplus z) - (y \oplus w),$$

and hence (6) holds for $x + y = y + x$ and $x \oplus y = y \oplus x$. □

Theorem 2.5 (Inequalities II) *If* X_p *is a positive Hilbert space, for* $x, y, z, w \in X$, *then*
(1) *if* $x \leq y$, $\theta \leq z$, *then* $\langle y, z\rangle \geq \langle x, z\rangle$;
(2) *if* $\theta \leq z$, *then* $\langle x \vee y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle$, $\langle x, z\rangle \wedge \langle y, z\rangle \geq \langle x \wedge y, z\rangle$;
(3) *if* $\theta \leq z$, *then* $\langle x + y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle + \langle x \wedge y, z\rangle$;
(4) *if* $\theta \leq z$, *then* $\langle x \vee y, z\rangle \geq \langle x, z\rangle + \langle y, z\rangle - \langle x, z\rangle \wedge \langle y, z\rangle$;
(5) *if* $\theta \leq z$, *then* $\langle x \oplus y, z\rangle \geq \langle x, z\rangle \oplus \langle y, z\rangle$.

Proof From Lemma 2.1, Definition 2.2 and Theorem 2.4 it follows that (1)-(4) hold. Let $\theta \leq z$, by (6) in Lemma 2.1 and (1)-(3) in Theorem 2.4, hold

$$\begin{aligned}\langle x \oplus y, z\rangle &= \big\langle (x - y) \vee (y - x), z\big\rangle \\ &= \big\langle (x - y), z\big\rangle + \big\langle (y - x), z\big\rangle - \big\langle (x - y) \wedge (y - x), z\big\rangle \\ &= -\big\langle (x - y) \wedge (y - x), z\big\rangle\end{aligned}$$

$$\begin{aligned}
&\geq -\big[\big\langle (x-y), z\big\rangle \wedge \big\langle (y-x), z\big\rangle\big] \\
&= -\big[\big(\langle x,z\rangle - \langle y,z\rangle\big) \wedge \big(\langle y,z\rangle - \langle x,z\rangle\big)\big] \\
&= \big(\langle x,z\rangle - \langle y,z\rangle\big) \vee \big(\langle y,z\rangle - \langle x,z\rangle\big) \\
&= \langle x,z\rangle \oplus \langle y,z\rangle.
\end{aligned}$$

It follows that (5) holds. □

3 Properties of (γ_G, λ)-weak-*GRD* set-valued mappings in positive Hilbert spaces

Definition 3.1 Let X be a real ordered Hilbert space, let $G : X \to X$ be a strong comparison and β-ordered compressed mapping [23], and let $M : X \to CB(X)$ be a set-valued mapping.

(1) [22] M is said to be an ordered rectangular mapping if for each $x, y \in X$, any $v_x \in M(x)$ and any $v_y \in M(y)$, $\langle v_x \odot v_y, -(x \oplus y)\rangle = 0$ holds;

(2) M is said to be a γ_G-ordered rectangular mapping with respect to G if there exists a constant $\gamma_G \geq 0$; for any $x, y \in X$, there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that

$$\big\langle v_x \odot v_y, -\big(G(x) \oplus (y)\big)\big\rangle \geq \gamma_G \big\| G(x) \oplus G(y)\big\|^2$$

holds, where v_x and v_y are said to be γ_G-elements, respectively;

(3) M is said to be a weak comparison mapping with respect to G if for any $x, y \in X$, $x \propto y$, then there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$, where v_x and v_y are said to be weak comparison elements, respectively;

(4) M with respect to G is said to be a λ-weak ordered different comparison mapping with respect to G if there exists a constant $\lambda > 0$ such that for any $x, y \in X$, there exist $v_x \in M(G(x))$, $v_y \in M(G(y))$, $\lambda(v_x - v_y) \propto x - y$ holds, where v_x and v_y are said to be λ-elements, respectively;

(5) A weak comparison mapping M with respect to B is said to be a (γ_G, λ)-weak-*GRD* mapping with respect to B if M is a γ_G-ordered rectangular and λ-weak ordered different comparison mapping with respect to B and $(G + \lambda M)(X) = X$ for $\lambda > 0$, and there exist $v_x \in M(G(x))$ and $v_y \in M(G(y))$ such that v_x and v_y are (γ_G, λ)-elements, respectively.

Remark 3.2 Let X be a real ordered Hilbert space, let $G : X \to X$ be a single-valued mapping, and let $M : X \to CB(X)$ be a set-valued mapping, then the following obviously hold:

(i) A λ-ordered monotone mapping must be λ-weak ordered different comparison [22];

(ii) If $G = I$ (identical mapping), then a γ_I-ordered rectangular mapping must be ordered rectangular in [22];

(iii) An ordered *RME* mapping must be λ-weak-*GRD* in [22].

Theorem 3.3 *Let X_P be a real positive Hilbert space with normal constant N, let G be a strong comparison and β-ordered compressed mapping, and let $M : X \to CB(X)$ be an α_I-weak ordered rectangular set-valued mapping that I is an identical mapping. Let a mapping $J^G_{M,\lambda} = (G + \lambda M)^{-1} : X \to 2^X$ be an inverse mapping of $(G + \lambda M)$.*

(1) *If* $\alpha_I\lambda > \beta > 0$, *then* $J^G_{M,\lambda}$ *is a single-valued mapping*;

(2) *If* $\lambda(\alpha_I \wedge \gamma_G) > \beta > 0$, *and* $M : X \to CB(X)$ *is a* (γ_G, λ)*-weak-GRD set-valued mapping with respect to* $J^G_{M,\lambda}$, *and* $v_x \in M(J^G_{M,\lambda}(x))$ *and* $v_y \in M(J^G_{M,\lambda}(y))$ *are* α_I, γ_G *and* λ*-elements, respectively, then the resolvent operator* $J^G_{M,\lambda}$ *of* M *is a comparison, and*

$$\left\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y) \right\| \le \frac{1}{\gamma_G\lambda - \beta}\|x \oplus y\|. \tag{3.1}$$

Proof Certificate (1): Let $u \in X$ and $x, y \in J^G_{M,\lambda}(u) = (G + \lambda M)^{-1}(u)$. Since M is an α_I-weak ordered rectangular mapping so that there exist $v_x = \frac{1}{\lambda}(u - G(x)) \in M(x)$ and $v_y = \frac{1}{\lambda}(u - G(y)) \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \ge \alpha\|x \oplus y\|^2,$$

where v_x and v_y are α_I-elements, respectively.

Since G is a β-ordered compressed mapping so that

$$\begin{aligned}
\langle v_x \odot v_y, -(x \oplus y)\rangle &= \left\langle \frac{1}{\lambda}\big(u - G(x)\big) \odot \frac{1}{\lambda}\big(u - G(y)\big), -(x \oplus y)\right\rangle \\
&= \frac{1}{\lambda}\big\langle -\big(G(x) \oplus G(y)\big), -(x \oplus y)\big\rangle \\
&\le \frac{1}{\lambda}\big\langle \beta(x \oplus y), (x \oplus y)\big\rangle = \frac{\beta}{\lambda}\|x \oplus y\|^2,
\end{aligned}$$

and $\alpha_I\|x \oplus y\|^2 \le \frac{\beta}{\lambda}\|x \oplus y\|^2$ for Theorems 2.4 and 2.5. It follows that $x = y = J^G_{M,\lambda}(u)$ and $J^G_{M,\lambda}(u)$ is a single-valued mapping from $\alpha_I\lambda > \beta > 0$.

Certificate (2): Since $M : X \to CB(X)$ still is an λ-weak ordered different comparison mapping so that $\lambda(v_x - v_y) \propto (x - y)$ and $x \propto J^G_{M,\lambda}(x)$, where v_x and v_y are α_I and λ-elements ($\forall x, y \in X$), such that $v_x = \frac{1}{\lambda}(x - G(J^G_{M,\lambda}(x))) \in M(J^G_{M,\lambda}(x))$ and $v_y = \frac{1}{\lambda}(y - G(J^G_{M,\lambda}(y))) \in M(J^G_{M,\lambda}(y))$, respectively, then

$$\lambda(v_x - v_y) - (x - y) = G\big(J^G_{M,\lambda}(x)\big) - G\big(J^G_{M,\lambda}(y)\big).$$

Hence, $G(J^G_{M,\lambda}(y)) \propto G(J^G_{M,\lambda}(x))$, and $J^G_{M,\lambda}(y) \propto J^G_{M,\lambda}(x)$ by strong comparability of G.

Let M be a (γ_G, λ)-weak-*GRD* mapping with respect to $J^G_{M,\lambda}(x)$, then for any $x, y \in X$ and $\lambda > 0$, $v_x = \frac{1}{\lambda}(x - G(J^G_{M,\lambda}(x))) \in M(J^G_{M,\lambda}(x))$ and $v_y = \frac{1}{\lambda}(y - G(J^G_{M,\lambda}(y))) \in M(J^G_{M,\lambda}(y))$ are α_I, λ and γ_G-elements, respectively. Hence, by Definition 3.1(4), Theorems 2.4 and 2.5 and the comparability of $J^G_{M,\lambda}$, we have

$$\begin{aligned}
&\gamma_G\left\|J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\right\|^2 \\
&\quad\le \left\langle \frac{1}{\lambda}\big(x - G\big(J^G_{M,\lambda}(x)\big)\big) \odot \frac{1}{\lambda}\big(y - G\big(J^G_{M,\lambda}(y)\big)\big), -\big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\right\rangle \\
&\quad= \left\langle \frac{1}{\lambda}\big[\big(x - G\big(J^G_{M,\lambda}(x)\big)\big) \oplus \big(y - G\big(J^G_{M,\lambda}(y)\big)\big)\big], \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\right\rangle \\
&\quad\le \left\langle \frac{1}{\lambda}\big[\big(G\big(J^G_{M,\lambda}(x)\big) \oplus G\big(J^G_{M,\lambda}(y)\big)\big) + (x \oplus y)\big], \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\right\rangle
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\lambda}\big\langle \big(G\big(J^G_{M,\lambda}(x)\big) \oplus G\big(J^G_{M,\lambda}(y)\big)\big), \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\big\rangle \\ &\quad + \frac{1}{\lambda}\big\langle (x \oplus y), \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\big\rangle \\ &\leq \frac{\beta}{\lambda}\big\langle \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big), J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\rangle \\ &\quad + \frac{1}{\lambda}\big\langle (x \oplus y), \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\big\rangle. \end{aligned}$$

It follows that

$$\lambda\gamma_G \big\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\|^2 \leq \beta \big\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\|^2 + \big\langle (x \oplus y), \big(J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big)\big\rangle$$

and

$$(\gamma_G\lambda - \beta)\big\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\|^2 \leq \|x \oplus y\| \big\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\|,$$

by the condition $\lambda(\alpha_I \wedge \gamma_G) > \beta > 0$, then there is

$$\big\| J^G_{M,\lambda}(x) \oplus J^G_{M,\lambda}(y)\big\| \leq \frac{1}{\gamma_G\lambda - \beta}\|x \oplus y\|. \qquad \square$$

4 Approximation solution for *GNOVI* frameworks

In this section, by using Theorems 2.4 and 2.5 and Theorem 3.3, we study a new class of *GNOVI* frameworks in positive Hilbert spaces.

Let X_P be a real positive Hilbert space with a normal constant N, a norm $\|\cdot\|$, an inner product $\langle\cdot,\cdot\rangle$ and zero θ. Let $M : X \to CB(X)$ and $\rho M(x) = \{\rho v | v \in M(x)\}$ be two set-valued mappings. We consider the problem: For $w \in X$ and $\rho > 0$, find $x \in X$ such that

$$w \in \rho M(x), \tag{4.1}$$

which is called a new class of general nonlinear ordered variational inclusion frameworks (*GNOVI*) in positive Hilbert spaces.

Remark 4.1 If $M(x) = A(g(x))$ is single-valued, $w = \theta$ and $\rho = 1$, then (4.1) reduces to (2.1) in [20]; when $M(x) = A(x) \oplus F(x, g(x))$, $w = \theta$ and $\rho = 1$, then (4.1) reduces to (1.1) in [21]; if $w = \theta$, then (1.1) in [22] or [23] can be obtained as special cases of (4.1) as $\rho = 1$.

Lemma 4.2 *Let X_P be a real positive Hilbert space with normal constant N, let G be a strong comparison and β-ordered compressed mapping, and let $M : X \to CB(X)$ be a (γ_G, λ)-weak ordered GRD set-valued mapping with respect to $J^G_{M,\lambda}$. Then the inclusion problem* (2) *has a solution x^* if and only if $x^* = J^G_{M,\lambda}(G(x^*) + \frac{\lambda}{\rho}w)$ in X.*

Proof For $\rho > 0$, take notice of the fact that $w \in \rho M(x)$ if and only if $\frac{w}{\rho} \in M(x)$, this directly follows from the definition of $J^G_{M,\lambda}$ and problem (4.1). $\square$

Theorem 4.3 *Let X_P be a real positive Hilbert space with normal constant N, let G be a strong comparison and β-ordered compressed mapping, and let $M : X \to CB(X)$ be an α_I-ordered rectangular and (γ_G, λ)-weak-GRD set-valued mapping with respect to $J^G_{M,\lambda}(x)$.*

Let $v_x \in M(J^G_{M,\lambda}(x))$ and $v_y \in M(J^G_{M,\lambda}(y))$ be α_I, λ and γ_G-elements, respectively. If β satisfies

$$0 < \beta < \lambda\left(\frac{\gamma_G}{2} \wedge \alpha_I\right) \wedge 1, \tag{4.2}$$

$$\beta + \frac{aN}{1-N(1-a)}\beta^2 < \gamma_G\lambda \quad (0 < a < 1), \tag{4.3}$$

then there exists a solution x^* of GNOVI (4.1), which is a fixed point of $J^G_{M,\lambda}$, that is converged strongly by a sequence $\{x_n\}_{n=0}^{\infty}$ generated by the following algorithm:

For any given $x_0 \in X$ and any $0 < a < 1$, set

$$x_{n+1} = (1-a)x_n + aJ_{M,\lambda}\left(G(x_n) + \frac{\lambda}{\rho}w\right) \quad (n = 1, 2, \ldots). \tag{4.4}$$

Proof Let X_P be a positive Hilbert space with normal constant N, let G be a strong comparison and β-ordered compression mapping, and let $M(x) = \{v | v \in M(x)\} : X \to CB(X)$ ($\rho > 0$) be a (γ_G, λ)-weak-*GRD* set-valued mapping with respect to $J^G_{M,\lambda}$.

Since $\alpha_I, \beta, \gamma_G, \lambda > 0$ and by condition (4.2) we have

$$\lambda(\alpha_I \wedge \gamma_G) \geq \lambda\left(\frac{\gamma_G}{2} \wedge \alpha\right) = \lambda\frac{\gamma_G}{2} \wedge \lambda\alpha_I > \beta > 0 \quad \text{and} \quad 1 > \frac{\beta}{\gamma_G\lambda - \beta} > 0.$$

By Theorem 3.3(1), if $x_1 \propto x_2$, then $J^G_{M,\lambda}(G + \frac{\lambda}{\rho}w)(x_1) \propto J^G_{M,\lambda}(G + \frac{\lambda}{\rho}w)(x_2)$ for $x_1, x_2 \in X$, and

$$\begin{aligned}
&\left\| J^G_{\rho M,\lambda}\left(G + \frac{\lambda}{\rho}w\right)(x_1) \oplus J^G_{\rho M,\lambda}\left(G + \frac{\lambda}{\rho}w\right)(x_2)\right\| \\
&\quad \leq \frac{1}{\gamma_G\lambda - \beta}\left\|\left(G + \frac{\lambda}{\rho}w\right)(x_1) \oplus \left(G + \frac{\lambda}{\rho}w\right)(x_2)\right\| \\
&\quad \leq \frac{1}{\gamma_G\lambda - \beta}\left\|G(x_1) \oplus G(x_2)\right\| \leq \frac{\beta}{\gamma_G\lambda - \beta}\|x_1 \oplus x_2\|.
\end{aligned} \tag{4.5}$$

It follows that $J^G_{M,\lambda}(G + \frac{\lambda}{\rho}w)$ has a fixed point x^*, which is a solution x^* for *GNOVI* (4.1), from Lemma 4.2 and (4.5).

For any $x_0 \in X$ and $0 < a < 1$, by using (4.4), (4.5) and Theorem 3.3, the following hold:

$$\begin{aligned}
\theta &\leq x_{n+1} \oplus x_n \\
&= \left((1-a)x_n + aJ_{M,\lambda}\left(G(x_n) + \frac{\lambda}{\rho}w\right)\right) \oplus \left((1-a)x_{n-1} + aJ_{M,\lambda}\left(G(x_{n-1}) + \frac{\lambda}{\rho}w\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &= \left\|(1-a)(x_n \oplus x_{n-1}) + a\left(J_{M,\lambda}\left(G(x_n) + \frac{\lambda}{\rho}w\right) \oplus J_{M,\lambda}\left(G(x_{n-1}) + \frac{\lambda}{\rho}w\right)\right)\right\| \\
&\leq N\left[(1-a)\|x_n \oplus x_{n-1}\| + a\frac{\beta}{\gamma_G\lambda - \beta}\left\|G(x_n) \oplus G(x_{n-1})\right\|\right] \\
&\leq \delta N\|x_n \oplus x_{n-1}\|,
\end{aligned} \tag{4.6}$$

where $\delta = 1 - a + a\frac{\beta^2}{\gamma_G\lambda - \beta}$. It follows that $\|x_m - x_n\| \leq \sum_{i=n}^{m-1}\|x_{i+1} - x_i\| \leq N\|x_1 - x_0\|\sum_{i=n}^{m-1}\delta^i \times N^i$ for any $m > n > 0$, $1 > \delta N > 0$ and (4.6), and hence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in a complete space X by condition (4.3) and $\|x_n - x_{n-1}\| \le \delta^n N^n \|x_1 - x_0\|$. Let $x_n \to x^*$ as $n \to \infty$ ($x^* \in X$), by (4.2) we get

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} J_{M,\lambda}\left(G(x_n) + \frac{\lambda}{\rho} w\right) = J_{M,\lambda}\left(G\left(x^*\right) + \frac{\lambda}{\rho} w\right),$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a solution x^* of problem (4.1), which is generated by (4.4). This completes the proof. □

Remark 4.4 (i) For a suitable choice of the mappings G, M and constant ρ, we can obtain several known results of [20] and [22] as special cases of Theorem 4.3.

(ii) There exists $\beta > 0$ satisfying (4.3). In fact, if we change (4.3) to $\beta + N\beta^2 < \gamma_G \lambda$ as $0 < a \uparrow 1$, then $\frac{\sqrt{1+4N\gamma_G\lambda}-1}{2} > \beta > 0$ holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by HGL, and HGL, XBP, ZYD and CYW prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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Acknowledgements
The authors acknowledge the support of the Educational Science Foundation of Chongqing (KJ1400426) and the support of the Scientific and Technological Research Program of Chongqing Municipal Education Commission of China (KJ120520).

Received: 4 March 2014 Accepted: 25 June 2014 Published: 22 Jul 2014

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10.1186/1687-1812-2014-146
Cite this article as: Li et al.: Solving *GNOVI* frameworks involving (γ_G, λ)-weak-*GRD* set-valued mappings in positive Hilbert spaces. *Fixed Point Theory and Applications* 2014, 2014:146