A new type fixed point theorem for a contraction on partially ordered generalized complete metric spaces with applications

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Abstract

In this paper, we prove a fixed point theorem for a contraction in generalized complete metric spaces endowed with partial order. As an application, we use the fixed point theorem to prove the Hyers-Ulam stability of the Cauchy functional equation in Banach spaces endowed with a partial order.
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1 Introduction

In 1940, Ulam gave a wide ranging talk in front of the mathematics club of University of Wisconsin in which he discussed a number of important unsolved problems (see [1]). One of the problems was the question concerning the stability of homomorphisms:

Let G_1 be a group and G_2 be a metric group with a metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta > 0$ such that, if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [2] affirmatively answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Taking this fact into account, the additive Cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to satisfy the *Hyers-Ulam stability*.

On the other hand, Banach's contraction principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Many kinds of generalizations of the above principle have been a heavily investigated branch of research. In particular, Diaz and Margolis [3] presented the following definition and fixed point theorem in a 'generalized complete metric space'.

Definition 1.1 Let X be an abstract (nonempty) set and assume that, in the Cartesian product $X \times X$, a distance function $d(x, y)$ ($0 \leq d(x, y) \leq \infty$ for all $x, y \in X$) is defined and satisfies the following conditions:

(D1) $d(x, y) = 0$ if and only if $x = y$;

(D2) $d(x, y) = d(y, x)$ (symmetry);

(D3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality);

(D4) every d-Cauchy sequence in X is d-convergent, *i.e.*, $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ for a sequence $\{x_n\}$ in X implies the existence of an element $x \in X$ with $\lim_{n\to\infty} d(x, x_n) = 0$ (the point x is unique by (D1) and (D3)).

Then we call (X, d) a *generalized complete metric space*.

Theorem 1.2 *Suppose that (X, d) is a generalized complete metric space and the function $T : X \to X$ is a contraction, that is, T satisfies the following condition*:

(CI) *There exists a constant q with $0 < q < 1$ such that, whenever $d(x, y) < \infty$,*

$$d(Tx, Ty) \leq qd(x, y).$$

Let $x_0 \in X$ and consider a sequence $\{T^l x_0\}$ of successive approximations with initial element x_0. Then the following alternative holds: *either*

(A) *for all $l \geq 0$, one has $d(T^l(x_0), T^{l+1}(x_0)) = \infty$*

or

(B) *the sequence $\{T^l x_0\}$ is d-convergent to a fixed point of T.*

Recently, Nieto and Rodriguez-Lopez [4] proved a fixed point theorem in partially ordered sets as follows.

Theorem 1.3 *Let $(X, \leq)$ be a partially ordered set. Suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $f : X \longrightarrow X$ be a continuous and nondecreasing mapping such that there exists $k \in [0, 1)$ with*

$$d\big(f(x), f(y)\big) \leq kd(x, y)$$

for all $x, y \in X$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

In 2003, Cădariu and Radu [5] applied the fixed point method to investigate the Jensen functional equation (see also [6–9]) and presented a short and simple proof (different from the *direct method* initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [5] for proving properties of generalized Hyers-Ulam stability for some functional equations in a single variable [7] for the stability of some nonlinear equations [6]. Recently, Brzdek [10], Brzdek and Cieplinski [11, 12] reported some interesting results in this direction (see also [13–16]).

In this paper, we prove a fixed point theorem for self-mappings on a partially ordered set X which has a generalized metric d. Moreover, we give a generalization of the Hyers-Ulam stability of the conditional Cauchy equation as an important result of our fixed point theorem.

2 Main results

We start our work by the following fixed point theorem in generalized complete metric spaces.

Theorem 2.1 *Let (X, d) be a generalized complete metric space and $\leq$ be a partial order on X. Let $f : X \to X$ be a continuous and nondecreasing mapping such that there exists*

$k \in [0,1)$ *with*

$$d\big(f(x),f(y)\big) \le kd(x,y)$$

for all $x,y \in X$ *with* $x \ge y$. *If there exists* $x_0 \le f(x_0)$, *then the following alternative holds*:
either

(A) *for all* $l \ge 0$, *one has*

$$d\big(f^l(x_0),f^{l+1}(x_0)\big) = \infty$$

or

(B) *the sequence of* $\{f^n(x_0)\}$ *is d-convergent to a fixed point of* f.

Proof Consider the sequence $\{d(f^l(x_0),f^{l+1}(x_0))\}$ of real numbers. Then we consider two
cases as follows:

(a) If, for all $l \ge 0$, $d(f^l(x_0),f^{l+1}(x_0)) = +\infty$, then (A) holds;

(b) If, for some integer l, $d(f^l(x_0),f^{l+1}(x_0)) < +\infty$, then $N = N(x_0)$ denotes the smallest
nonnegative integer such that $d(f^N(x_0),f^{N+1}(x_0)) < +\infty$.

We see that $d(f^{N+1}(x_0),f^{N+2}(x_0)) < d(f^N(x_0),f^{N+1}(x_0)) < +\infty$ and, by induction, we have

$$d\big(f^{N+l}(x_0),f^{N+l+1}(x_0)\big) < +\infty$$

for all $l \ge 1$.

Note that f is nondecreasing. Then we have

$$x_0 \le f(x_0) \le f^2(x_0) \le \cdots \le f^n(x_0) \le \cdots$$

and we can write $d(f^n(x_0),f^{n+1}(x_0)) \le kd(f^n(x_0),f^{n-1}(x_0))$ for all $n \ge 1$.

Now, by induction, we show that

$$d\big(f^{n+1}(x_0),f^N(x_0)\big) \le k^n d\big(f(x_0),x_0\big) \tag{Ω}$$

for all $n > N(x_0)$. For $n = 1$, since $x_0 \le f(x_0)$, we have

$$d\big(f^2(x_0),f(x_0)\big) \le kd\big(f(x_0),x_0\big).$$

Supposing that (Ω) holds for some n and using that $f^n(x_0) \le f^{n+1}(x_0)$, we obtain

$$d\big(f^{n+2}(x_0),f^{n+1}(x_0)\big) \le kd\big(f^{n+1}(x_0),f^n(x_0)\big) \le kk^n d\big(f(x_0),x_0\big) = k^{n+1}d\big(f(x_0),x_0\big).$$

Thus it follows that $\{f^n(x_0)\}$ is a Cauchy sequence in X. Indeed, let $m > n > N(x_0)$. Then
we have

$$\begin{aligned}
d\big(f^m(x_0),f^n(x_0)\big) &\le d\big(f^m(x_0),f^{m-1}(x_0)\big) + \cdots + d\big(f^{n+1}(x_0),f^n(x_0)\big) \\
&\le \big(k^{m-1} + k^{m-2} + \cdots + k^n\big)d\big(f(x_0),x_0\big) \\
&= \frac{k^n - k^m}{1-k}d\big(f(x_0),x_0\big).
\end{aligned}$$

On the other hand, since X is a complete generalized metric space, there exists $y \in X$ such that $\lim_{n \to \infty} f^n(x_0) = y$.

Finally, we prove that $y \in X$ is a fixed point of f, that is, $f(y) = y$. Let $\epsilon > 0$ be a positive real number. Using the continuity of f at y, for $\epsilon/2$, there exists $\delta > 0$ such that $d(z, y) < \delta$ implies that $d(f(z), f(y)) < \epsilon/2$. Now, by the convergence of $\{f^n(x_0)\}$ to y and $\eta = \min\{\epsilon/2, \delta\} > 0$, there exists $n_0 \in \mathbb{N}$ such that $n_0 > N(x_0)$ and, for all $n \geq n_0$, $d(f^{n+1}(x_0), y) < \eta$. Therefore, for all $n > n_0$, we have

$$d\big(f(y), y\big) \leq d\big(f(y), f\big(f^n(x_0)\big)\big) + d\big(f^{n+1}(x_0), y\big) < \epsilon/2 + \eta \leq \epsilon,$$

and hence $f(y) = y$. This completes the proof. □

Theorem 2.2 *In Theorem* 2.1, *we can replace the following condition with the continuity of f:*

If $\{x_n\}$ is a nondecreasing sequence and $x_n \to x$ in X, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point.

Proof In Theorem 2.1, we just showed that y is a fixed point of f. Let $\epsilon > 0$ be given. Since $f^n(x_0) \to y$ and $\{f^n(x_0)\}$ is a nondecreasing sequence, we have $f^n(x_0) \leq y$. For any $\epsilon/2 > 0$, there exists $n_0 \in \mathbb{N}$ such that $n_0 \geq l$ and, for all $n \geq n_0$, $d(f^{n_0}(x_0), y) < \epsilon/2$. Therefore, we have

$$
\begin{aligned}
d\big(f(y), y\big) &\leq d\big(f(y), f^{n_0+1}(x_0)\big) + d\big(f^{n_0+1}(x_0), y\big) \\
&\leq k d\big(y, f^{n_0}(x_0)\big) + d\big(f^{n_0+1}(x_0), y\big) \\
&< k\epsilon/2 + \epsilon/2 \\
&< \epsilon.
\end{aligned}
$$

This shows that $f(y) = y$. This completes the proof. □

Remark 2.3 In Theorem 2.2, since $f^n(x_0) \to y$ and $\{f^n(x_0)\}$ is nondecreasing, we have $x_0 \leq y$ and

$$d(x_0, y) \leq d\big(x_0, f(x_0)\big) + d\big(f(x_0), f(y)\big).$$

Therefore, we have

$$d(x_0, y) \leq d\big(x_0, f(x_0)\big) + k d(x_0, y).$$

Thus it follows that

$$d(x_0, y) \leq \frac{1}{1-k} d\big(x_0, f(x_0)\big).$$

Theorem 2.4 *If, for all $x, y \in X$, there exists z which is comparable to x and y and $d(z, x) < \infty$, $d(z, y) < \infty$, then, in Theorems 2.1 and 2.2, the uniqueness of the fixed point of f follows.*

Proof If $x \in X$ is another fixed point of f, then we prove that $d(x,y) = 0$, where $y = \lim_{n \to \infty} f^n(x_0)$. Since there exists $z \in X$ which is comparable to x and y, $f^n(z)$ is comparable to $f^n(x) = x$ and $f^n(y) = y$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$d(x,y) \leq d\big(f^n(x), f^n(z)\big) + d\big(f^n(z) + f^n(y)\big) \leq k^n d(x,z) + k^n d(z,y) \to 0$$

whenever $n \to \infty$ and so we have $d(x,z) = 0$. This completes the proof. □

3 Application

In this section, we suppose that $(E_1, \| \cdot \|_1, \leq_1)$ is a partially ordered normed space with the following conditions:

(a) for all $x, y \in E_1$, $x \leq_1 y \Longrightarrow rx \leq_1 ry$ for all $r \in \mathbb{R}^+$;

(b) for all $x, y \in E_1$, there exists $z \in E_1$ such that z is comparable to x and y.

Also, we suppose that $(E_2, \| \cdot \|_2, \leq_2)$ is a partially ordered Banach space with the condition (i) and satisfies the following:

(c) for all $x, y \in E_1$, there exists $z \in E_1$ such that z is an upper bound of $\{x, y\}$;

(d) if $\{x_n\}$ is a nondecreasing sequence in E_2 and $x_n \to x$, then $x \geq x_n$ for all $n \in \mathbb{N}$.

As a simple example, we can show that $\mathbb{R}$ satisfies the conditions (a), (b), (c) and (d). Also, in this section, we consider $0 \times \infty = 0$.

Now, we prove the main result of this section as follows.

Theorem 3.1 *Suppose that $f : E_1 \to E_2$ is a mapping satisfying*

$$2f(x) \leq_2 f(2x) \tag{3.1}$$

and

$$\big\| f(x + y + z - w) - f(x) - f(y) - f(z) + f(w) \big\|_2 \leq \phi(x,z) + \phi(y,w) \tag{3.2}$$

for all $x, y, z, w \in E_1$, where x is comparable to z, y is comparable to w, where $\phi : E_1 \times E_1 \to [0, \infty)$ is a function satisfying $\phi(0,0) = 0$ and the following condition:

$$\phi(x,y) \leq 2L\phi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.3}$$

for all $x, y \in E_1$, where x is comparable to y and $L \in (0,1)$ is a constant. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\big\| T(x) - f(x) \big\|_2 \leq \frac{1}{1 - L} \phi(x,x) \tag{3.4}$$

for all $x \in E_1$.

Proof It is clear that $f(0) = 0$. Putting $z := x$ and $y = w := 0$ in (3.2), we get

$$\big\| f(2x) - 2f(x) \big\|_2 \leq \phi(x,x)$$

for all $x \in E_1$. Hence we have

$$\left\| \frac{f(2x)}{2} - f(x) \right\|_2 \le \frac{1}{2}\phi(x,x) \le \phi(x,x) \tag{3.5}$$

for all $x \in E_1$. Consider $X := \{g : g : E_1 \to E_2\}$ and introduce the generalized metric d on X by

$$d(h,g) := \inf\left\{ C \in \mathbb{R}^+ : \left\| h(x) - g(x) \right\|_2 \le C\phi(x,x), \forall x \in E_1 \right\}$$

for all $h,g \in X$. It is easy to show that (X,d) is a complete generalized metric space.

Now, we put the partial order $\le$ on X as follows: for all $h,g \in X$,

$$h \le g \quad \Longleftrightarrow \quad h(x) \le_2 g(x)$$

for all $x \in E_1$. Now, we define a mapping $J : X \to X$ by

$$J(h)(x) := \frac{1}{2}h(2x)$$

for all $x \in E_1$. For any $g,h \in X$ with $g \le h$, it follows that, for all $x \in E_1$,

$$d(g,h) < C \quad \Longrightarrow \quad \left\| g(x) - h(x) \right\|_2 \le C\phi(x,x)$$
$$\Longrightarrow \quad \left\| \frac{g(2x)}{2} - \frac{h(2x)}{2} \right\|_2 \le C\frac{\phi(2x,2x)}{2}$$
$$\Longrightarrow \quad \left\| J(g)(x) - J(h)(x) \right\|_2 \le LC\phi(x,x).$$

It follows that

$$d\left(J(g),J(h)\right) \le Ld(g,h).$$

It is easy to show that J is a nondecreasing mapping.

Now, we show that J is a continuous function. To this end, let $\{h_n\}$ be a sequence in (X,d) which converges to $h \in X$ and let $\epsilon > 0$ be given. Then there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}^+$ with $C \le \epsilon$ such that

$$\left\| h_n(x) - h(x) \right\|_2 \le C\phi(x,x)$$

for all $x \in E_1$ and $n \ge N$ and so

$$\left\| h_n(2x) - h(2x) \right\|_2 \le C\phi(2x,2x)$$

for all $x \in E_1$ and $n \ge N$. By inequality (3.3) and the definition of J, we get

$$\left\| J(h_n)(x) - J(h)(x) \right\|_2 \le LC\phi(x,x)$$

for all $x \in E_1$ and $n \geq N$. Hence

$$d\big(J(h_n), J(h)\big) \leq LC < \epsilon$$

for all $n \geq N$. It follows that J is continuous. On the other hand, by (3.1), we have $f \leq J(f)$ and, by applying inequality (3.5), we see that $d(J(f), f) \leq 1$. Applying Theorem 2.1, it follows that J has a fixed point $T \in X$ such that $\lim_{n \to \infty} d(J^n(f), T) = 0$. It follows that

$$T(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{3.6}$$

for all $x \in E_1$. For any $x \in E_1$, it follows from (3.1) that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a nondecreasing sequence in E_2 and so, by (3.6), we have $\frac{f(2^n x)}{2^n} \leq T(x)$ for all $n \geq 0$. In particular, $f(x) \leq T(x)$. This shows that $f \leq T$. Now, we can see that

$$d\big(J(f), J(T)\big) \leq L d(f, T)$$

and hence

$$d(f, T) \leq \frac{1}{1 - L}.$$

This implies inequality (3.4).

On the other hand, by using inequality (3.3), we have

$$\phi\big(2^n x, 2^n y\big) \leq L^n \phi(x, y) \tag{3.7}$$

for all $x, y \in E_1$ and $n \in \mathbb{N}$, where x is comparable to y. Let $x, y \in E_1$ be arbitrary elements. Then there exists $z \in E_1$ such that z is comparable to x and y. This implies that $2^n z$ is comparable to $2^n x$ and $2^n y$ for all $n \in \mathbb{N}$. It follows from (3.2) that

$$\big\| f\big(2^n(x + y)\big) - f\big(2^n x\big) - f\big(2^n y\big) \big\|_2$$
$$= \big\| f\big(2^n x + 2^n y + 2^n z - 2^n z\big) - f\big(2^n x\big) - f\big(2^n y\big) - f\big(2^n z\big) + f\big(2^n z\big) \big\|_2$$
$$\leq \phi\big(2^n x, 2^n z\big) + \phi\big(2^n y, 2^n z\big)$$

for all $n \in \mathbb{N}$. By using (3.6) and (3.7), it follows that T is a Cauchy mapping.

To prove the uniqueness property of T, suppose that T_1 is another additive function satisfying (3.4). It is clear that $J(T_1) = T_1$. Then, for any $x \in E_1$, there exists $g(x) \in E_2$ such that $g(x)$ is an upper bound of $\{T(x), T_1(x)\}$. This shows that $g : E_1 \to E_2$ is a mapping which is comparable to T and T_1. Hence we have

$$d(T, T_1) \leq d\big(T, J^n(g)\big) + d\big(J^n(g), T_1\big)$$
$$= d\big(J^n(T), J^n(g)\big) + d\big(J^n(g), J^n(T_1)\big)$$
$$\leq L^n d(T, g) + L^n d(g, T_1)$$

for all $n \in \mathbb{N}$. Since $L < 1$, $T = T_1$. This completes the proof. $\qquad\square$

Corollary 3.2 *Let $\epsilon \in [0, \infty)$ and $f : E_1 \to E_2$ be a function such that $f(0) = 0$ and*

$$2f(x) \leq_2 f(2x)$$

and

$$\left\| f(x + y + z - w) - f(x) - f(y) - f(z) + f(w) \right\|_2 \leq \epsilon$$

for all $x, y, z, w \in E_1$, where x is comparable to z and y is comparable to w. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\left\| T(x) - f(x) \right\|_2 \leq \epsilon$$

for all $x \in E_1$.

Proof Set $\phi(x, y) = \frac{\epsilon}{2}$ for all $x, y \in E_1$ with $x, y \neq 0$, $\phi(0, 0) = 0$, and let $L = \frac{1}{2}$ in Theorem 3.1. Then we get the desired result. □

Corollary 3.3 *Let $p \in (0, 1)$ and $\epsilon \in [0, \infty)$. Suppose that $f : E_1 \to E_2$ is a mapping such that*

$$2f(x) \leq_2 f(2x)$$

and

$$\left\| f(x + y + z - w) - f(x) - f(y) - f(z) + f(w) \right\|_2 \leq \epsilon \left(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p \right)$$

for all $x, y, z, w \in E_1$, where x is comparable to z and y is comparable to w. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\left\| T(x) - f(x) \right\|_2 \leq \frac{2^{2-p}}{2^{1-p} - 1} \epsilon \|x\|^p$$

for all $x \in E_1$.

Proof Set $\phi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$, and let $L = 2^{p-1}$ in Theorem 3.1. Then we get the desired result. □

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