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# Common fixed point theorems for generalized multivalued contractions on cone metric spaces over a non-normal solid cone

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**Abstract**

In this paper we define a new class of multivalued generalized contractions on cone metric spaces. Then, by using a necessary new technique, we prove two common fixed point theorems for a pair of those mappings on complete cone metric spaces over solid, not necessarily normal cone. Our main theorems are generalizations of the theorem of Wardowski (Appl. Math. Lett. 24:275-278, 2011) and many existing theorems in the literature. By using our main theorems, we can obtained some important corollaries which are generalizations of the well-known metric fixed point theorems to setting of cone metric spaces over a solid non-normal cone.

**MSC:** 47H10; 54H25

**Keywords:** multivalued mapping; common fixed point; multivalued contraction; generalized multivalued contraction; cone metric space; non-normal cone

## 1 Introduction and preliminaries

There exist many generalizations of the concept of metric spaces in the literature. Fixed point theory in abstract (cone) metric, or in $K$-metric spaces over a Banach space, was developed in the mid-1970s. Huang and Zhang [1] reintroduced cone metric spaces and defined the convergence via interior points of the cone which determines an order on $E$. Although they considered and proved several fixed point theorems only in cone metric spaces over a normal cone, their approach enables the investigation of cone metric spaces over a cone which is not necessarily normal. It is well known that many fixed point results in the setting of cone metric spaces can be obtained from the corresponding results in metric spaces (see [2–4]). The results in the setting of cone metric spaces are appropriate only if the underlying cone is not necessarily normal (see [3]).

**Definition 1.1** Let $E$ be a topological vector space and $P$ be a subset of $E$. The set $P$ is called a cone if

($\mathrm{P}_1$) $P$ is closed, nonempty, and $P \neq \{\theta\}$, where $\theta$ is the zero vector of $E$;
($\mathrm{P}_2$) $a, b \in R$, $a, b \geq 0$, $x, y \in P \implies ax + by \in P$;
($\mathrm{P}_3$) $P \cap (-P) = \{\theta\}$.

A cone $P$ is called solid [5] if $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ is the interior of $P$.

Each cone $P$ of $E$ determines a partial order $\preceq$ on $E$ by $x \preceq y$ if and only if $y - x \in P$ for each $x, y \in X$. We write $x \prec y$ if $x \preceq y$ but $x \neq y$, while $x \ll y$ will denote that $y - x \in \operatorname{int} P$. This relation is compatible with the vector structure.

**Definition 1.2** Let $P$ be a cone in a real Banach space $E$. The cone $P$ is called normal, if there exists a constant $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|,$$

or, equivalently, if

$$x_n \preccurlyeq y_n \preccurlyeq z_n \quad \text{and} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x, \quad \text{then} \lim_{n\to\infty} y_n = x. \tag{1.1}$$

The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

The following example shows that there are non-normal cones.

**Example 1.3** Let $E = C_R^1([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and consider the cone $P = \{f \in E : f(t) \geq 0\}$. For each $n \geq 1$, put $f(x) = x$ and $g(x) = x^{2n}$. Then $\theta \preceq g \preceq f$, $\|f\| = 2$ and $\|g\| = 2n + 1$. Since for each $K > 0$ there exists $n \in N$ such that $2n + 1 > 2K$, we have $g \preceq f$, but $\|g\| \npreceq K\|f\|$ for any $K > 0$. Therefore, the cone $P$ is non-normal.

**Definition 1.4** ([1, 6]) Let $E$ be a Banach space and $\theta$ be the zero vector of $E$. Let $P$ be a cone in $E$ with $\operatorname{int}(P) \neq \emptyset$ and let $\preceq$ be a partial ordering with respect to $P$. A mapping $d : X \times X \to E$ is called a cone metric on the nonempty set $X$ if the following axioms are satisfied:

(d1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

The pair $(X, d)$, where $X$ is a nonempty set and $d$ is a cone metric, is called a cone metric space.

**Example 1.5** Let $E = R^2$, $P = \{(x,y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x,y) = (|x - y|, c|x - y|)$, where $c \geq 0$ is a constant. Then $(X, d)$ is a cone metric space with the normal cone $P$, where the normal constant $K = 1$.

**Definition 1.6** (Huang and Zhang [1]) Let $(X, d)$ be a cone metric space. We say that a sequence $\{x_n\}$ in $X$ is

(i) a convergent sequence if, for every $c$ in $E$ with $\theta \ll c$, there is an $N$ such that $d(x_n, x) \ll c$ for all $n > N$ and for some fixed $x$ in $X$;
(ii) a Cauchy sequence if, for every $c$ in $E$ with $\theta \ll c$, there is an $N$ such that $d(x_n, x_m) \ll c$ for all $n, m > N$.

A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

In the following lemma we suppose that $E$ is a Banach space, $P$ is a cone in $E$ with $\operatorname{int}(P) \neq \emptyset$, without the assumption of normality of cone $P$.

**Lemma 1.7** ([3, 7]) *Let $(X,d)$ be a cone metric space. Then the following properties are often used (particularly when dealing with cone metric spaces in which the cone need not be normal).*

($p_1$) *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*
($p_2$) *If $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.*
($p_3$) *If $a \preceq b + c$ for each $c \in \operatorname{int} P$, then $a \preceq b$.*
($p_4$) *If $\theta \preceq x \preceq y$ and $a \geq 0$, then $0 \preceq ax \preceq ay$.*
($p_5$) *If $E$ is a real Banach space with a cone $P$ and if $a \preceq \lambda a$, where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.*
($p_6$) *If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists $n_0$ such that, for all $n > n_0$, we have $a_n \ll c$.*

From ($p_6$) it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n,x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n,x_m) \to \theta$ as $n,m \to \infty$. In the situation with a non-normal cone we have only one part of Lemmas 1 and 4 from [1]. Also, in this case from $x_n \to x$ and $y_n \to y$ it need not follow that $d(x_n,y_n) \to d(x,y)$, as well as from $x_n \preccurlyeq y_n \preccurlyeq z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ it need not follow that $\lim_{n\to\infty} y_n = x$.

**Example 1.8** Let $E = C_R^1([0,1])$, $P \subset E$ and the norm $\|\cdot\|$ be as in Example 1.3. Consider the sequences $x_n(t) = t^n/n$ and $y_n(t) = 2/n$. Then $0 \preceq x_n \prec y_n$ and $\lim_{n\to\infty} y_n = 0$, but

$$\|x_n\| = \max_{t\in[0,1]}\left|\frac{t^n}{n}\right| + \max_{t\in[0,1]}\left|t^{n-1}\right| = \frac{1}{n} + 1 > 1.$$

Therefore, $\{x_n\}$ does not converge to 0, although $0 \leq x_n(t) < 2/n$. Thus it follows by (1.1) that $P$ is a non-normal cone.

The study of fixed points of multivalued mappings satisfying certain contractive conditions has many applications and studied by many researchers (see [8–12]). An element $x \in X$ is said to be a fixed point of a multivalued map $T : X \to 2^X$ if $x \in Tx$. Recently many authors proved fixed point theorems for multivalued mappings on complete cone metric spaces assuming that the corresponding cone is regular or normal (see [13–20]). For a cone metric space $(X,d)$ let $\tilde{A}$ be a family of subsets of $X$. Wardowski [19, Definition 3.1] introduced a new cone metric $H : \tilde{A} \times \tilde{A} \to E$. Then he introduced the concept of set-valued contraction of Nadler type [21] and proved a fixed point theorem by assumption that a cone $P$ of $E$ is solid and normal. But, as noted in [3], most of the fixed points results in cone metric spaces over a normal cone can be obtained as a consequences from the corresponding results in metric spaces. Very recently Arshad and Ahmad [22] modified Wardowski's [19] idea of $H$-cone metric. They introduced the following notion of $H$-cone metric.

**Definition 1.9** (Arshad and Ahmad [22]) Let $(X,d)$ be a cone metric space and let $\tilde{A}$ be a family of all nonempty, closed, bounded subsets of $X$. A map $H : \tilde{A} \times \tilde{A} \to E$ is called an $H$-cone metric on $\tilde{A}$ induced by $d$ if the following conditions hold:

($H_1$) $\theta \preccurlyeq H(A,B)$ for all $A,B \in \tilde{A}$ and $H(A,B) = \theta$ if and only if $A = B$;
($H_2$) $H(A,B) = H(B,A)$ for all $A,B \in \tilde{A}$;
($H_3$) $H(A,B) \preccurlyeq H(A,C) + H(C,B)$ for all $A,B,C \in \tilde{A}$;
($H_4$) If $A,B \in \tilde{A}$, $\theta \prec \epsilon \in E$ with $H(A,B) \prec \epsilon$, then for each $a \in A$ there exists $b \in B$ such that $d(a,b) \prec \epsilon$.

**Example 1.10** Let $(X,d)$ be a metric space and let $\tilde{A}$ be a family of all nonempty, closed, bounded subsets of $X$. Then the mapping $\mathcal{H} : \tilde{A} \times \tilde{A} \to R^+$ given by the formula

$$\mathcal{H}(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\}, \quad A,B \in \tilde{A}, \tag{1.2}$$

which is called a Hausdorff metric induced by the metric $d$, is an $H$-cone metric induced by $d$.

Arshad and Ahmad [22] extended the theorem of Wardowski [19] to a complete cone metric space without the assumption that a cone $P$ is normal. They proved the following theorem.

**Theorem 1.11** (Arshad and Ahmad [22]) *Let $(X,d)$ be a complete cone metric space. Let $\tilde{A}$ be a collection of nonempty, closed, and bounded subsets of $X$ and let $H : \tilde{A} \times \tilde{A} \to E$ be an $H$-cone metric induced by $d$. If for a map $T : X \to \tilde{A}$ there exists $\lambda \in (0,1)$ such that, for all $x,y \in X$,*

$$H(Tx,Ty) \le \lambda \cdot d(x,y), \tag{1.3}$$

*then $T$ has a fixed point.*

Clearly, Theorem 1.11 is a generalization of the classical theorem of multivalued contractive mappings (Nadler [21]). Recall that some of the initial generalizations of the theorem of Nadler are given in [23] and in [24]. In 1971 Ćirić in [25] introduced the concept of a generalized single-valued contraction, and then in 1972 in [23] he used the following concept of a generalized multivalued contraction.

**Definition 1.12** (Ćirić [23]) Let $(X,d)$ be a metric space and let $\tilde{A}$ be a family of nonempty, closed, and bounded subsets of $X$. A mapping $T : X \to \tilde{A}$ is said to be a generalized multivalued contraction if and only if there exists $\lambda \in [0,1)$ such that, for all $x,y \in X$,

$$H(Tx,Ty) \le \lambda \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}, \tag{1.4}$$

where $H(A,B)$ for $A,B \in \tilde{A}$ is the Hausdorff metric (1.2) induced by metric $d$.

In the present paper we will introduce the concept of a generalized multivalued contraction on cone metric spaces and then, using a new technique of proof, we prove two common fixed point theorems for a pair of those multivalued mappings on cone metric spaces over solid non-normal cones. As a consequence, we also obtain some important corollaries which are generalizations of the well-known metric fixed point theorems.

## 2 Main results

Inspired by Definition 1.12 of Ćirić we shall introduce the notion of the cone generalized multivalued contraction.

**Definition 2.1** Let $E$ be a Banach space and let $(X,d)$ be a cone metric space over $E$. Let $\tilde{A}$ be a family of nonempty, closed, and bounded subsets of $X$ and let there exists an $H$-cone metric $H:\tilde{A}\times\tilde{A}\to E$ induced by $d$. A mapping $T:X\to\tilde{A}$ is said to be a cone generalized multivalued contraction if and only if there exists $\lambda\in[0,1)$ such that, for all $x,y\in X$, a mapping $T$ satisfies one of the following contractive conditions:

(D1) $H(Tx,Ty)\preccurlyeq\lambda\cdot d(x,y)$;
(D2) $H(Tx,Ty)\preccurlyeq\lambda\cdot d(x,u)$ for each fixed $u\in Tx$;
(D3) $H(Tx,Ty)\preccurlyeq\lambda\cdot d(y,v)$ for each fixed $v\in Ty$;
(D4) $H(Tx,Ty)\preccurlyeq\lambda\cdot\frac{d(x,v)+d(y,u)}{2}$ for each fixed $v\in Ty$ and each fixed $u\in Tx$.

It is easy to show that the generalized multivalued contraction defined in Definition 1.12 is an example of the cone generalized multivalued contraction defined in Definition 2.1.

**Example 2.2** Let $X=R$ and $(X,d)$ be the usual metric space ordered by a usual ordering $\leq$. Let $\tilde{A}$ be a family of all nonempty, closed, bounded subsets of $X$ and $H:\tilde{A}\times\tilde{A}\to E$ be a Hausdorff metric. Suppose that a mapping $T:X\to\tilde{A}$ is a generalized multivalued contraction defined in Definition 1.12. If we set $E=R$, $\theta=0$, $P=\{x\in E:x\geq 0\}=R^+$ and for $x,y\in E$, we define $x\preceq y$ if and only if $x\leq y$, then $(X,d)$ is a cone metric space over cone $P$ and $T:X\to\tilde{A}$ is a cone generalized multivalued contraction.

Now we prove our main theorem.

**Theorem 2.3** *Let $E$ be a Banach space, let $P$ be a solid not necessarily normal cone of $E$ and let $(X,d)$ be a cone metric space over $E$. Let $\tilde{A}$ be a family of nonempty, closed, and bounded subsets of $X$ and let there exists an $H$-cone metric $H:\tilde{A}\times\tilde{A}\to E$ induced by $d$. Suppose that $T,S:X\to\tilde{A}$ are two cone multivalued mappings and suppose that there is $\lambda\in(0,1)$ such that, for all $x,y\in X$, at least one of the following conditions holds*:

(C1) $H(Tx,Sy)\preccurlyeq\lambda\cdot d(x,y)$;
(C2) $H(Tx,Sy)\preccurlyeq\lambda\cdot d(x,u)$ *for each fixed* $u\in Tx$;
(C3) $H(Tx,Sy)\preccurlyeq\lambda\cdot d(y,v)$ *for each fixed* $v\in Sy$;
(C4) $H(Tx,Sy)\preccurlyeq\lambda\cdot\frac{d(x,v)+d(y,u)}{2}$ *for each fixed* $v\in Sy$, $u\in Tx$.

*Then $T$ and $S$ have a common fixed point.*

*Proof* Let $x_0\in X$ and $x_1\in Sx_0$ be arbitrary. Consider the element $H(Tx_1,Sx_0)\in E$. If each right hand side of (C1), (C2), (C3), and (C4) with $x=x_1$ and $y=x_0$ is $\theta$ in $E$, then $d(x_1,x_0)=\theta$ and hence from the property $(\mathrm{d}_1)$ of the metric $d$ it follows $x_1=x_0$. This and $x_1\in Sx_0$ imply $x_0\in Sx_0$. Further, $d(x_1,u)=d(x_0,u)=\theta$ for each fixed $u\in Tx_1$ implies $x_0=u\in Tx_1=Tx_0$. Hence $x_0\in Tx_0$. Therefore, in this case $x_0$ is a common fixed point of $S$ and $T$ and proof is done.

Consider now the element $H(Tx_1,Sx_0)\in E$ in the case that, in the one of the inequalities (C1), (C2), (C3) or (C4) which holds, the right hand side is not $\theta$. Let $e\in P$ be a fixed element. Since $\lambda>0$, we have $\theta\prec\lambda e$. Thus we have $H(Tx_1,Sx_0)\prec\epsilon$, where

$\epsilon = H(Tx_1, Sx_0) + \lambda e$. Then from $H(Tx_1, Sx_0) \prec \epsilon$ and from the property ($H_4$) of the $H$-cone metric in Definition 1.9 we find, as $x_1 \in Sx_0$, that there exists $x_2 \in Tx_1$ such that

$$d(x_2, x_1) \prec \epsilon = H(Tx_1, Sx_0) + \lambda e.$$

Consider now the element $H(Sx_2, Tx_1)$. Clearly, $H(Sx_2, Tx_1) \prec H(Sx_2, Tx_1) + \lambda^2 e$. Again from ($H_4$) with $\epsilon = H(Sx_2, Tx_1) + \lambda^2 e$, as $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \prec H(Sx_2, Tx_1) + \lambda^2 e.$$

Continuing this process we can construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} \in Sx_{2n}$, $x_{2n+2} \in Tx_{2n+1}$ and

$$d(x_{2n+1}, x_{2n}) \prec H(Sx_{2n}, Tx_{2n-1}) + \lambda^{2n} e, \tag{2.1}$$

$$d(x_{2n+2}, x_{2n+1}) \prec H(Tx_{2n+1}, Sx_{2n}) + \lambda^{2n+1} e. \tag{2.2}$$

According to the inequality (2.2) and the inequalities (C1), (C2), (C3), and (C4) with $x = x_{2n+1}$ and $y = x_{2n}$, we have to consider four cases.

(1) If $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot d(x_{2n+1}, x_{2n})$, then from (2.2) we have

$$d(x_{2n+2}, x_{2n+1}) \prec \lambda \cdot d(_{2n+1}, x_{2n}) + \lambda^{2n+1} e. \tag{2.3}$$

(2) If $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot d(x_{2n+1}, u)$ for any $u \in Tx_{2n+1}$, then we can take $u = x_{2n+2} \in Tx_{2n+1}$. So, we obtain $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot d(x_{2n+1}, x_{2n+2})$ and from (2.2) we get

$$d(x_{2n+2}, x_{2n+1}) \prec \lambda \cdot d(x_{2n+1}, x_{2n+2}) + \lambda^{2n+1} e.$$

Hence

$$d(x_{2n+2}, x_{2n+1}) \preccurlyeq \lambda^{2n+1}(1-\lambda)^{-1} e. \tag{2.4}$$

(3) If $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot d(x_{2n}, v)$ for any $v \in Sx_{2n}$, then we may take $v = x_{2n+1} \in Sx_{2n}$ and we obtain $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot d(x_{2n}, x_{2n+1})$. Then from (2.2) we again have (2.3).

(4) If $H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot \frac{d(x_{2n+1}, v) + d(x_{2n}, u)}{2}$ for any $v \in Sx_{2n}$ and $u \in Tx_{2n+1}$, then we may take $v = x_{2n+1} \in Sx_{2n}$, $u = x_{2n+2} \in Tx_{2n+1}$. So we obtain

$$H(Tx_{2n+1}, Sx_{2n}) \preceq \lambda \cdot \frac{d(x_{2n}, x_{2n+2})}{2}.$$

Then from (2.2) and by the triangle inequality we have

$$d(x_{2n+2}, x_{2n+1}) \prec \lambda \cdot \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} + \lambda^{2n+1} e,$$

which implies that

$$d(x_{2n+2}, x_{2n+1}) \prec \frac{\lambda}{2-\lambda} \cdot d(x_{2n}, x_{2n+1}) + \frac{\lambda^{2n+1}}{2-\lambda} 2e. \tag{2.5}$$

Since $\lambda < 1$, we have $1/(2-\lambda) < 1$, and from (2.5) we obtain

$$d(x_{2n+2}, x_{2n+1}) \prec \lambda \cdot d(x_{2n}, x_{2n+1}) + \lambda^{2n+1} 2e. \tag{2.6}$$

It is easy see that from (2.3), (2.4), and (2.6) we get

$$d(x_{2n+2}, x_{2n+1}) \prec \lambda \cdot d(x_{2n+1}, x_{2n}) + \lambda^{2n+1}(1-\lambda)^{-1} 2e. \tag{2.7}$$

Using similar arguments to (2.1) we can prove that

$$d(x_{2n+1}, x_{2n}) \prec \lambda \cdot d(x_{2n}, x_{2n-1}) + \lambda^{2n} 2(1-\lambda)^{-1} e. \tag{2.8}$$

From (2.7) and (2.8) we conclude that

$$d(x_{n+1}, x_n) \preccurlyeq \lambda \cdot d(x_n, x_{n-1}) + \lambda^n 2(1-\lambda)^{-1} e \tag{2.9}$$

for all $n \geq 1$. From (2.9) we get

$$\begin{aligned} d(x_{n+1}, x_n) &\preccurlyeq \lambda\bigl[\lambda \cdot d(x_{n-1}, x_{n-2}) + \lambda^{n-1} 2(1-\lambda)^{-1} e\bigr] + \lambda^n 2(1-\lambda)^{-1} e \\ &= \lambda^2 \cdot d(x_{n-1}, x_{n-2}) + 2\lambda^n 2(1-\lambda)^{-1} e. \end{aligned}$$

Using mathematical induction it is easy to prove that

$$d(x_{n+1}, x_n) \preccurlyeq \lambda^n \cdot d(x_1, x_0) + n\lambda^n 2(1-\lambda)^{-1} e. \tag{2.10}$$

By the triangle inequality and (2.10) for any $m > n$ we have

$$\begin{aligned} d(x_n, x_m) &\preccurlyeq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preccurlyeq \bigl[\lambda^n \cdot d(x_0, x_1) + n\lambda^n 2(1-\lambda)^{-1} e\bigr] \\ &\quad + \bigl[\lambda^{n+1} \cdot d(x_0, x_1) + (n+1)\lambda^{n+1} 2(1-\lambda)^{-1} e\bigr] \\ &\quad + \cdots \\ &\quad + \bigl[\lambda^{m-1} \cdot d(x_0, x_1) + (m-1)\lambda^{m-1} 2(1-\lambda)^{-1} e\bigr]. \end{aligned}$$

Hence we get

$$\begin{aligned} d(x_n, x_m) &\preccurlyeq \bigl(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\bigr) d(x_0, x_1) + R_n(\lambda) 2(1-\lambda)^{-1} e \\ &\preccurlyeq \frac{\lambda^n}{1-\lambda} \cdot d(x_0, x_1) + R_n(\lambda) 2(1-\lambda)^{-1} e, \end{aligned} \tag{2.11}$$

where $R_n(\lambda)$ is the remainder of the convergent series $\sum_{n=1}^{\infty} n \cdot \lambda^n$. Since $\lambda^n \to 0$ and $R_n(\lambda) \to 0$ as $n \to \infty$, we get

$$\lambda^n (1-\lambda)^{-1} \cdot d(x_0, x_1) + R_n(\lambda) 2(1-\lambda)^{-1} e \to \theta \quad \text{as } n \to \infty. \tag{2.12}$$

Let $c \in E$ with $\theta \ll c$ be arbitrary. From (2.12) and ($p_6$) in Lemma 1.7 it follows that we can choose a natural number $n_1$ such that

$$\lambda^n(1-\lambda)^{-1} \cdot d(x_0, x_1) + R_n(\lambda)2(1-\lambda)^{-1}e \ll c$$

for all $n \geq n_1$. Thus, by (2.11), $d(x_n, x_m) \preccurlyeq c$ for all $m > n \geq n_1$. Therefore, by (ii) in Definition 1.6, we conclude that $\{x_n\}$ is Cauchy sequence. Since $X$ is complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = \theta$.

Now we shall show that $z$ is a common fixed point of $T$ and $S$. Since

$$H(Tx_{2n+1}, Sz) \prec H(Tx_{2n+1}, Sz) + \lambda^{2n+1}e,$$

from the property ($H_4$) of the $H$-cone metric in Definition 1.9 we see, as $x_{2n+2} \in Tx_{2n+1}$, that there exists $y_{2n+1} \in Sz$ such that

$$d(x_{2n+2}, y_{2n+1}) \prec H(Tx_{2n+1}, Sz) + \lambda^{2n+1}e. \tag{2.13}$$

According to (2.13) and the inequalities (C1), (C2), (C3), and (C4) with $x = x_{2n+1}$ and $y = z$ we have to consider four cases.

(1) If $H(Tx_{2n+1}, Sz) \preceq \lambda \cdot d(x_{2n+1}, z)$, then from (2.13) we have

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot d(x_{2n+1}, z) + \lambda^{2n+1}e. \tag{2.14}$$

(2) If $H(Tx_{2n+1}, Sz) \preceq \lambda \cdot d(x_{2n+1}, u)$ for any fixed $u \in Tx_{2n+1}$, then we can take $u = x_{2n+2} \in Tx_{2n+1}$. Thus from (2.13) we get

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot d(x_{2n+1}, x_{2n+2}) + \lambda^{2n+1}e. \tag{2.15}$$

(3) If $H(Tx_{2n+1}, Sz) \preceq \lambda \cdot d(z, v)$ for any fixed $v \in Sz$, then we can take $v = y_{2n+1} \in Sz$. Thus from (2.13) and by the triangle inequality we get

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot d(z, x_{2n+2}) + \lambda \cdot d(x_{2n+2}, y_{2n+1}) + \lambda^{2n+1}e,$$

which implies that

$$d(x_{2n+2}, y_{2n+1}) \prec \frac{\lambda}{1-\lambda} \cdot d(z, x_{2n+2}) + \frac{\lambda^{2n+1}}{1-\lambda}e. \tag{2.16}$$

(4) If $H(Tx_{2n+1}, Sz) \preceq \lambda \cdot \frac{d(x_{2n+1}, v) + d(z, u)}{2}$ for any $v \in Sz$ and $u \in Tx_{2n+1}$, then we can take $v = y_{2n+1} \in Sz$ and $u = x_{2n+2} \in Tx_{2n+1}$. Thus from (2.13) and by the triangle inequality we get

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot \frac{d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, y_{2n+1}) + d(z, x_{2n+2})}{2} + \lambda^{2n+1}e,$$

which implies that

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot \frac{d(x_{2n+1}, x_{2n+2}) + d(z, x_{2n+2})}{2-\lambda} + \frac{\lambda^{2n+1}}{2-\lambda}2e.$$

Since $1/(2-\lambda) < 1$ for $\lambda \in (0,1)$, we have

$$d(x_{2n+2}, y_{2n+1}) \prec \lambda \cdot d(x_{2n+1}, x_{2n+2}) + \lambda \cdot d(z, x_{2n+2}) + \lambda^{2n+1} 2e. \tag{2.17}$$

Thus, from (2.14), (2.15), (2.16), and (2.17) we have

$$\begin{aligned} d(x_{2n+2}, y_{2n+1}) &\prec \lambda \bigl[ d(x_{2n+1}, z) + d(x_{2n+1}, x_{2n+2}) + d(z, x_{2n+2})(1-\lambda)^{-1} \bigr] \\ &\quad + \lambda^{2n+1} 2(1-\lambda)^{-1} e. \end{aligned} \tag{2.18}$$

By the triangle inequality and (2.18) we get

$$\begin{aligned} d(z, y_{2n+1}) &\preccurlyeq d(z, x_{2n+2}) + d(x_{2n+2}, y_{2n+1}) \\ &\prec d(z, x_{2n+2}) + \lambda^{2n+1} 2(1-\lambda)^{-1} e \\ &\quad + \frac{\lambda}{1-\lambda} \cdot \bigl[ d(x_{2n+1}, z) + d(z, x_{2n+2}) + d(x_{2n+1}, x_{2n+2}) \bigr]. \end{aligned} \tag{2.19}$$

Since $\{x_n\}$ converges to $z$ and since $\lambda^{2n+1} \to 0$ and by (2.10) $d(x_{2n+1}, x_{2n+2}) \to \theta$ as $n \to \infty$, the right hand side of the inequality (2.19) converges to $\theta$ as $n \to \infty$. Therefore, from ($p_6$) in Lemma 1.7 and (2.19) we can choose a natural number $n_2$ such that $d(z, y_{2n+1}) \preccurlyeq c$ for all $n \geq n_2$, where $c \in E$ with $\theta \ll c$ is arbitrary. By (i) in Definition 1.6 we conclude that $\{y_n\}$ converges to $z$. Since $y_{2n+1} \in Sz$ and $Sz$ is closed, we get $z \in Sz$.

Analogously, we can get $z \in Tz$. So, we proved that $z$ is a common fixed point of $T$ and $S$. □

If we take $S = T$ in Theorem 2.3, then we obtain the following fixed point theorem in complete non-normal cone metric spaces.

**Theorem 2.4** *Let $(X, d)$ be a complete cone metric space over a solid non-normal cone and let $\tilde{A}$ be a family of nonempty, closed, and bounded subsets of $X$. Suppose that there exists an $H$-cone metric $H : \tilde{A} \times \tilde{A} \to E$ induced by $d$ and suppose that $T : X \to \tilde{A}$ is a cone generalized multivalued contraction. Then $T$ has a fixed point.*

From Theorem 2.4 we can obtain Theorem 3.1 of Arshad and Ahmad [22] and Theorem 3.1 of Wardowski [19].

Now we shall present an example where Theorem 2.4 can be applied, but the theorem of Arshad and Ahmad [22] (Theorem 1.11) and the theorem of Wardowski [19] cannot be applied.

**Example 2.5** Let $X = [0,1]$ and let $E = C^1_R([0,1])$, $P \subset E$ and the norm $\|\cdot\|$ be as in Example 1.3. Define $d : X \times X \to E$ by

$$d(x,y)(t) = |x-y| \cdot e^t,$$

where $0 \leq t \leq 1$. Then $d$ is a cone metric on $X$. Let $\tilde{A}$ be a family of all nonempty, closed, bounded subsets of $X$ and let the mapping $T : X \to \tilde{A}$ be defined by

$$T(x) = \begin{cases} [0, \frac{x}{3}] & \text{for } 0 \leq x < 1, \\ [0, \frac{1}{8}] & \text{for } x = 1. \end{cases}$$

Let $H : \tilde{A} \times \tilde{A} \to E$ be defined by

$$H(A,B)(t) = \mathcal{H}(A,B) \cdot e^t \quad \text{for } A,B \in \tilde{A},$$

where $\mathcal{H}$ is the usual Hausdorff metric on $X$ induced by the metric $d(x,y) = |x-y|$.

Now we can show that $T$ satisfies all conditions of Theorem 2.4 with $\lambda = \frac{1}{2}$.

(1) If $0 \le x, y < 1$ and $x \neq y$, then we have

$$H\bigl(T(x), T(y)\bigr)(t) = \mathcal{H}\left(\left[0, \frac{x}{3}\right], \left[0, \frac{y}{3}\right]\right) \cdot e^t = \left|\frac{x}{3} - \frac{y}{3}\right| \cdot e^t < \frac{1}{2}|x-y| \cdot e^t = \lambda d(x,y)(t).$$

Thus $T$ satisfies the contractive condition (D1) in Definition 1.12 with $\lambda = 1/2$.

(2) If $0 \le x < 1$ and $y = 1$, then we have

$$H\bigl(T(x), T(1)\bigr)(t) < \left|\frac{1}{3} - \frac{1}{8}\right| \cdot e^t = \frac{5}{24} \cdot e^t < \frac{1}{2} \cdot \frac{7}{8} e^t \le \lambda d(1,u)(t)$$

for all $u \in [0, 1/8] = T(1)$. Therefore, in this case $T$ satisfies the contractive condition (D2) in Definition 1.12 with $\lambda = 1/2$.

From (1) and (2) we see that the mapping $T$ satisfies all of the conditions of Theorem 2.4 and has a fixed point $x = 0$.

Now we shall show that in this example the theorems of Arshad and Ahmad [22] and Wardowski [19], as well as other theorems known in the literature, cannot be applied. Let $y = 1$ and $\frac{6}{7} < x < 1$. Then

$$H\bigl(T(x), T(1)\bigr)(t) = \left|\frac{x}{3} - \frac{1}{8}\right| \cdot e^t > \left|\frac{2}{7} - \frac{1}{8}\right| \cdot e^t > \left|1 - \frac{6}{7}\right| \cdot e^t > |1-x| \cdot e^t = d(x,1)(t).$$

Clearly, there does not exist $\lambda < 1$ such that $H(T(x), T(1))(t) \le \lambda \cdot d(x,1)(t)$. Therefore, Theorem 3.1 of Arshad and Ahmad [22] (Theorem 1.11) and Theorem 3.1 of Wardowski [19] cannot be applied in this example.

In a cone $P$ of an ordered Hausdorff topological vector space $(E,P)$, from $a, b \in P$ it does not need to follow that $(1/2)(a+b) \preceq a$ nor $(1/2)(a+b) \preceq b$. Thus in addition to the conditions (C1)-(C4) of Theorem 2.3 we can consider the condition

(C5) $H(Tx, Sy) \preccurlyeq \lambda \cdot \frac{d(x,u)+d(y,v)}{2}$ for each fixed $u \in Tx$ and $v \in Sy$.

The following theorem is a generalization of Theorem 2.3.

**Theorem 2.6** *Let* $(X,d)$ *be a complete cone metric space over a solid non-normal cone, let* $\tilde{A}$ *be a family of non-empty, closed, and bounded subsets of* $X$ *and let there exists an* $H$*-cone metric* $H : \tilde{A} \times \tilde{A} \to E$ *induced by* $d$*. Suppose that* $T, S : X \to \tilde{A}$ *are two cone multivalued mappings and suppose that there is* $\lambda \in (0,1)$ *such that, for all* $x, y \in X$*, at least one of the conditions* (C1)-(C5) *holds. Then* $T$ *and* $S$ *have a common fixed point.*

We shall omit the proof of this theorem since it is similar to the proof of Theorem 2.3.

By using Theorem 2.3 and Theorem 2.6 we can obtain corollaries which are generalizations of the well-known metric fixed point theorems of Kannan [26], Reich [27], Chatterjea

[28] and Ćirić [25] to non-normal cone metric spaces. For example, the following corollary is a cone multivalued version of Kannan's fixed point theorem, and it easily follows from Theorem 2.6.

**Corollary 2.7** *Let $(X,d)$ be a complete cone metric space over a solid non-normal cone, let $\tilde{A}$ be a family of non-empty, closed, and bounded subsets of $X$ and let there exists an $H$-cone metric $H:\tilde{A}\times\tilde{A}\to E$ induced by $d$. Suppose that $T,S:X\to\tilde{A}$ are two cone multivalued mappings and suppose that there is $\gamma\in(0,1/2)$ such that, for all $x,y\in X$, the mappings $T$ and $S$ satisfy the condition*

$$H(Tx,Sy)\preccurlyeq\gamma\big(d(x,u)+d(y,v)\big)$$

*for each $u\in Tx$ and for each $v\in Sy$. Then $T$ and $S$ have a common fixed point.*

**Competing interests**
The author declares that he has no competing interests.

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