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Coincidence and common fixed point theorems for Suzuki type hybrid contractions and applications

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Abstract
Coincidence and common fixed point theorems for a class of Suzuki hybrid contractions involving two pairs of single-valued and multivalued maps in a metric space are obtained. In addition, the existence of a common solution for a certain class of functional equations arising in a dynamic programming is also discussed.
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1 Introduction

Consistent with [1] (see also [2]), Y denotes an arbitrary nonempty set, (X,d) a metric space and $CL(X)$ (resp. $CB(X)$), the collection of all nonempty closed (resp. closed bounded) subsets of X. The hyperspace $(CL(X),H)$ (resp. $(CB(X),H)$) is called the generalized Hausdorff (resp. the Hausdorff) metric space induced by the metric d on X.

For nonempty subsets A, B of X, $d(A,B)$ denotes the gap between the subsets A and B, while

$$\rho(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$BN(X) = \{A : \varnothing \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}.$$

As usual, we write $d(x,B)$ (resp. $\rho(x,B)$) for $d(A,B)$ (resp. $\rho(A,B)$) when $A = \{x\}$.

For the sake of brevity, we choose the following notations, wherein S, T, f, and g are maps to be defined specifically in a particular context, while x and y are elements of some specific domain:

$$M(S,T;fx,gy) = \max\left\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(Ty,fx)}{2}\right\};$$
$$M(Sx,Ty) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(Sx,y)+d(Ty,x)}{2}\right\}.$$

Let $CB(X)$ denote the class of all nonempty closed bounded subsets of X.

A map $T : X \to CB(X)$ is called a Nadler multivalued contraction if there exists $k \in [0,1)$ such that, for every $x,y \in X$, $H(Tx,Ty) \leq kd(x,y)$.

The classical multivalued contraction theorem due to Nadler [1] states that Nadler multivalued contraction on a complete metric space X has a fixed point in X, that is, there exists $z \in X$ such that $z \in Tz$. For a detailed discussion of this theorem on generalized Hausdorff metric spaces and applications, one may refer to [3–13], and [14].

Nadler's multivalued contraction theorem [1] has led to a rich fixed point theory for multivalued maps in nonlinear analysis (see, for instance [6, 9–12, 15–22], and [13, 14, 23, 24]). It has various applications in mathematical sciences (see, for instance, [2, 5, 7–9], and [25]).

The following important result involving two pairs of hybrid maps on an arbitrary nonempty set with values in a metric space is due to Singh and Mishra [12] (see also [21]).

Theorem 1.1 *Let $S, T : Y \to CL(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq g(Y)$ and $T(Y) \subseteq f(Y)$ and one of $S(Y)$, $T(Y)$, $f(Y)$ or $g(Y)$ is a complete subspace of X. Assume there exists $r \in [0,1)$ such that, for every $x, y \in Y$,*

$$H(Sx, Ty) \le rM(S, T; fx, gy).$$

Then

(i) *S and f have a coincidence point v in Y;*
(ii) *T and g have a coincidence point w in Y.*

Further, if $Y = X$, then

(iii) *S and f have a common fixed point v provided that fv is a fixed point of f, and f and S commute at v;*
(iv) *T and g have a common fixed point w provided that gw is a fixed point of g, and g and T commute at w;*
(v) *S, T, f, and g have a common fixed point provided that* (iii) *and* (iv) *both are true.*

The following result due to Kikkawa and Suzuki [26] (see also [13, 14]) generalizes Nadler's multivalued contraction theorem.

Theorem 1.2 *Let X be a complete metric space and $T : X \to CB(X)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in X$,*

$$d(x, Tx) \le (1 + r)d(x, y) \tag{1.1}$$

implies

$$H(Tx, Ty) \le rd(x, y). \tag{1.2}$$

Then T has a fixed point in X.

Subsequently, some interesting extensions and generalizations of Theorem 1.2 have recently been obtained among others by Abbas *et al.* [27], Dhompongsa and Yingtaweesittikul [18], Dorič and Lazovič [28], Kamal *et al.* [29], Moț and Petrușel [19], Singh and Mishra [13, 14] and Singh *et al.* [10, 30], and [23].

The importance of Suzuki contraction theorem [24, Theorem 2], Theorem 1.2 and subsequently obtained coincidence and fixed point theorems (*cf.* [13, 14, 18, 19, 23, 26–28], and others) for maps in metric spaces satisfying Suzuki type contractive conditions is that the contractive conditions are required to be satisfied not for all points of the domain. For example, the condition (1.1) of Theorem 1.2 puts some restrictions on the domain of the map T.

In all that follows, we take a nonincreasing function φ from $[0,1)$ onto $(0,1]$ defined by

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ 1-r & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Recently, Singh *et al.* [10] obtained the following coincidence and common fixed point theorem which generalizes a result of Dorič and Lazovič [28] and some other results from [3, 26], and [21].

Theorem 1.3 *Let $S, T : Y \to CL(X)$ and $f : Y \to X$ be such that $S(Y) \subseteq f(Y)$ and $T(Y) \subseteq f(Y)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in Y$,*

$$\varphi(r) \min\{d(fx, Sx), d(fy, Ty)\} \leq d(fx, fy)$$

implies

$$H(Sx, Ty) \leq rM(Sx, Ty; fx, fy).$$

If one of $S(Y)$, $T(Y)$ or $f(Y)$ is a complete subspace of X, then there exists a point $z \in Y$ such that $fz \in Sz \cap Tz$.

Further, if $Y = X$, and fz is a fixed point of f, then fz is common fixed point of S and T provided that f is IT (Itoh-Takahashi)-commuting [13] with S and T at z.

Now a natural question arises whether Theorem 1.1 can further be generalized. In this paper, we answer this question affirmatively under tight minimal conditions. Our main result (Theorem 2.2) also presents an extension of Theorem 1.3 for a quadruplet of maps. Some recent results are discussed as special cases. Further, using two corollaries of the main result (Theorem 2.2), we obtain other common fixed point theorems for multivalued and single-valued maps on metric spaces. We also deduce the existence of common solution for a certain class of functional equations arising in dynamic programming. Examples are given to justify applications.

2 Main results

The following definition is due to Itoh and Takahashi [31] (see also [13]).

Definition 2.1 Let $T : X \to CL(X)$ and $f : X \to X$. Then the hybrid pair (T, f) is IT-commuting at $z \in X$ if $fTz \subseteq Tfz$.

Evidently a pair of commuting multivalued map $T : X \to CL(X)$ and a single-valued map $f : X \to X$ are IT-commuting but the reverse implication is not true [32, p.2]. However, a pair of single-valued maps $f, g : X \to X$ are IT-commuting (also called weakly compatible by Jungck and Rhoades [33]) at $x \in X$ if $fgx = gfx$ when $fx = gx$.

We shall need the following lemma, essentially due to Nadler [1] (see also [3], [2, p.61], [9, p.76].

Lemma 2.1 *If $A, B \in CL(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$.*

Let $C(S,f)$ denote the collection of all coincidence points of S and f, that is, $C(S,f) = \{z \in Y : fz \in Sz\}$ when $S : Y \to CL(X)$ and $f : Y \to X$; and $C(S,f) = \{z \in Y : fz = Sz\}$ when $S,f : Y \to X$. The following is the main result of this section.

Theorem 2.2 *Let $S, T : Y \to CL(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq g(Y)$ and $T(Y) \subseteq f(Y)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in Y$,*

$$\varphi(r)\min\{d(fx,Sx), d(gy,Ty)\} \leq d(fx,gy)$$

implies

$$H(Sx,Ty) \leq rM(S,T;fx,gy).$$

If one of $S(Y)$, $T(Y)$, $f(Y)$ or $g(Y)$ is a complete subspace of X, then
(I) *$C(S,f)$ is nonempty, i.e. there exists a point $z \in Y$ such that $fz \in Sz$;*
(II) *$C(T,g)$ is nonempty, i.e. there exists a point $z_1 \in Y$ such that $gz_1 \in Tz_1$.*
Furthermore, if $Y = X$, then
(III) *S and f have a common fixed point provided that the maps S and f are IT-commuting just at coincidence point z and fz is fixed point of f;*
(IV) *T and g have a common fixed point provided that the maps T and g are IT-commuting just at coincidence point z_1 and gz_1 is fixed point of g;*
(V) *S, T, f, and g have a common fixed point provided that both* (III) *and* (IV) *are true.*

Proof Without loss of generality, we may take $r > 0$ and f, g non-constant maps.

Let $\varepsilon > 0$ be such that $\beta = r + \varepsilon < 1$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ in Y as follows.

Let $x_0 \in Y$ and $y_0 = gx_1 \in Sx_0$. By Lemma 2.1, there exists $y_1 = fx_2 \in Tx_1$ such that

$$d(fx_2, gx_1) \leq H(Sx_0, Tx_1) + \varepsilon M(S,T;fx_0,gx_1).$$

Similarly, there exists $y_2 = gx_3 \in Sx_2$ such that

$$d(fx_2, gx_3) \leq H(Sx_2, Tx_1) + \varepsilon M(S,T;fx_2,gx_1).$$

Continuing in this manner, we find a sequence $\{y_n\}$ in Y such that

$$y_{2n} = gx_{2n+1} \in Sx_{2n}, \qquad y_{2n+1} = fx_{2n+2} \in Tx_{2n+1}$$

and

$$d(fx_{2n}, gx_{2n+1}) \leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S,T;fx_{2n},gx_{2n-1}),$$

$$d(fx_{2n+2}, gx_{2n+1}) \leq H(Sx_{2n}, Tx_{2n+1}) + \varepsilon M(S,T;fx_{2n},gx_{2n+1}).$$

Now, we show that, for any $n \in N$,

$$d(y_{2n}, y_{2n-1}) \leq \beta d(y_{2n-1}, y_{2n-2}). \tag{2.1}$$

Suppose if $d(gx_{2n-1}, Tx_{2n-1}) \geq d(fx_{2n}, Sx_{2n})$, then

$$\varphi(r) \min\{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1})\} \leq d(fx_{2n}, gx_{2n-1}).$$

Therefore by the assumption,

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &\leq rM(S, T; fx_{2n}, gx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta \max\biggl\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1}), \\ &\quad \frac{d(gx_{2n-1}, Sx_{2n}) + d(fx_{2n}, Tx_{2n-1})}{2}\biggr\}. \end{aligned}$$

This yields (2.1). Suppose if $d(fx_{2n}, Sx_{2n}) \geq d(gx_{2n-1}, Tx_{2n-1})$, then

$$\varphi(r) \min\{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1})\} \leq d(fx_{2n}, gx_{2n-1}).$$

Therefore by the assumption,

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &\leq rM(S, T; fx_{2n}, gx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta \max\biggl\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1}), \\ &\quad \frac{d(gx_{2n-1}, Sx_{2n}) + d(fx_{2n}, Tx_{2n-1})}{2}\biggr\} \\ &\leq \beta \max\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, gx_{2n+1})\}, \end{aligned}$$

yielding (2.1). So, in both cases we obtain (2.1). In an analogous manner, we show that

$$d(y_{2n+1}, y_{2n}) \leq \beta d(y_{2n}, y_{2n-1}). \tag{2.2}$$

We conclude from (2.1) and (2.2) that, for any $n \in N$,

$$d(y_{n+1}, y_n) \leq \beta d(y_n, y_{n-1}).$$

Therefore the sequence $\{y_n\}$ is Cauchy. Assume that the subspace $g(Y)$ is complete. Notice that the sequence $\{y_{2n}\}$ is contained in $g(Y)$ and has a limit in $g(Y)$. Call it u. Let $z \in f^{-1}u$. Then $z \in Y$ and $fz = u$. The subsequence $\{y_{2n+1}\}$ also converges to u. Let $z_1 \in g^{-1}u$. Then

$$gz_1 = u. \tag{2.3}$$

Now we show that, for any $gy \in X - \{fz\}$,

$$d(u, Ty) \leq r \max\{d(u, gy), d(gy, Ty)\}, \tag{2.4}$$

and for any $fy \in X - \{gz\}$,

$$d(u, Sy) \leq r \max\{d(u, fy), d(fy, Sy)\}. \tag{2.5}$$

Since $fx_{2n} \to fz$, there exists $n_0 \in N$ (natural numbers) such that

$$d(fx_{2n}, fz) \leq \frac{1}{3} d(fz, gy)q \quad \text{for } gy \neq fz \text{ and all } n \geq n_0.$$

Also $gx_{2n+1} \to fz$, there exists $n_1 \in N$ such that

$$d(gx_{2n+1}, fz) \leq \frac{1}{3} d(fz, gy) \quad \text{for } gy \neq fz \text{ and all } n \geq n_1.$$

Then as in [24, p.1862] (see also [28]),

$$\begin{aligned}
\varphi(r) d(fx_{2n}, Sx_{2n}) \leq d(fx_{2n}, Sx_{2n}) &\leq d(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{2}{3} d(fz, gy) \\
&= d(fz, gy) - \frac{1}{3} d(fz, gy) \\
&\leq d(fz, gy) - d(fx_{2n}, fz) \\
&\leq d(fx_{2n}, gy).
\end{aligned}$$

Therefore

$$\varphi(r) d(fx_{2n}, Sx_{2n}) \leq d(fx_{2n}, gy). \tag{2.6}$$

Now, either $d(fx_{2n}, Sx_{2n}) \leq d(gy, Ty)$ or $d(gy, Ty) \leq d(fx_{2n}, Sx_{2n})$.

In either case, by (2.6) and the assumption,

$$\begin{aligned}
d(fx_{2n+1}, Ty) \leq H(Sx_{2n}, Ty) &\leq rM(S, T; fx_{2n}, gy) \\
&\leq r \max\left\{d(fx_{2n}, gy), d(fx_{2n}, Sx_{2n}), d(gy, Ty), \right. \\
&\qquad \left. \frac{d(fx_{2n}, Ty) + d(gy, Sx_{2n})}{2}\right\}.
\end{aligned}$$

Making $n \to \infty$,

$$\begin{aligned}
d(u, Ty) &\leq r \max\left\{d(u, gy), d(u, u), d(gy, Ty), \frac{d(u, Ty) + d(u, gy)}{2}\right\}, \\
&\leq r \max\left\{d(u, gy), d(gy, Ty), \frac{d(u, Ty) + d(u, gy)}{2}\right\},
\end{aligned}$$

that is, $d(u, Ty) \leq r \max\{d(u, gy), d(gy, Ty)\}$.

This yields (2.4), that is,

$$d(fz, Ty) \le r \max\{d(fz, gy), d(gy, Ty)\}.$$

Analogously, we can prove (2.5), that is,

$$d(gz_1, Sy) \le r \max\{d(gz_1, fy), d(fy, Sy)\}.$$

Now, we show that $C(S,f)$ is nonempty.

First we consider the case $0 \le r < \frac{1}{2}$.

Suppose $fz \notin Sz$. Then as in [18, p.6], let $ga \in Sz$ be such that $2rd(ga, fz) < d(Sz, fz)$.

Since $ga \in Sz$ implies $ga \neq fz$, we have from (2.4) and (2.5),

$$d(fz, Ta) \le r \max\{d(fz, ga), d(ga, Ta)\}. \tag{2.7}$$

On the other hand, since $\varphi(r)d(fz, Sz) \le d(fz, Sz) \le d(fz, ga)$,

$$\varphi(r) \min\{d(fz, Sz), d(ga, Ta)\} \le d(fz, ga).$$

Therefore, by the given assumption,

$$\begin{aligned} d(ga, Ta) &\le H(Sz, Ta) \\ &\le r \max\left\{d(fz, ga), d(fz, Sz), d(ga, Ta), \frac{d(fz, Ta) + d(ga, Sz)}{2}\right\} \\ &= r \max\{d(fz, ga), d(ga, Ta)\}. \end{aligned}$$

This gives $d(ga, Ta) \le H(Sz, Ta) \le rd(fz, ga) < d(fz, ga)$.

So by (2.7), $d(fz, Ta) \le rd(fz, ga)$.

Therefore,

$$\begin{aligned} d(fz, Sz) &\le d(fz, Ta) + H(Sz, Ta) \le rd(fz, ga) + rd(fz, ga) \\ &= 2rd(fz, ga) < d(fz, Sz). \end{aligned}$$

This contradicts $fz \notin Sz$. Consequently $fz \in Sz$, and $C(S,f)$ is nonempty.

In an analogous manner, we can prove in the case $0 \le r < \frac{1}{2}$ that $C(T,g)$ is nonempty.

Now we consider the case $\frac{1}{2} \le r < 1$.

We first show that

$$H(Sz, Ty) \le r \max\left\{d(fz, gy), d(fz, Sz), d(gy, Ty), \frac{d(gy, Sz) + d(fz, Ty)}{2}\right\}.$$

Assume that $fz \neq gy$. Then for every $n \in N$, there exists $z_n \in Ty$ such that

$$d(fz, z_n) \le d(fz, Ty) + \frac{1}{n} d(fz, gy).$$

Therefore

$$\begin{aligned} d(gy,Ty) &\le d(gy,z_n) \\ &\le d(gy,fz)+d(fz,z_n) \\ &\le d(gy,fz)+d(fz,Ty)+\frac{1}{n}d(fz,gy). \end{aligned} \tag{2.8}$$

So, using (2.5), the inequality (2.8) implies

$$d(gy,Ty) \le d(fz,gy)+r\max\bigl\{d(fz,gy),d(gy,Ty)\bigr\}+\frac{1}{n}d(fz,gy). \tag{2.9}$$

If $d(fz,gy) \ge d(gy,Ty)$, then (2.9) gives

$$\begin{aligned} d(gy,Ty) &\le d(fz,gy)+rd(fz,gy)+\frac{1}{n}d(fz,gy) \\ &= \left(1+r+\frac{1}{n}\right)d(fz,gy). \end{aligned}$$

Making $n\to\infty$,

$$d(gy,Ty) \le (1+r)d(fz,gy).$$

Thus

$$\varphi(r)d(gy,Ty)=(1-r)d(gy,Ty) \le \left(\frac{1}{1+r}\right)d(gy,Ty) \le d(fz,gy).$$

Then

$$\varphi(r)\min\bigl\{d(fz,Sz),d(gy,Ty)\bigr\} \le d(fz,gy),$$

and by the assumption,

$$H(Sz,Ty) \le r\max\left\{d(fz,gy),d(fz,Sz),d(gy,Ty),\frac{d(gy,Sz)+d(fz,Ty)}{2}\right\}. \tag{2.10}$$

If $d(fz,gy) < d(gy,Ty)$, then (2.9) gives

$$d(gy,Ty) \le d(fz,gy)+rd(gy,Ty)+\frac{1}{n}d(fz,gy),$$

that is, $(1-r)d(gy,Ty) \le (1+\frac{1}{n})d(fz,gy)$.

Making $n\to\infty$,

$$\varphi(r)d(gy,Ty) \le d(fz,gy).$$

Then $\varphi(r)\min\{d(fz,Sz),d(gy,Ty)\} \le d(fz,gy)$, and by the assumption, we get (2.10).

Now taking $y=u_{2n+1}$ in (2.10) and passing to the limit, we obtain $d(fz,Sz) \le rd(fz,Sz)$.

This gives $fz \in Sz$, that is, z is a coincidence point of f and S. Analogously, $fz \in Tz$. Thus (I) and (II) are completely proved.

Further, if $Y = X$, and fz is a fixed point of f, and S and f are IT-commuting at z, then $fSz \subseteq Sfz$. Therefore, $fz \in Sz$ implies $ffz \in fSz \subseteq Sfz$, so $fz \in Sfz$. This proves that $u = fz$ is a common fixed point of f and S. Therefore (2.3) implies that u is a common fixed point of f and S. This proves (III). Analogously, T and g have a common fixed point gz_1. Therefore (2.3) implies that u is a common fixed point of T and g. This proves (IV). Now (V) is immediate. □

Remark 2.1 In Theorem 2.2, the hypothesis 'fz is a fixed point of f' is essential for the existence of a common fixed point of S and f (see [22, 34] and the following example). Similarly, the hypothesis 'gz_1 is a fixed point of g' is essential for the existence of a common fixed point of T and g.

Example 2.3 Let $X = R^+$ (nonnegative reals) be endowed with the usual metric. Define for $x \in X$, $fx = 2x^2$, $gx = 2x^3$, $Sx = [\frac{1}{4}, x^2 + \frac{1}{4}]$ and $Tx = [\frac{1}{4}, x^3 + \frac{1}{4}]$. Then $S(X) = T(X) = [\frac{1}{4}, \infty) \subset X = f(X) = g(X)$, and all other hypotheses of Theorem 2.2 with $Y = X = R^+$ are satisfied for $r = \frac{1}{2} = \varphi(r)$. Notice that $gz_1 = Tz_1 = \frac{1}{2}$, where $z_1 = 4^{-1/3}$. Thus g and T have a coincidence at z_1, but $gz_1 = \frac{1}{2}$ is not a fixed point of g and hence not a common fixed point of g and T. Note that $z = \frac{1}{2}$ is a coincidence point of f and S, and $Sf(z) = [\frac{1}{8}, \frac{1}{2}] \subset [\frac{1}{4}, \frac{1}{2}] = fS(z)$, that is, f and S are IT-commuting at z. Evidently, $z = f(z)$ is a common fixed point of f and S.

The following result due to Singh *et al.* [35] extends and generalizes certain results of [10, 12, 26] and others.

Corollary 2.4 *Let $S : Y \to CL(X)$ and $f, g : Y \to X$ be such that $S(Y) \subseteq f(Y) \cap g(Y)$. Assume there exists $r \in [0, 1)$ such that, for every $x, y \in Y$,*

$$\varphi(r) \min\{d(fx, Sx), d(gy, Sy)\} \le d(fx, gy)$$

implies

$$H(Sx, Sy) \le rM(S; fx, gy).$$

If one of $S(Y)$, $f(Y)$ or $g(Y)$ is a complete subspace of X, then

(I) *$C(S, f)$ is nonempty, i.e. there exists a point $z \in Y$ such that $fz \in Sz$;*

(II) *$C(S, g)$ is nonempty, i.e. there exists a point $z_1 \in Y$ such that $gz_1 \in Sz_1$.*

Furthermore, if $Y = X$, then

(III) *S and f have a common fixed point provided that the maps S and f are IT-commuting just at coincidence point z and fz is fixed point of f;*

(IV) *S and g have a common fixed point provided that the maps S and g are IT-commuting just at coincidence point z_1 and gz_1 is fixed point of g;*

(V) *S, f, and g have a common fixed point provided that both* (III) *and* (IV) *are true.*

Proof It follows from Theorem 2.2 when $T = S$. □

We remark that in general the coincidence points z and z_1 guaranteed by Theorem 2.2 or Corollary 2.4 may be different. However, if we take $f = g$ in Theorem 2.2, the maps S, T, and f have a common coincidence point. So we have a slightly sharp result.

Corollary 2.5 *Theorem* 1.3.

Proof It follows from Theorem 2.2 when $g = f$. □

The following result extends and generalizes certain results of [28, 36] and others.

Corollary 2.6 [23] *Let X be a complete metric space and $S, T : X \to CL(X)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in X$,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \quad \textit{implies} \quad H(Sx, Ty) \leq rM(Sx, Ty).$$

Then there exists an element $z \in X$ such that $z \in Sz \cap Tz$.

Proof It follows from Theorem 2.2 when $Y = X$ and f and g are the identity maps on $Y = X$. □

The following result due to Doriċ and Lazoviċ [28] generalizes many fixed point theorems from [13, 26] and [37].

Corollary 2.7 *Let X be a complete metric space and $S : X \to CL(X)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in X$,*

$$\varphi(r) d(x, Sx) \leq d(x, y) \quad \textit{implies} \quad H(Sx, Sy) \leq rM(Sx, Sy).$$

Then there exists an element $z \in X$ such that $z \in Sz$.

Proof It follows from Theorem 2.2 when $Y = X$, $T = S$, and f, g are the identity maps on X. □

The following result extends a common fixed point theorem of [10, Theorem 2.8].

Corollary 2.8 *Let $f, g, P, Q : Y \to X$ be such that $P(Y) \subseteq g(Y)$, $Q(Y) \subseteq f(Y)$, and one of $P(Y)$ or $Q(Y)$ or $f(Y)$ or $g(Y)$ is complete subspace of X. Assume there exists $r \in [0,1)$ such that, for every $x, y \in Y$,*

$$\varphi(r) \min\{d(fx, Px), d(gy, Qy)\} \leq d(fx, gy)$$

implies

$$d(Px, Qy) \leq rM(P, Q; fx, gy).$$

Then $C(P,f)$ and $C(Q,g)$ are nonempty. Further, if $Y = X$, and if f, g, P, and Q are commuting at a common coincidence point, then f, g, P, and Q have a unique common fixed point, that is, there exists a unique point $z \in X$ such that $fz = gz = Pz = Qz = z$.

Proof Set $Sx = \{Px\}$ and $Tx = \{Qx\}$ for every $x \in Y$. Then it easily comes from Theorem 2.2 that $C(P,f)$ and $C(Q,g)$ are nonempty. Furthermore, if $Y = X$ and f and g commute, respectively, with P and Q at z, then $ffz = fPz = Pfz$, $ffz = fQz = Qfz$, $ggz = gPz = Pgz$, and $ggz = gQz = Qgz$.

Also $\varphi(r)\min\{d(fz,Pz), d(ffz,Qfz)\} = 0 \leq d(fz,ffz)$, and this implies

$$\begin{aligned} d(Pz,Qfz) &\leq r\max\left\{d(fz,ffz), d(fz,Pz), d(ffz,Qfz), \right. \\ &\qquad \left. \frac{d(fz,Qfz)+d(ffz,Pz)}{2}\right\} \\ &= rd(Pz,Qfz). \end{aligned}$$

This says that fz is fixed point of f and P. Analogously gz is fixed point of g and Q. The uniqueness of the common fixed point follows easily. □

The following result extends and generalizes coincidence and common fixed point theorems of Goebel [38], Jungck [39], Fisher [40], and others.

Corollary 2.9 [35] *Let $f,g,P: Y \to X$ be such that $P(Y) \subseteq f(Y) \cap g(Y)$. Let $P(Y)$ or $f(Y)$ or $g(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that, for every $x,y \in Y$,*

$$\varphi(r)\min\{d(fx,Px), d(gy,Py)\} \leq d(fx,gy)$$

implies

$$d(Px,Py) \leq rM(P;fx,gy).$$

Then $C(P,f)$ and $C(P,g)$ are nonempty. Further, if $Y = X$ and if P commutes with f and g at a common coincidence point, then f, g, and P have a unique common fixed point, that is, there exists a unique point $z \in X$ such that $fz = gz = Pz = z$.

Proof It follows from Corollary 2.8 when $Q = P$. □

Corollary 2.10 *Let (X,d) be a complete metric space and $f,g: X \to X$ be onto maps. Assume there exists $r \in [0,1)$ such that, for every $x,y \in X$,*

$$\varphi(r)\min\{d(x,fx), d(y,gy)\} \leq d(fx,gy) \quad \textit{implies} \quad d(x,y) \leq rM_1(fx,gy).$$

Then f and g have a unique common fixed point.

Proof It follows from Corollary 2.8 when $Y = X$ and P, Q both are the identity maps on X. □

Corollary 2.11 *Let (X,d) be a complete metric space and $f: X \to X$ be an onto map. Assume there exists $r \in [0,1)$ such that, for every $x,y \in X$,*

$$\varphi(r)d(x,fx) \leq d(fx,fy) \quad \textit{implies} \quad d(x,y) \leq rM(fx,fy).$$

Then f has a unique fixed point.

Proof It follows from Corollary 2.10 when $f = g$. □

The following example shows that Theorem 2.2 is indeed more general than Theorem 1.1.

Example 2.12 Consider a metric space $X = \{(0,0),(0,1),(1,0),(1,2),(2,1)\}$, where d is defined by

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let S, T, f and $g : X \to X$ be such that

$$S(x_1,x_2) = \begin{cases} (0,0) & \text{if } (x_1,x_2) \neq (1,2),(2,1), \\ (1,0) & \text{if } (x_1,x_2) = (1,2), \\ (0,1) & \text{if } (x_1,x_2) = (2,1), \end{cases}$$

$$T(x_1,x_2) = \begin{cases} (0,0) & \text{if } (x_1,x_2) \neq (1,2),(2,1), \\ (0,1) & \text{if } (x_1,x_2) = (1,2), \\ (1,0) & \text{if } (x_1,x_2) = (2,1), \end{cases}$$

$$f(x_1,x_2) = \begin{cases} (x_2,x_1) & \text{if } (x_1,x_2) \neq (1,2),(2,1), \\ (x_1,x_2) & \text{if } (x_1,x_2) = (1,2),(2,1) \end{cases}$$

and

$$g(x_1,x_2) = (x_1,x_2) \quad \text{for all } (x_1,x_2) \in X.$$

Then S, T, f, and g do not satisfy the assumption in Theorem 1.1 at $x = (1,2)$, $y = (1,2)$ or at $x = (2,1)$, $y = (2,1)$. However,

$$d(Sx, Ty) \leq \frac{1}{2} \max\left\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy) + d(Ty,fx)}{2}\right\}$$

if $(x,y) \neq ((1,2),(1,2))$ and $(x,y) \neq ((2,1),(2,1))$.

Since at $(x,y) = ((1,2),(1,2))$, $\varphi(r)\min\{d(fx,Sx), d(gy,Ty)\} = \varphi(r)\min\{d(f(1,2),S(1,2)), d(g(1,2),T(1,2))\} = \varphi(r)\min\{2,2\} = 2\varphi(r)$.

Here we note that the value of r is 1/2, so by definition, $\varphi(r) = 1/2$, so $\varphi(r)\min\{d(fx,Sx), d(gy,Ty)\} = 1 > 0 = d(fx,gy)$.

Thus S, T, f, and g satisfy the assumption of Theorem 2.2 (and also Corollary 2.8).

In the following example, we show that two multivalued maps and two single-valued maps satisfy all the hypotheses of Theorem 2.2 to ensure common coincidence points of pairwise maps.

Example 2.13 Let $Y = \{a,b,c,d\}$ and $X = \{2,3,4,5,7\}$. Let d be the usual metric on X, and S, T, f, and g be defined on Y with values in X as

$$S(x) = \begin{cases} \{2,3,4\} & \text{if } x = a,b,c, \\ \{2\} & \text{if } x = d, \end{cases}$$

$$T(x) = \begin{cases} \{2,3,4\} & \text{if } x = a,b,c, \\ \{3\} & \text{if } x = d, \end{cases}$$

$$f(x) = \begin{cases} 4 & \text{if } x = a, \\ 2 & \text{if } x = b, \\ 3 & \text{if } x = c, \\ 7 & \text{if } x = d \end{cases}$$

and

$$g(x) = \begin{cases} 2 & \text{if } x = a, \\ 4 & \text{if } x = b, \\ 3 & \text{if } x = c, \\ 5 & \text{if } x = d. \end{cases}$$

Notice that $S(Y) \subset g(Y)$ and $T(Y) \subset f(Y)$. Further, all other conditions of Theorem 2.2 are readily verified with $r = 2/3$ and $\varphi(r) = 1/3$. Evidently, $fa \in Sa$, $fb \in Sb$, $fc \in Sc$, and $ga \in Ta$, $gb \in Tb$, $gc \in Tc$. Moreover, $C(f,S) = C(g,T) = \{b,c,d\}$.

Now we give an application of Corollary 2.8.

Theorem 2.14 *Let $S,T : Y \to BN(X)$ and $f,g : Y \to X$ be such that $S(Y) \subseteq g(Y)$, $T(Y) \subseteq f(Y)$, and let one of $S(Y)$, $T(Y)$, $f(Y)$ or $g(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that, for every $x,y \in Y$,*

$$\varphi(r)\min\bigl\{\rho(fx,Sx),\rho(gy,Ty)\bigr\} \le d(fx,gy) \tag{2.11}$$

implies

$$\rho(Sx,Ty) \le r\max\biggl\{d(fx,gy),\rho(fx,Sx),\rho(gy,Ty),\frac{d(fx,Ty)+d(gy,Sx)}{2}\biggr\}. \tag{2.12}$$

Then $C(S,f)$ and $C(T,g)$ are nonempty.

Proof Choose $\lambda \in (0,1)$. Define single-valued maps $h_1,h_2 : X \to X$ as follows. For each $x \in X$, let h_1x be a point of Sx which satisfies

$$d(fx,h_1x) \ge r^{\lambda}\rho(fx,Sx).$$

Similarly, for each $y \in X$, let h_2y be a point of Ty such that

$$d(gy,h_2y) \ge r^{\lambda}\rho(gy,Ty).$$

Since $h_1x \in Sx$ and $h_2y \in Ty$,

$$d(fx,h_1x) \le \rho(fx,Sx) \quad \text{and} \quad d(gy,h_2y) \le \rho(gy,Ty).$$

So (2.11) gives

$$\varphi(r)\min\bigl\{d(fx,h_1x),d(gy,h_2y)\bigr\}\le\varphi(r)\min\bigl\{\rho(fx,Sx),\rho(gy,Ty)\bigr\}\le d(fx,gy), \tag{2.13}$$

and this implies (2.12). Therefore

$$\begin{aligned}d(h_1x,h_2y)&\le\rho(Sx,Ty)\\&\le r\cdot r^{-\lambda}\max\biggl\{r^{\lambda}d(fx,gy),r^{\lambda}\rho(fx,Sx),r^{\lambda}\rho(gy,Ty),\\&\qquad\frac{r^{\lambda}d(fx,Ty)+r^{\lambda}d(gy,Sx)}{2}\biggr\}\\&\le r^{1-\lambda}\max\biggl\{d(fx,gy),d(fx,h_1x),d(gy,h_2y),\\&\qquad\frac{d(fx,h_2y)+d(gy,h_1x)}{2}\biggr\}.\end{aligned}$$

So (2.13), *viz.*, $\varphi(r')\min\{d(fx,h_1x),d(gy,h_2y)\}\le d(fx,gy)$ implies

$$\begin{aligned}d(h_1x,h_2y)\le r'\max\biggl\{&d(fx,gy),d(fx,h_1x),d(gy,h_2y),\\&\frac{d(fx,h_2y)+d(gy,h_1x)}{2}\biggr\},\end{aligned}$$

where $r'=r^{1-\lambda}<1$.

Hence by Corollary 2.8, there exist $z_1,z_2\in Y$ such that $h_1z_1=fz_1$ and $h_2z_2=gz_2$. This implies that z_1 is a coincidence point of f and S, and z_2 is a coincidence point of g and T. □

Corollary 2.15 *Let $S:Y\to BN(X)$ and $f,g:Y\to X$ be such that $S(Y)\subseteq f(Y)\cap g(Y)$, and let one of $S(Y)$, $f(Y)$ or $g(Y)$ be a complete subspace of X. Assume there exists $r\in[0,1)$ such that, for every $x,y\in Y$,*

$$\varphi(r)\min\bigl\{\rho(fx,Sx),\rho(gy,Sy)\bigr\}\le d(fx,gy) \tag{2.14}$$

implies

$$\rho(Sx,Sy)\le r\max\biggl\{d(fx,gy),\rho(fx,Sx),\rho(gy,Sy),\frac{d(fx,Sy)+d(gy,Sx)}{2}\biggr\}. \tag{2.15}$$

Then $C(S,f)$ and $C(S,g)$ are nonempty.

Proof It follows from Theorem 2.14 when $T=S$. □

Corollary 2.16 [10] *Let $S,T:Y\to BN(X)$ and $f:Y\to X$ be such that $S(Y)\subseteq f(Y)$, $T(Y)\subseteq f(Y)$ and let $S(Y)$ or $T(Y)$ or $f(Y)$ be a complete subspace of X. Assume there exists $r\in[0,1)$ such that, for every $x,y\in X$,*

$$\varphi(r)\min\bigl\{\rho(fx,Sx),\rho(fy,Ty)\bigr\}\le d(fx,fy)$$

implies

$$\rho(Sx, Ty) \leq r \max\left\{d(fx, fy), \rho(fx, Sx), \rho(fy, Ty), \frac{d(fx, Ty) + d(fy, Sx)}{2}\right\}.$$

Then there exists $z \in Y$ such that $fz \in Sz \cap Tz$.

Proof It follows from Theorem 2.14 when $g = f$. □

Corollary 2.17 [23] *Let X be a complete metric space and let $S, T : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in X$,*

$$\varphi(r) \min\{\rho(x, Sx), \rho(y, Ty)\} \leq d(x, y)$$

implies

$$\rho(Sx, Ty) \leq r \max\left\{d(x, y), \rho(x, Sx), \rho(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\right\}.$$

Then there exists a unique point $z \in X$ such that $z \in Sz \cap Tz$.

Proof It follows from Theorem 2.14 when f and g are the identity maps on X. □

Corollary 2.18 *Let $S : Y \to BN(X)$ and $f : Y \to X$ be such that $S(Y) \subseteq f(Y)$, and let $S(Y)$ or $f(Y)$ be a complete subspace of X. Assume there exists $r \in [0,1)$ such that, for every $x, y \in Y$,*

$$\varphi(r)\rho(fx, Sx) \leq d(fx, fy)$$

implies

$$\rho(Sx, Sy) \leq r \max\left\{d(fx, fy), \rho(fx, Sx), \rho(fy, Sy), \frac{d(fx, Sy) + d(fy, Sx)}{2}\right\}.$$

Then there exists $z \in Y$ such that $fz \in Sz$.

Proof It follows from Theorem 2.14 when $g = f$ and $T = S$. □

Corollary 2.19 *Let X be a complete metric space and let $S : X \to BN(X)$. Assume there exists $r \in [0,1)$ such that, for every $x, y \in X$,*

$$\varphi(r)\rho(x, Sx) \leq d(x, y)$$

implies

$$\rho(Sx, Sy) \leq r \max\left\{d(x, y), \rho(x, Sx), \rho(y, Sy), \frac{d(x, Sy) + d(y, Sx)}{2}\right\}.$$

Then there exists a unique point $z \in X$ such that $z \in Sz$.

Proof It follows from Theorem 2.14 that S has a fixed point when $f = g$ is the identity map on X and $T = S$. The uniqueness of the fixed point follows easily. □

3 Applications

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Let R denote the field of reals, $\tau : W \times D \to W$, $g, g' : W \times D \to R$ and $G_1, G_2, F_1, F_2 : W \times D \times R \to R$. Considering W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p_i = \sup_{y \in D} \{ g(x,y) + G_i(x,y,p(\tau(x,y))) \}, \quad x \in W, i = 1,2, \tag{3.1a}$$

$$q_i = \sup_{y \in D} \{ g'(x,y) + F_i(x,y,q(\tau(x,y))) \}, \quad x \in W, i = 1,2. \tag{3.1b}$$

Indeed, in the multistage process, some functional equations arise in a natural way (*cf.* Bellman [41] and Bellman and Lee [42]; see also [10, 43–47], and [23]). In this section, we study the existence of a common solution of the functional equations (3.1a) and (3.1b) arising in dynamic programming.

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold:

(DP-1) G_1, G_2, F_1, F_2, g, and g' are bounded.

(DP-2) Let $\varphi(r)$ be defined as in the previous sections. Assume that there exists $r \in [0,1)$ such that, for every $(x,y) \in W \times D$, $h,k \in B(W)$, and $t \in W$,

$$\varphi(r) \min\{ |J_1h(t) - A_1h(t)|, |J_2k(t) - A_2k(t)| \} \le |J_1h(t) - J_2k(t)|$$

implies

$$|G_1(x,y,h(t)) - G_2(x,y,k(t))| \le rM(A_1, A_2; J_1h, J_2k),$$

where

$$\begin{aligned} &M(A_1, A_2; J_1h, J_2k) \\ &\quad = \max\Big\{ |J_1h(t) - J_2k(t)|, |J_1h(t) - A_1h(t)|, |J_2k(t) - A_2k(t)|, \\ &\qquad \frac{|J_1h(t) - A_2k(t)| + |J_2k(t) - A_1h(t)|}{2} \Big\}, \end{aligned}$$

and A_1, A_2, J_1, and J_2 are defined as follows:

$$A_ih(x) = \sup_{y \in D} \{ g(x,y) + G_i(x,y,h(\tau(x,y))) \}, \quad x \in W, h \in B(W), i = 1,2,$$

$$J_ih(x) = q = \sup_{y \in D} \{ g'(x,y) + F_i(x,y,h(\tau(x,y))) \}, \quad x \in W, h \in B(W), i = 1,2.$$

(DP-3) For any $h, k \in B(W)$, there exist $u, v \in B(W)$ such that

$$A_1h(x) = J_1u(x) \quad \text{and} \quad A_2k(x) = J_2v(x), \quad x \in W.$$

(DP-4) There exist $h, k \in B(W)$ such that

$$J_1 h(x) = A_1 h(x) \quad \text{implies} \quad J_1 A_1 h(x) = A_1 J_1 h(x)$$

and

$$J_2 k(x) = A_2 k(x) \quad \text{implies} \quad J_2 A_2 k(x) = A_2 J_2 k(x).$$

Theorem 3.1 *Assume the conditions* (DP-1)-(DP-4) *hold. Let* $J(B(W))$ *be a closed convex subspace of* $B(W)$. *Then the functional equations* (3.1a) *and* (3.1b), $i = 1, 2$, *have a unique bounded common solution in* $B(W)$.

Proof For any $h, k \in B(W)$, let $d(h,k) = \sup\{|h(x) - k(x)| : x \in W\}$. Then $(B(W), d)$ is a complete metric space.

Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Pick $x \in W$, and choose $y_1, y_2 \in D$ such that

$$A_j h_j < g(x, y_j) + G_j\big(x, y_j, h_j(x_j)\big) + \lambda, \qquad x_i = (x, y_i), \quad i = 1, 2, \tag{3.1}$$

where $x_j = \tau(x, y_j)$.

Further,

$$A_1 h_1 \geq g(x, y_2) + G_1\big(x, y_2, h_1(x_2)\big), \tag{3.2}$$

$$A_2 h_2 \geq g(x, y_1) + G_2\big(x, y_1, h_2(x_1)\big). \tag{3.3}$$

Therefore, the first inequality in (DP-2) becomes

$$\begin{aligned} &\varphi(r) \min\big\{\big|J_1 h_1(x) - A_1 h_1(x)\big|, \big|J_2 h_2(x) - A_2 h_2(x)\big|\big\} \\ &\quad \leq \big|J_1 h_1(x) - J_2 h_2(x)\big|, \end{aligned} \tag{3.4}$$

and this together with (3.1), (3.3), and (3.4) implies

$$\begin{aligned} A_1 h_1 - A_2 h_2 &< G_1\big(x, y_1, h_1(x_1)\big) - G_2\big(x, y, h_2(x_1)\big) + \lambda \\ &\leq \big|G_1\big(x, y_1, h_1(x_1)\big) - G_2\big(x, y_1, h_2(x_1)\big)\big| + \lambda \\ &\leq rM(A_1, A_2; J_1 h_1, J_2 h_2) + \lambda. \end{aligned} \tag{3.5}$$

Similarly, (3.1), (3.2), and (3.4) imply

$$A_2 h_2(x) - A_1 h_1(x) \leq rM(A_1, A_2; J_1 h_1, J_2 h_2) + \lambda. \tag{3.6}$$

So, from (3.5) and (3.6), we obtain

$$\big|A_1 h_1(x) - A_2 h_2(x)\big| \leq rM(A_1, A_2; J_1 h_1, J_2 h_2) + \lambda. \tag{3.7}$$

As $\lambda > 0$ is arbitrary and (3.7) is true for any $x \in W$, taking supremum, we find from (3.4) and (3.7) that

$$\varphi(r)\min\{d(J_1h_1, A_1h_1), d(J_2h_2, A_2h_2)\} \le d(J_1h_1, J_2h_2)$$

implies

$$d(A_1h_1, A_2h_2) \le rM(A_1, A_2; J_1h_1, J_2h_2).$$

Therefore, Corollary 2.8 applies, wherein A_1, A_2, J_1, and J_2 correspond, respectively, to the maps P, Q, f, and g. So A_1, A_2, J_1, and J_2 have a unique common fixed point h^*, that is, $h^*(x)$ is the unique bounded common solution of the functional equations (3.1a) and (3.1b), $i = 1, 2$. □

Now we furnish an example in support of Theorem 3.1.

Example 3.2 Let $X = Y = R$ be a Banach space endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$, for all $x \in X$. Suppose $W = [0,1] \subset X$ be the state space, and $D = [0,\infty) \subset Y$ be the decision space.

Define $\tau : W \times D \to W$ by

$$\tau(x,y) = \frac{x}{y^2+1}, \quad x \in W, y \in D.$$

For any $h, k \in B(W)$, and $i = 1, 2$, define $p_i, q_i : W \to R$ by

$$p_i(x) = q_i(x) = x^2 + \frac{1}{2}.$$

Define $G_i, F : W \times D \times R \to R$ by

$$G_1(x,y,t) = \frac{1}{4}\left\{\frac{x}{(x+1)(y+1)}\sin\frac{y}{y+1} + 2\right\};$$

$$G_2(x,y,t) = \frac{1}{4}\left\{\frac{x}{(x+1)(2y+1)}\sin\frac{y}{y+1} + 2\right\};$$

$$F_1(x,y,t) = \frac{1}{2x+y+1} + \frac{1}{2}\sin t;$$

$$F_2(x,y,t) = \frac{1}{2x+3y+1} + \frac{1}{2}\sin t;$$

$$g(x,y) = \frac{x^2y^2}{x+y^2} \quad \text{and} \quad g'(x,y) = \frac{x^2y^5}{x+y^5}.$$

Notice that G_1, G_2, F_1, F_2, g, and g' are bounded. Also

$$J_1h(x) = \sup_{y\in D}\{g'(x,y) + F_1(x,y,h(\tau(x,y)))\} = x^2 + \frac{1}{2} = q_1(x), \quad x \in W, h \in B(W);$$

$$J_2k(x) = \sup_{y\in D}\{g'(x,y) + F_2(x,y,k(\tau(x,y)))\} = x^2 + \frac{1}{2} = q_2(x), \quad x \in W, h \in B(W);$$

$$A_1h(x) = \sup_{y\in D}\{g(x,y) + G_1(x,y,h(\tau(x,y)))\} = x^2 + \frac{1}{2} = p_1(x), \quad x \in W, h \in B(W);$$

$$A_2k(x) = \sup_{y\in D}\{g(x,y) + G_2(x,y,k(\tau(x,y)))\} = x^2 + \frac{1}{2} = p_2(x), \quad x \in W, h \in B(W).$$

We see that

$$\begin{aligned}&\varphi(r)\min\{|J_1h(t) - A_1h(t)|, |J_2k(t) - A_2k(t)|\}\\ &\quad= \varphi(r)\min\{|q_1(x) - p_1(x)|, |q_2(x) - p_2(x)|\}\\ &\quad= 0 = |J_1h(t) - J_2k(t)|.\end{aligned}$$

Thus

$$\varphi(r)\min\{|Jh(t) - A_1h(t)|, |Jk(t) - A_2k(t)|\} = |Jh(t) - Jk(t)|,$$

and this implies

$$|G_1(x,y,h(t)) - G_2(x,y,k(t))| = 0 \le rM(A_1,A_2;Jh(t),Jk(t)).$$

Finally for any $h,k \in B(W)$ with $A_1h = Jh$, we have $A_1Jh = p_1(x) = q(x) = JJh = JA_1h$, that is, $JA_1h = A_1Jh$, and with $A_2k = Jk$, we have $A_2Jk = p_2(x) = q(x) = JJk = JA_2k$, that is, $JA_2k = A_2Jk$.

Thus all the assumptions of Theorem 3.1 are satisfied. So the system of equations (3.1a) and (3.1b) has a unique solution in $B(W)$.

Corollary 3.3 *Suppose that the following conditions hold*:

(i) *G, F_1, F_2, g, and g' are bounded.*

(ii) *Let $\varphi(r)$ be defined as in the previous sections. Assume that there exists $r \in [0,1)$ such that, for every $(x,y) \in W \times D$, $h,k \in B(W)$, and $t \in W$,*

$$\varphi(r)\min\{|J_1h(t) - Ah(t)|, |J_2k(t) - Ak(t)|\} \le |J_1h(t) - J_2k(t)|$$

implies

$$|G(x,y,h(t)) - G(x,y,k(t))| \le rM(A;J_1h,J_2k),$$

where

$$M(A;J_1h,J_2k) = \max\left\{|J_1h(t) - J_2k(t)|, |J_1h(t) - Ah(t)|, |J_2k(t) - Ak(t)|, \frac{|J_1h(t) - Ak(t)| + |J_2k(t) - Ah(t)|}{2}\right\},$$

and A, J_1, and J_2 are defined as follows:

$$Ah(x) = \sup_{y\in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\}, \quad x \in W, h \in B(W),$$

$$J_ih(x) = q = \sup_{y\in D}\{g'(x,y) + F_i(x,y,h(\tau(x,y)))\}, \quad x \in W, h \in B(W), i = 1,2.$$

(iii) *For any* $h, k \in B(W)$, *there exist* $u, v \in B(W)$ *such that*

$$Ah(x) = J_1 u(x) \quad \text{and} \quad Ak(x) = J_2 v(x), \quad x \in W.$$

(iv) *There exist* $h, k \in B(W)$ *such that*

$$J_1 h(x) = Ah(x) \quad \text{implies} \quad J_1 Ah(x) = AJ_1 h(x)$$

and

$$J_2 k(x) = Ak(x) \quad \text{implies} \quad J_2 Ak(x) = AJ_2 k(x).$$

Then the functional equations (3.1a) *and* (3.1b), $i = 1, 2$, *have a unique bounded common solution in* $B(W)$.

Proof It follows from Theorem 3.1 when $G_1 = G_2 = G$. □

Corollary 3.4 [10] *Suppose that the following conditions hold*:

(i) G_1, G_2, F, g, *and* g' *are bounded.*

(ii) *Assume there exists* $r \in [0, 1)$ *such that, for every* $(x, y) \in W \times D$, $h, k \in B(W)$ *and* $t \in W$,

$$\varphi(r) \min\{|Jh(t) - A_1 h(t)|, |Jk(t) - A_2 k(t)|\} \le |Jh(t) - Jk(t)|$$

implies

$$\begin{aligned} &\bigl|G_1\bigl(x, y, h(t)\bigr) - G_2\bigl(x, y, k(t)\bigr)\bigr| \\ &\quad \le r \max\biggl\{|Jh(t) - Jk(t)|, |Jh(t) - A_1 h(t)|, |Jk(t) - A_2 k(t)|, \\ &\qquad \frac{|Jh(t) - A_2 k(t)| + |Jk(t) - A_1 h(t)|}{2}\biggr\}, \end{aligned}$$

where A_1, A_2, *and* J *are defined as follows*:

$$A_i h(x) = \sup_{y \in D}\bigl\{g(x, y) + G_i\bigl(x, y, h\bigl(\tau(x, y)\bigr)\bigr)\bigr\}, \quad x \in W, h \in B(W), i = 1, 2,$$

$$Jh(x) = q = \sup_{y \in D}\bigl\{g'(x, y) + F\bigl(x, y, h\bigl(\tau(x, y)\bigr)\bigr)\bigr\}, \quad x \in W, h \in B(W).$$

(iii) *For any* $h, k \in B(W)$, *there exist* $u, v \in B(W)$ *such that*

$$A_1 h(x) = Ju(x) \quad \text{and} \quad A_2 k(x) = Jv(x), \quad x \in W.$$

(iv) *There exist* $h, k \in B(W)$ *such that*

$$Jh(x) = A_1 h(x) \quad \text{implies} \quad JA_1 h(x) = A_1 Jh(x)$$

and

$$Jk(x) = A_2k(x) \quad \text{implies} \quad JA_2k(x) = A_2Jk(x).$$

Then the functional equations (3.1a) *and* (3.1b) *with* $F_1 = F_2 = F$ *possesses a unique bounded common solution in* W.

Proof It follows from Theorem 3.1 when $F_1 = F_2 = F$. □

As an immediate consequence of Theorem 3.1 and Corollary 2.6, we obtain the following.

Corollary 3.5 [23] *Suppose that the following conditions hold*:

(i) G_1, G_2, *and* g *are bounded.*

(ii) *There exists* $r \in [0,1)$ *such that, for every* $(x,y) \in W \times D$, $h,k \in B(W)$, *and* $t \in W$,

$$\varphi(r)\min\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|\} \le |h(t) - k(t)|$$

implies

$$\begin{aligned}&|G_1(x,y,h(t)) - G_2(x,y,k(t))|\\&\quad\le r\max\biggl\{|h(t)-k(t)|, |h(t)-A_1h(t)|, |k(t)-A_2k(t)|,\\&\qquad\frac{|h(t)-A_2k(t)| + |k(t)-A_1h(t)|}{2}\biggr\},\end{aligned}$$

where A_1 *and* A_2 *are defined as follows*:

$$A_ih(x) = \sup_{y\in D}\{g(x,y) + G_i(x,y,h(\tau(x,y)))\}, \quad x \in W, h \in B(W), i = 1,2.$$

Then the functional equation (3.1a) *possesses a unique bounded solution in* W.

Proof It follows from Corollary 3.4 when $g = 0$, $\tau(x,y) = x$, and $F(x,y,t) = t$ as the assumption (DP-3) becomes redundant in this context. □

The following result generalizes a recent result of Singh and Mishra [11, Corollary 4.2], which in turn extends certain results from [42] and [43].

Corollary 3.6 *Suppose that the following conditions hold*:

(i) G *and* g *are bounded.*

(ii) *There exists* $r \in [0,1)$ *such that, for every* $(x,y) \in W \times D$, $h,k \in B(W)$, *and* $t \in W$,

$$\varphi(r)|h(t) - Kh(t)| \le |h(t) - k(t)|$$

implies

$$|G(x,y,h(t)) - G(x,y,k(t))| \le r\max M(K,K;h(t),k(t)),$$

where K is defined as

$$Ah(x) = \sup_{y \in D} \{ g(x,y) + G(x,y,h(\tau(x,y))) \}, \quad x \in W, h \in B(W).$$

Then the functional equation (3.1a) *with $G_1 = G_2 = G$ possesses a unique bounded solution in W.*

Proof It follows from Corollary 3.5 when $G_1 = G_2 = G$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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