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Hybrid iterative method for systems of generalized equilibria with constraints of variational inclusion and fixed point problems

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Abstract
In this paper, we introduce and analyze an iterative algorithm by the hybrid iterative method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. Weak convergence result under mild assumptions will be established.
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1 Introduction and formulations

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, let C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S: C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma} \in (0,1]$ such that

$$\langle Vx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

A mapping $S: C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, S is called a nonexpansive mapping if $L = 1$ and A is called a contraction if $L \in [0,1)$.

Let $\varphi: C \to \mathbf{R}$ be a real-valued function, $A: H \to H$ be a nonlinear mapping and $\Theta: C \times C \to \mathbf{R}$ be a bifunction. Peng and Yao [1] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of GMEP (1.2) by $\mathrm{GMEP}(\Theta, \varphi, A)$. GMEP (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others.

Throughout this paper, we assume as in [1] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (H5), where

(H1) $\Theta(x,x) = 0$ for all $x \in C$;
(H2) Θ is monotone, *i.e.*, $\Theta(x,y) + \Theta(y,x) \le 0$ for any $x, y \in C$;
(H3) Θ is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta\bigl(tz + (1-t)x, y\bigr) \le \Theta(x,y);$$

(H4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;
(H5) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

Given a positive number $r > 0$, let $S_r^{(\Theta,\varphi)} : H \to C$ be a solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$S_r^{(\Theta,\varphi)}(x) := \biggl\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\bigl\langle K'(y) - K'(x), z - y\bigr\rangle \ge 0, \forall z \in C \biggr\}.$$

In particular, whenever $K(x) = \frac{1}{2}\|x\|^2$, $\forall x \in H$, $S_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{(\Theta,\varphi)}$.

Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions and $A_1, A_2 : C \to H$ be two nonlinear mappings. Consider the following system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Theta_1(x^*, x) + \langle A_1 y^*, x - x^*\rangle + \frac{1}{\nu_1}\langle x^* - y^*, x - x^*\rangle \ge 0, & \forall x \in C, \\ \Theta_2(y^*, y) + \langle A_2 x^*, y - y^*\rangle + \frac{1}{\nu_2}\langle y^* - x^*, y - y^*\rangle \ge 0, & \forall y \in C, \end{cases} \tag{1.3}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It is introduced and studied in [2]. When $\Theta_1 \equiv \Theta_2 \equiv 0$, the SGEP reduces to a system of variational inequalities, which is considered and studied in [3]. It is worth to mention that the system of variational inequalities is a tool to solve the Nash equilibrium problem for noncooperative games.

In 2010, Ceng and Yao [2] transformed the SGEP into a fixed point problem in the following way.

Proposition 1.1 (see [2]) *Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions* (H1)-(H4), *and let $A_k : C \to H$ be ζ_k-inverse-strongly monotone for $k = 1, 2$. Let $\nu_k \in (0, 2\zeta_k)$ for $k = 1, 2$. Then $(x^*, y^*) \in C \times C$ is a solution of SGEP* (1.3) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $G = T_{\nu_1}^{\Theta_1}(I - \nu_1 A_1) T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)$, where $y^* = T_{\nu_2}^{\Theta_2}(I - \nu_2 A_2)x^*$. Here, we denote the fixed point set of G by* $\mathrm{SGEP}(G)$.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n on H as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \ldots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \ldots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{1.4}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In 2011, for the case where $C = H$, Yao *et al.* [4] proposed the following hybrid iterative algorithm:

$$\begin{cases} \Theta(y_n, z) + \varphi(z) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), z - y_n\rangle \geq 0, & z \in H, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n y_n, & \forall n \geq 1, \end{cases} \tag{1.5}$$

where $f : H \to H$ is a contraction, $K : H \to \mathbf{R}$ is differentiable and strongly convex, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $x_0, u \in H$ are given, for finding a common element of the set $\mathrm{MEP}(\Theta, \varphi)$ and the fixed point set $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H. They proved the strong convergence of the sequence generated by the hybrid iterative algorithm (1.5) to a point $x^* \in \Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{MEP}(\Theta, \varphi)$ under some appropriate conditions. This point x^* also solves the following optimization problem:

$$\min_{x \in \Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{OP0}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

Let $f : H \to H$ be a contraction and V be a strongly positive bounded linear operator on H. Assume that $\varphi : H \to \mathbf{R}$ is a lower semicontinuous and convex functional, that $\Theta, \Theta_1, \Theta_2 : H \times H \to \mathbf{R}$ satisfy conditions (H1)-(H4), and that $A, A_1, A_2 : H \to H$ are inverse-strongly monotone. Let the mapping G be defined as in Proposition 1.1. Very recently, Ceng *et al.* [5] introduced the following hybrid extragradient-like iterative algorithm:

$$\begin{cases} z_n = S_{r_n}^{(\Theta,\varphi)}(x_n - r_n A x_n), \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu V))W_n G z_n, & \forall n \geq 0, \end{cases} \tag{1.6}$$

for finding a common solution of GMEP (1.2), SGEP (1.3) and the fixed point problem of an infinite family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ on H, where $\{r_n\} \subset (0, \infty)$,

$\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\nu_k \in (0, 2\zeta_k)$, $k = 1, 2$, and $x_0, u \in H$ are given. The authors proved the strong convergence of the sequence generated by the hybrid iterative algorithm (1.6) to a point $x^* \in \Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G)$ under some suitable conditions. This point x^* also solves the following optimization problem:

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{OP1}$$

where $h : H \to \mathbf{R}$ is the potential function of γf.

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with domain $D(R) = C$. Consider the following variational inclusion [6]: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{1.7}$$

We denote by $\mathrm{I}(B, R)$ the solution set of the variational inclusion (1.7). It is known that problem (1.7) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, *etc.* Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H,$$

where λ is a positive number.

In 2011, for the case where $C = H$, Yao *et al.* [7] introduced and analyzed the following iterative algorithms for finding an element of the intersection $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{I}(B, R)$ of the solution set of GMEP (1.2), the solution set of the variational inclusion (1.7) and the fixed point set of a countable family $\{T_n\}_{n=1}^{\infty}$ of nonexpansive mappings: for arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Ax_n\rangle + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n V] W_n J_{R,\lambda}(u_n - \lambda B u_n), & \forall n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0,1]$ and W_n is the W-mapping defined by (1.4). It is proven that under appropriate conditions the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $x^* = P_\Omega(\gamma f(x^*) + (I - V)x^*)$ is a unique solution of the VIP:

$$\big\langle (\gamma f - V)x^*, y - x^*\big\rangle \leq 0, \quad \forall y \in \Omega. \tag{1.9}$$

Next, we recall some concepts. Let C be a nonempty subset of a normed space X. A mapping $S : C \to C$ is called uniformly Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$\big\|S^n x - S^n y\big\| \leq \mathcal{L}\|x - y\|, \quad \forall n \geq 1, \forall x, y \in C.$$

Recently, Kim and Xu [8] introduced the concept of asymptotically k-strict pseudocontractive mappings in a Hilbert space as follows.

Definition 1.1 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $k \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\left\|S^n x - S^n y\right\|^2 \leq (1+\gamma_n)\|x-y\|^2 + k\left\|x - S^n x - \left(y - S^n y\right)\right\|^2, \quad \forall n \geq 1, \forall x,y \in C.$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \geq 1\}$. Subsequently, Sahu *et al.* [9] considered the concept of asymptotically k-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 1.2 Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\|S^n x - S^n y\right\|^2 - (1+\gamma_n)\|x-y\|^2 - k\left\|x - S^n x - \left(y - S^n y\right)\right\|^2\right) \leq 0.$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \geq 0$ ($\forall n \geq 1$), $c_n \to 0$ ($n \to \infty$) and we get the relation

$$\begin{aligned} \left\|S^n x - S^n y\right\|^2 &\leq (1+\gamma_n)\|x-y\|^2 \\ &\quad + k\left\|x - S^n x - \left(y - S^n y\right)\right\|^2 + c_n, \quad \forall n \geq 1, \forall x,y \in C. \end{aligned} \tag{1.10}$$

Whenever $c_n = 0$ for all $n \geq 1$ in (1.10), then S is an asymptotically k-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. In 2009, Sahu *et al.* [9] derived the weak and strong convergence of the modified Mann iteration processes for an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. More precisely, they first established one weak convergence theorem for the following iterative scheme:

$$\begin{cases} x_1 = x \in C & \text{chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n x_n, & \forall n \geq 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1 - k - \delta$, $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$; and then obtained another strong convergence theorem for the following iterative scheme:

$$\begin{cases} x_1 = x \in C & \text{chosen arbitrarily,} \\ y_n = (1-\alpha_n)x_n + \alpha_n S^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, & \forall n \geq 1, \end{cases}$$

where $0 < \delta \leq \alpha_n \leq 1 - k$, $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$.

Motivated and inspired by the above results and the method in [10], we introduce and analyze an iterative algorithm by the hybrid iterative method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. A weak convergence theorem for the iterative algorithm will be established under mild conditions.

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We use the notations $x_n \rightharpoonup x$ and $x_n \to x$ to indicate the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x, respectively. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \bigl\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\bigr\}.$$

Definition 2.1 A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2.2 A differentiable function $K : H \to \mathbf{R}$ is called:

(i) convex if

$$K(y) - K(x) \geq \bigl\langle K'(x), y - x\bigr\rangle, \quad \forall x, y \in H,$$

where $K'(x)$ is the Frechet derivative of K at x;

(ii) strongly convex if there exists a constant $\sigma > 0$ such that

$$K(y) - K(x) - \bigl\langle K'(x), y - x\bigr\rangle \geq \frac{\sigma}{2}\|x - y\|^2, \quad \forall x, y \in H.$$

It is easy to see that if $K : H \to \mathbf{R}$ is a differentiable strongly convex function with constant $\sigma > 0$, then $K' : H \to H$ is strongly monotone with constant $\sigma > 0$.

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are listed in the following proposition.

Proposition 2.1 ([11, p.17]) *For given $x \in H$ and $z \in C$,*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$, $\forall y \in C$;
(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall y \in H$. (*This implies that P_C is nonexpansive and monotone*.)

By using the technique of [12], we can readily obtain the following elementary result where $\mathrm{MEP}(\Theta, \varphi)$ is the solution set of the mixed equilibrium problem [5].

Proposition 2.2 (see [5, Lemma 1 and Proposition 1]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions* (H1)-(H4). *Assume that*

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$*;
(ii) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,*

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\bigl\langle K'(z) - K'(x), y_x - z\bigr\rangle < 0.$$

Then the following hold:

(a) *for each $x \in H$, $S_r^{(\Theta,\varphi)}(x) \neq \emptyset$*;
(b) *$S_r^{(\Theta,\varphi)}$ is single-valued*;
(c) *$S_r^{(\Theta,\varphi)}$ is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ and*

$$\bigl\langle K'(x_1) - K'(x_2), u_1 - u_2\bigr\rangle \leq \bigl\langle K'(u_1) - K'(u_2), u_1 - u_2\bigr\rangle, \quad \forall (x_1, x_2) \in H \times H,$$

where $u_i = S_r^{(\Theta,\varphi)}(x_i)$ for $i = 1, 2$;
(d) *for all $s, t > 0$ and $x \in H$,*

$$\bigl\langle K'\bigl(S_s^{(\Theta,\varphi)}x\bigr) - K'\bigl(S_t^{(\Theta,\varphi)}x\bigr), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\bigr\rangle \leq \frac{s-t}{s}\bigl\langle K'\bigl(S_s^{(\Theta,\varphi)}x\bigr) - K'(x), S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\bigr\rangle;$$

(e) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;
(f) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex*.

Remark 2.1 In Proposition 2.2, whenever $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (H1)-(H4) and $K(x) = \frac{1}{2}\|x\|^2$, $\forall x \in H$, we have, for any $x, y \in H$,

$$\bigl\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\bigr\|^2 \leq \bigl\langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\bigr\rangle$$

($S_r^{(\Theta,\varphi)}$ is firmly nonexpansive) and

$$\left\|S_s^{(\Theta,\varphi)}x - S_t^{(\Theta,\varphi)}x\right\| \le \frac{|s-t|}{s}\left\|S_s^{(\Theta,\varphi)}x - x\right\|, \quad \forall s,t > 0, x \in H.$$

If, in addition, $\varphi \equiv 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}; see [2, Lemma 2.1] for more details.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in X.$$

Lemma 2.2 ([11, p.20]) *Let H be a real Hilbert space. Then the following hold*:
(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all* $x,y \in H$ *and* $\lambda,\mu \in [0,1]$ *with* $\lambda + \mu = 1$;
(c) *If* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

We have the following crucial lemmas concerning the W-mappings defined by (1.4).

Lemma 2.3 (see [13, Lemma 3.3]) *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H such that $\bigcap_{n=1}^{\infty}\operatorname{Fix}(T_n) \ne \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then* $\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty}\operatorname{Fix}(T_n)$.

Lemma 2.4 (see [14, Demiclosedness principle]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 2.5 ([9, Lemma 2.6]) *Let C be a nonempty subset of a Hilbert space H and $S : C \to C$ be an asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\left\|S^n x - S^n y\right\| \le \frac{1}{1-k}\left(k\|x-y\| + \sqrt{\left(1 + (1-k)\gamma_n\right)\|x-y\|^2 + (1-k)c_n}\right)$$

for all $x,y \in C$ and $n \ge 1$.

Lemma 2.6 (Demiclosedness principle [9, Proposition 3.1]) *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to C$ be a continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and* $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, *then* $(I-S)x = 0$.

Recall that a Banach space X is said to satisfy the Opial condition [4] if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, the following inequality holds:

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well known in [4] that every Hilbert space H satisfies the Opial condition.

Lemma 2.7 (see [15, Proposition 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \leq (1 + \lambda_n)\|x_n - p\|^2 + \delta_n, \quad \forall p \in C, n \geq 1,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

3 Weak convergence theorem

In this section, we will prove weak convergence of another iterative algorithm by the hybrid Mann-type viscosity method for finding a solution of the system of generalized equilibrium problems with constraints of several problems: a generalized mixed equilibrium problem, finitely many variational inclusions, and the common fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense and infinitely many nonexpansive mappings in a real Hilbert space. This iterative algorithm is based on the extragradient method, viscosity approximation method and Mann-type iterative method.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let N be an integer. Let Θ, Θ_1, Θ_2 be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping, and let $A, A_k : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone, and η_i-inverse strongly monotone, respectively, where $k \in \{1, 2\}$ and $i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \leq k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{c_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\bar{\gamma}$. Let W_n be the W-mapping defined by* (1.4). *Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i) \cap \mathrm{Fix}(S)$ is nonempty, where G is defined as in Proposition* 1.1. *Let $\{r_n\}$ be a sequence in $[0, 2\zeta]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $0 < k + \epsilon \leq \delta_n \leq d < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_N,\lambda_{N,n}}(I - \lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}}(I - \lambda_{N-1,n}B_{N-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n(I + \mu V)]W_n G k_n, \quad \forall n \geq 1. \end{cases} \tag{3.1}$$

Assume that the following conditions are satisfied:

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\bigl\langle K'(y) - K'(x), z_x - y\bigr\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta$;

(iv) $\nu_k \in (0, 2\zeta_k)$, $k \in \{1, 2\}$ *and* $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\forall i \in \{1, 2, \dots, N\}$.

If $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, then* $\{x_n\}$ *converges weakly to* $w = \lim_{n\to\infty} P_\Omega x_n$.

Proof First, let us show that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in \Omega$. Put

$$\Lambda_n^i = J_{R_i,\lambda_{i,n}}(I - \lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}}(I - \lambda_{i-1,n}B_{i-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)$$

for all $i \in \{1, 2, \dots, N\}$, $n \ge 1$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we get $z_n = \Lambda_n^N u_n$. Take $p \in \Omega$ arbitrarily. Repeating the same arguments as in the proof of [16, Theorem 3.1], we can obtain that

$$\bigl\|(1 - \beta_n)I - \alpha_n(I + \mu V)\bigr\| \le 1 - \beta_n - \alpha_n - \alpha_n\mu\bar{\gamma}, \tag{3.2}$$

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + r_n(r_n - 2\zeta)\|Ax_n - Ap\|^2 \le \|x_n - p\|^2, \tag{3.3}$$

$$\|z_n - p\| \le \|u_n - p\|, \tag{3.4}$$

$$\begin{aligned}
\|Gk_n - p\|^2 &\le \bigl\|T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)k_n - T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)p\bigr\|^2 \\
&\quad + \nu_1(\nu_1 - 2\zeta_1)\bigl\|A_1T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)k_n - A_1T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)p\bigr\|^2 \\
&\le \bigl\|T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)k_n - T_{\nu_2}^{\Theta_2}(I - \nu_2A_2)p\bigr\|^2 \\
&\le \|k_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|A_2k_n - A_2p\|^2 \\
&\le \|k_n - p\|^2,
\end{aligned} \tag{3.5}$$

$$\bigl\|\Lambda_n^iu_n - p\bigr\|^2 \le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\bigl\|B_i\Lambda_n^{i-1}u_n - B_ip\bigr\|^2, \quad i \in \{1, 2, \dots, N\}, \tag{3.6}$$

$$\begin{aligned}
\bigl\|\Lambda_n^iu_n - p\bigr\|^2 &\le \|x_n - p\|^2 - \bigl\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\bigr\|^2 \\
&\quad + 2\lambda_{i,n}\bigl\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\bigr\|\bigl\|B_i\Lambda_n^{i-1}u_n - B_ip\bigr\|, \quad i \in \{1, 2, \dots, N\}.
\end{aligned} \tag{3.7}$$

We observe that

$$\begin{aligned}
\|k_n - p\|^2 &= \bigl\|\delta_n(z_n - p) + (1 - \delta_n)\bigl(S^nz_n - p\bigr)\bigr\|^2 \\
&= \delta_n\|z_n - p\|^2 + (1 - \delta_n)\bigl\|S^nz_n - p\bigr\|^2 - \delta_n(1 - \delta_n)\bigl\|z_n - S^nz_n\bigr\|^2 \\
&\le \delta_n\|z_n - p\|^2 + (1 - \delta_n)\bigl[(1 + \gamma_n)\|z_n - p\|^2 + k\bigl\|z_n - S^nz_n\bigr\|^2 + c_n\bigr] \\
&\quad - \delta_n(1 - \delta_n)\bigl\|z_n - S^nz_n\bigr\|^2 \\
&= \bigl[1 + \gamma_n(1 - \delta_n)\bigr]\|z_n - p\|^2 + (1 - \delta_n)(k - \delta_n)\bigl\|z_n - S^nz_n\bigr\|^2 + (1 - \delta_n)c_n
\end{aligned}$$

$$\begin{aligned}
&\leq (1+\gamma_n)\|z_n-p\|^2+(1-\delta_n)(k-\delta_n)\big\|z_n-S^nz_n\big\|^2+c_n \\
&\leq (1+\gamma_n)\|z_n-p\|^2+c_n. \qquad (3.8)
\end{aligned}$$

Set $\bar{V}=I+\mu V$. Then by Lemma 2.1 we deduce from (3.2)-(3.5) and (3.8) and $0\leq\gamma l\leq(1+\mu)\bar{\gamma}$ that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad=\big\|\alpha_n\gamma\big(f(x_n)-f(p)\big)+\beta_n(k_n-p)+\big((1-\beta_n)I-\alpha_n\bar{V}\big)(W_nGk_n-p) \\
&\qquad+\alpha_n\big(u+(\gamma f-\bar{V})p\big)\big\|^2 \\
&\quad\leq\big\|\alpha_n\gamma\big(f(x_n)-f(p)\big)+\beta_n(k_n-p)+\big((1-\beta_n)I-\alpha_n\bar{V}\big)(W_nGk_n-p)\big\|^2 \\
&\qquad+2\alpha_n\big\langle u+(\gamma f-\bar{V})p,x_{n+1}-p\big\rangle \\
&\quad\leq\big[\alpha_n\gamma\big\|f(x_n)-f(p)\big\|+\beta_n\|k_n-p\|+\big\|(1-\beta_n)I-\alpha_n\bar{V}\big\|\|W_nGk_n-p\|\big]^2 \\
&\qquad+2\alpha_n\big\|u+(\gamma f-\bar{V})p\big\|\|x_{n+1}-p\| \\
&\quad\leq\big[\alpha_n\gamma l\|x_n-p\|+\beta_n\|k_n-p\|+(1-\beta_n-\alpha_n-\alpha_n\mu\bar{\gamma})\|k_n-p\|\big]^2 \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad\leq\big[\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|+\beta_n\|k_n-p\|+\big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big)\|k_n-p\|\big]^2 \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad=\big[\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\|k_n-p\|\big]^2 \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad\leq\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\|k_n-p\|^2 \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad\leq\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|z_n-p\|^2+c_n\big) \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad\leq\alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|x_n-p\|^2+c_n\big) \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad=\|x_n-p\|^2+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)\gamma_n\|x_n-p\|^2+\big(1-\alpha_n(1+\mu)\bar{\gamma}\big)c_n \\
&\qquad+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big) \\
&\quad\leq(1+\gamma_n)\|x_n-p\|^2+c_n+\alpha_n\big(\big\|u+(\gamma f-\bar{V})p\big\|^2+\|x_{n+1}-p\|^2\big),
\end{aligned}$$

which hence yields

$$\begin{aligned}
\|x_{n+1}-p\|^2&\leq\frac{1+\gamma_n}{1-\alpha_n}\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\big\|u+(\gamma f-\bar{V})p\big\|^2+\frac{1}{1-\alpha_n}c_n \\
&=\left(1+\frac{\alpha_n+\gamma_n}{1-\alpha_n}\right)\|x_n-p\|^2+\frac{\alpha_n}{1-\alpha_n}\big\|u+(\gamma f-\bar{V})p\big\|^2+\frac{1}{1-\alpha_n}c_n \\
&\leq\big[1+(\alpha_n+\gamma_n)\varrho\big]\|x_n-p\|^2+\alpha_n\varrho\big\|u+(\gamma f-\bar{V})p\big\|^2+\varrho c_n, \qquad (3.9)
\end{aligned}$$

where $\varrho = \frac{1}{1-\sup_{n\geq 1}\alpha_n} < \infty$ (due to $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$). Since $\sum_{n=1}^{\infty}\alpha_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}c_n < \infty$, by Lemma 2.7 we have that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$, $\{z_n\}$ and $\{k_n\}$.

Also, utilizing Lemmas 2.1 and 2.2(b), we obtain from (3.3)-(3.5) and (3.8) that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr) + \beta_n(k_n - p) + (1-\beta_n)(W_nGk_n - p)\bigr\|^2 \\
&\quad \leq \bigl\|\beta_n(k_n - p) + (1-\beta_n)(W_nGk_n - p)\bigr\|^2 + 2\alpha_n\bigl\langle u + \gamma f(x_n) - \bar{V}W_nGk_n, x_{n+1} - p\bigr\rangle \\
&\quad = \beta_n\|k_n - p\|^2 + (1-\beta_n)\|W_nGk_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\| \\
&\quad \leq \beta_n\|k_n - p\|^2 + (1-\beta_n)\|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\| \\
&\quad = \|k_n - p\|^2 - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\| \\
&\quad \leq (1+\gamma_n)\|z_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\| \\
&\quad \leq (1+\gamma_n)\|x_n - p\|^2 + c_n - \beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\|,
\end{aligned} \tag{3.10}$$

which leads to

$$\begin{aligned}
&\beta_n(1-\beta_n)\|k_n - W_nGk_n\|^2 \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\
&\qquad + 2\alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\|\|x_{n+1} - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, it follows from the existence of $\lim_{n\to\infty}\|x_n - p\|$ and condition (iii) that

$$\lim_{n\to\infty}\|k_n - W_nGk_n\| = 0. \tag{3.11}$$

Note that

$$x_{n+1} - k_n = \alpha_n\bigl(u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr) + (1-\beta_n)(W_nGk_n - k_n),$$

which yields

$$\begin{aligned}
\|x_{n+1} - k_n\| &\leq \alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\| + (1-\beta_n)\|W_nGk_n - k_n\| \\
&\leq \alpha_n\bigl\|u + \gamma f(x_n) - \bar{V}W_nGk_n\bigr\| + \|W_nGk_n - k_n\|.
\end{aligned}$$

So, from (3.11) and $\lim_{n\to\infty}\alpha_n = 0$, we get

$$\lim_{n\to\infty}\|x_{n+1} - k_n\| = 0. \tag{3.12}$$

In the meantime, we conclude from (3.3), (3.4), (3.8) and (3.10) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \|k_n-p\|^2-\beta_n(1-\beta_n)\|k_n-W_nGk_n\|^2+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le \|k_n-p\|^2+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le (1+\gamma_n)\|z_n-p\|^2+(1-\delta_n)(k-\delta_n)\big\|z_n-S^nz_n\big\|^2+c_n \\
&\qquad+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le (1+\gamma_n)\|x_n-p\|^2+(1-\delta_n)(k-\delta_n)\big\|z_n-S^nz_n\big\|^2+c_n \\
&\qquad+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\|,
\end{aligned}$$

which together with $0<k+\epsilon\le\delta_n\le d<1$ implies that

$$\begin{aligned}
&(1-d)\epsilon\big\|z_n-S^nz_n\big\|^2 \\
&\quad\le (1-\delta_n)(\delta_n-k)\big\|z_n-S^nz_n\big\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\qquad+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\|.
\end{aligned}$$

Consequently, from $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\gamma_n=0$, $\lim_{n\to\infty}c_n=0$ and the existence of $\lim_{n\to\infty}\|x_n-p\|$, we get

$$\lim_{n\to\infty}\big\|z_n-S^nz_n\big\|=0. \tag{3.13}$$

Since $k_n-z_n=(1-\delta_n)(S^nz_n-z_n)$, from (3.13) we have

$$\lim_{n\to\infty}\|k_n-z_n\|=0. \tag{3.14}$$

Combining (3.3), (3.4), (3.8) and (3.10), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \|k_n-p\|^2+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le \|z_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le \|u_n-p\|^2+\gamma_n\|z_n-p\|^2+c_n+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\| \\
&\quad\le \|x_n-p\|^2+r_n(r_n-2\zeta)\|Ax_n-Ap\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\qquad+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&r_n(2\zeta-r_n)\|Ax_n-Ap\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|x_n-p\|^2+c_n \\
&\qquad+2\alpha_n\big\|u+\gamma f(x_n)-\bar{V}W_nGk_n\big\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, from condition (iii) and the existence of $\lim_{n\to\infty}\|x_n - p\|$ we get

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.15}$$

Repeating the same arguments as those of (3.17) in the proof of [16, Theorem 3.1], we can get

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\|. \tag{3.16}$$

Combining (3.8), (3.10) and (3.16), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|k_n - p\|^2 + 2\alpha_n\|u + \gamma f(x_n) - \bar{V}W_nGk_n\|\|x_{n+1} - p\| \\ &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\| \\ &\le \|u_n - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\| \\ &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| + \gamma_n\|x_n - p\|^2 + c_n \\ &\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\|,\end{aligned}$$

which implies

$$\begin{aligned}\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n\|Ax_n - Ap\|\|x_n - u_n\| + \gamma_n\|x_n - p\|^2 + c_n \\ &\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, from (3.15) and the existence of $\lim_{n\to\infty}\|x_n - p\|$ we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.17}$$

Combining (3.6), (3.8) and (3.10), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|k_n - p\|^2 + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\| \\ &\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\| \\ &\le \big\|\Lambda_n^i u_n - p\big\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\| \\ &\le \|x_n - p\|^2 + \lambda_{i,n}(\lambda_{i,n} - 2\eta_i)\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 \\ &\quad + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\|,\end{aligned}$$

where $i \in \{1, 2, \ldots, N\}$, which implies

$$\begin{aligned}&\lambda_{i,n}(2\eta_i - \lambda_{i,n})\big\|B_i\Lambda_n^{i-1}u_n - B_ip\big\|^2 \\ &\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \\ &\quad\quad + 2\alpha_n\big\|u + \gamma f(x_n) - \bar{V}W_nGk_n\big\|\|x_{n+1} - p\|.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, from $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1,2,\ldots,N\}$ and the existence of $\lim_{n\to\infty}\|x_n - p\|$ we obtain

$$\lim_{n\to\infty}\left\|B_i\Lambda_n^{i-1}u_n - B_ip\right\| = 0, \quad i \in \{1,2,\ldots,N\}. \tag{3.18}$$

Combining (3.7), (3.8) and (3.10), we get

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \|k_n - p\|^2 + 2\alpha_n\left\|u + \gamma f(x_n) - \bar{V}W_nGk_n\right\|\|x_{n+1} - p\| \\
&\quad \le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n + 2\alpha_n\left\|u + \gamma f(x_n) - \bar{V}W_nGk_n\right\|\|x_{n+1} - p\| \\
&\quad \le \left\|\Lambda_n^iu_n - p\right\|^2 + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\left\|u + \gamma f(x_n) - \bar{V}W_nGk_n\right\|\|x_{n+1} - p\| \\
&\quad \le \|x_n - p\|^2 - \left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\right\|^2 + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_ip\right\| \\
&\qquad + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\left\|u + \gamma f(x_n) - \bar{V}W_nGk_n\right\|\|x_{n+1} - p\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\right\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\right\|\left\|B_i\Lambda_n^{i-1}u_n - B_ip\right\| \\
&\qquad + \gamma_n\|x_n - p\|^2 + c_n + 2\alpha_n\left\|u + \gamma f(x_n) - \bar{V}W_nGk_n\right\|\|x_{n+1} - p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$ and $\lim_{n\to\infty}c_n = 0$, from (3.18) and the existence of $\lim_{n\to\infty}\|x_n - p\|$ we obtain

$$\lim_{n\to\infty}\left\|\Lambda_n^{i-1}u_n - \Lambda_n^iu_n\right\| = 0, \quad i \in \{1,2,\ldots,N\}. \tag{3.19}$$

By (3.19), we have

$$\begin{aligned}
\|u_n - z_n\| &= \left\|\Lambda_n^0u_n - \Lambda_n^Nu_n\right\| \\
&\le \left\|\Lambda_n^0u_n - \Lambda_n^1u_n\right\| + \left\|\Lambda_n^1u_n - \Lambda_n^2u_n\right\| + \cdots + \left\|\Lambda_n^{N-1}u_n - \Lambda_n^Nu_n\right\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.20}$$

From (3.17) and (3.20), we have

$$\begin{aligned}
\|x_n - z_n\| &\le \|x_n - u_n\| + \|u_n - z_n\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.21}$$

By (3.14) and (3.21), we obtain

$$\begin{aligned}
\|k_n - x_n\| &\le \|k_n - z_n\| + \|z_n - x_n\| \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.22}$$

which together with (3.12) and (3.22) implies that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_{n+1}-k_n\| + \|k_n - x_n\| \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{3.23}$$

On the other hand, we observe that

$$\|z_{n+1}-z_n\| \le \|z_{n+1}-x_{n+1}\| + \|x_{n+1}-x_n\| + \|x_n - z_n\|.$$

By (3.21) and (3.23), we have

$$\lim_{n\to\infty} \|z_{n+1}-z_n\| = 0. \tag{3.24}$$

We note that

$$\|z_n - Sz_n\| \le \|z_n - z_{n+1}\| + \left\|z_{n+1} - S^{n+1}z_{n+1}\right\| + \left\|S^{n+1}z_{n+1} - S^{n+1}z_n\right\| + \left\|S^{n+1}z_n - Sz_n\right\|.$$

From (3.13), (3.24), Lemma 2.5 and the uniform continuity of S, we obtain

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{3.25}$$

On the other hand, for simplicity, we write $\tilde{p} = T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p$, $\nu_n = T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)k_n$ and $\tilde{\nu}_n = Gk_n = T_{\nu_1}^{\Theta_1}(I-\nu_1 A_1)\nu_n$ for all $n\ge 1$. Then

$$p = Gp = T_{\nu_1}^{\Theta_1}(I-\nu_1 A_1)\tilde{p} = T_{\nu_1}^{\Theta_1}(I-\nu_1 A_1)T_{\nu_2}^{\Theta_2}(I-\nu_2 A_2)p.$$

We now show that $\lim_{n\to\infty}\|Gk_n - k_n\| = 0$, *i.e.*, $\lim_{n\to\infty}\|\tilde{\nu}_n - k_n\| = 0$. As a matter of fact, utilizing the arguments similar to those of (3.29) in the proof of [16, Theorem 3.1], we deduce from (3.1)-(3.5) and (3.8) that for $p\in\Omega$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \left\|\alpha_n\gamma\big(f(x_n)-f(p)\big) + \beta_n(k_n-p) + \big((1-\beta_n)I - \alpha_n\bar{V}\big)(W_nGk_n - p)\right\|^2 \\
&\qquad + 2\alpha_n\big\langle u + (\gamma f - \bar{V})p, x_{n+1}-p\big\rangle \\
&\quad\le \alpha_n(1+\mu)\bar{\gamma}\|x_n-p\|^2 + \beta_n\|k_n-p\|^2 + \big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big)\|\tilde{\nu}_n - p\|^2 \\
&\qquad + 2\alpha_n\left\|u+(\gamma f-\bar{V})p\right\|\|y_n-p\| \\
&\qquad + \big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big) \\
&\qquad \times \left[\nu_2(\nu_2-2\zeta_2)\|A_2k_n - A_2p\|^2 + \nu_1(\nu_1-2\zeta_1)\|A_1\nu_n - A_1\tilde{p}\|^2\right],
\end{aligned} \tag{3.26}$$

which immediately leads to

$$\begin{aligned}
&\big(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\big)\left[\nu_2(2\zeta_2-\nu_2)\|A_2k_n-A_2p\|^2 + \nu_1(2\zeta_1-\nu_1)\|A_1\nu_n - A_1\tilde{p}\|^2\right] \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \gamma_n\|x_n-p\|^2 + c_n + 2\alpha_n\left\|u+(\gamma f-\bar{V})p\right\|\|x_{n+1}-p\| \\
&\quad\le \|x_n - x_{n+1}\|\big(\|x_n-p\| + \|x_{n+1}-p\|\big) + \gamma_n\|x_n-p\|^2 + c_n \\
&\qquad + 2\alpha_n\left\|u+(\gamma f-\bar{V})p\right\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$, we conclude from (3.23) and condition (iv) that

$$\lim_{n\to\infty}\|A_2k_n - A_2p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|A_1v_n - A_1\tilde{p}\| = 0. \tag{3.27}$$

Utilizing the arguments similar to those of (3.31) and (3.32) in the proof of [16, Theorem 3.1], we can obtain

$$\begin{aligned}\|v_n - \tilde{p}\|^2 &\le \|k_n - p\|^2 - \big\|(k_n - v_n) - (p - \tilde{p})\big\|^2 \\ &\quad + 2\nu_2\big\langle(k_n - v_n) - (p - \tilde{p}), A_2k_n - A_2p\big\rangle - \nu_2^2\|A_2k_n - A_2p\|^2,\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\|\tilde{v}_n - p\|^2 &\le \|k_n - p\|^2 - \big\|(v_n - \tilde{v}_n) + (p - \tilde{p})\big\|^2 \\ &\quad + 2\nu_1\|A_1v_n - A_1\tilde{p}\|\big\|(v_n - \tilde{v}_n) + (p - \tilde{p})\big\|.\end{aligned} \tag{3.29}$$

Consequently, from (3.3), (3.4), (3.8), (3.26) and (3.28) it follows that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|\tilde{v}_n - p\|^2 \\ &\qquad + 2\alpha_n\big\|u + (\gamma f - \bar{V})p\big\|\|x_{n+1} - p\| \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\|v_n - \tilde{p}\|^2 \\ &\qquad + 2\alpha_n\big\|u + (\gamma f - \bar{V})p\big\|\|x_{n+1} - p\| \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \beta_n\|k_n - p\|^2 \\ &\qquad + \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big) \\ &\qquad \times \big[\|k_n - p\|^2 - \big\|(k_n - v_n) - (p - \tilde{p})\big\|^2 + 2\nu_2\big\|(k_n - v_n) - (p - \tilde{p})\big\|\|A_2k_n - A_2p\|\big] \\ &\qquad + 2\alpha_n\big\|u + (\gamma f - \bar{V})p\big\|\|x_{n+1} - p\| \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \big(1 - \alpha_n(1+\mu)\bar{\gamma}\big)\|k_n - p\|^2 \\ &\qquad - \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\big\|(k_n - v_n) - (p - \tilde{p})\big\|^2 \\ &\qquad + 2\nu_2\big\|(k_n - v_n) - (p - \tilde{p})\big\|\|A_2k_n - A_2p\| \\ &\qquad + 2\alpha_n\big\|u + (\gamma f - \bar{V})p\big\|\|x_{n+1} - p\| \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \big(1 - \alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|z_n - p\|^2 + c_n\big) \\ &\qquad - \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\big\|(k_n - v_n) - (p - \tilde{p})\big\|^2 \\ &\qquad + 2\nu_2\big\|(k_n - v_n) - (p - \tilde{p})\big\|\|A_2k_n - A_2p\| \\ &\qquad + 2\alpha_n\big\|u + (\gamma f - \bar{V})p\big\|\|x_{n+1} - p\| \\ &\quad \le \alpha_n(1+\mu)\bar{\gamma}\|x_n - p\|^2 + \big(1 - \alpha_n(1+\mu)\bar{\gamma}\big)\big((1+\gamma_n)\|x_n - p\|^2 + c_n\big) \\ &\qquad - \big(1 - \beta_n - \alpha_n(1+\mu)\bar{\gamma}\big)\big\|(k_n - v_n) - (p - \tilde{p})\big\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 2\nu_2 \big\| (k_n - v_n) - (p - \tilde{p}) \big\| \|A_2 k_n - A_2 p\| \\
&+ 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
\le{}& (1 + \gamma_n)\|x_n - p\|^2 + c_n - \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \big\| (k_n - v_n) - (p - \tilde{p}) \big\|^2 \\
&+ 2\nu_2 \big\| (k_n - v_n) - (p - \tilde{p}) \big\| \|A_2 k_n - A_2 p\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\|,
\end{aligned}
$$

which hence leads to

$$
\begin{aligned}
&\big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \big\| (k_n - v_n) - (p - \tilde{p}) \big\|^2 \\
&\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n \|x_n - p\|^2 + c_n \\
&\qquad + 2\nu_2 \big\| (k_n - v_n) - (p - \tilde{p}) \big\| \|A_2 k_n - A_2 p\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
&\quad\le \|x_n - x_{n+1}\| \big(\|x_n - p\| + \|x_{n+1} - p\|\big) + \gamma_n \|x_n - p\|^2 + c_n \\
&\qquad + 2\nu_2 \big\| (k_n - v_n) - (p - \tilde{p}) \big\| \|A_2 k_n - A_2 p\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$, $\lim_{n\to\infty} c_n = 0$ and $\limsup_{n\to\infty} \beta_n < 1$, from (3.23) and (3.27) we have

$$
\lim_{n\to\infty} \big\| (k_n - v_n) - (p - \tilde{p}) \big\| = 0. \tag{3.30}
$$

Furthermore, from (3.3), (3.4), (3.8), (3.26) and (3.29) it follows that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\le \alpha_n(1 + \mu)\bar{\gamma} \|x_n - p\|^2 + \beta_n \|k_n - p\|^2 + \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \|\tilde{v}_n - p\|^2 \\
&\qquad + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
&\quad\le \alpha_n(1 + \mu)\bar{\gamma} \|x_n - p\|^2 + \beta_n \|k_n - p\|^2 \\
&\qquad + \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \\
&\qquad \times \big[\|k_n - p\|^2 - \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\|^2 + 2\nu_1 \|A_1 v_n - A_1 \tilde{p}\| \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\|\big] \\
&\qquad + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
&\quad\le \alpha_n(1 + \mu)\bar{\gamma} \|x_n - p\|^2 + \big(1 - \alpha_n(1 + \mu)\bar{\gamma}\big) \|k_n - p\|^2 \\
&\qquad - \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\|^2 \\
&\qquad + 2\nu_1 \|A_1 v_n - A_1 \tilde{p}\| \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
&\quad\le \alpha_n(1 + \mu)\bar{\gamma} \|x_n - p\|^2 + \big(1 - \alpha_n(1 + \mu)\bar{\gamma}\big) \big((1 + \gamma_n)\|z_n - p\|^2 + c_n\big) \\
&\qquad - \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\|^2 \\
&\qquad + 2\nu_1 \|A_1 v_n - A_1 \tilde{p}\| \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\| \\
&\quad\le \alpha_n(1 + \mu)\bar{\gamma} \|x_n - p\|^2 + \big(1 - \alpha_n(1 + \mu)\bar{\gamma}\big) \big((1 + \gamma_n)\|x_n - p\|^2 + c_n\big) \\
&\qquad - \big(1 - \beta_n - \alpha_n(1 + \mu)\bar{\gamma}\big) \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\|^2 \\
&\qquad + 2\nu_1 \|A_1 v_n - A_1 \tilde{p}\| \big\| (v_n - \tilde{v}_n) + (p - \tilde{p}) \big\| + 2\alpha_n \big\| u + (\gamma f - \bar{V})p \big\| \|x_{n+1} - p\|
\end{aligned}
$$

$$\begin{aligned}
&\le (1+\gamma_n)\|x_n-p\|^2 + c_n - \bigl(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\bigr)\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\|^2 \\
&\quad + 2v_1\|A_1v_n - A_1\tilde{p}\|\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\| + 2\alpha_n\bigl\|u+(\gamma f-\bar{V})p\bigr\|\|x_{n+1}-p\|,
\end{aligned}$$

which hence yields

$$\begin{aligned}
&\bigl(1-\beta_n-\alpha_n(1+\mu)\bar{\gamma}\bigr)\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\|^2 \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \gamma_n\|x_n-p\|^2 + c_n \\
&\qquad + 2v_1\|A_1v_n - A_1\tilde{p}\|\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\| + 2\alpha_n\bigl\|u+(\gamma f-\bar{V})p\bigr\|\|x_{n+1}-p\| \\
&\quad\le \|x_n-x_{n+1}\|\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr) + \gamma_n\|x_n-p\|^2 + c_n \\
&\qquad + 2v_1\|A_1v_n - A_1\tilde{p}\|\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\| + 2\alpha_n\bigl\|u+(\gamma f-\bar{V})p\bigr\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\gamma_n = 0$, $\lim_{n\to\infty}c_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$, from (3.23) and (3.27) we have

$$\lim_{n\to\infty}\bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\| = 0. \tag{3.31}$$

Note that

$$\|k_n-\tilde{v}_n\| \le \bigl\|(k_n-v_n)-(p-\tilde{p})\bigr\| + \bigl\|(v_n-\tilde{v}_n)+(p-\tilde{p})\bigr\|.$$

Hence from (3.30) and (3.31) we get

$$\lim_{n\to\infty}\|k_n-\tilde{v}_n\| = \lim_{n\to\infty}\|k_n-Gk_n\| = 0, \tag{3.32}$$

which together with (3.11) and (3.32) implies that

$$\begin{aligned}
\|k_n-W_nk_n\| &\le \|k_n-W_nGk_n\| + \|W_nGk_n-W_nk_n\| \\
&\le \|k_n-W_nGk_n\| + \|Gk_n-k_n\| \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.33}$$

Also, observe that

$$\|k_n-Wk_n\| \le \|k_n-W_nk_n\| + \|W_nk_n-Wk_n\|.$$

From (3.33), [17, Remark 2.3] and the boundedness of $\{k_n\}$ we immediately obtain

$$\lim_{n\to\infty}\|k_n-Wk_n\| = 0. \tag{3.34}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From (3.21) and (3.22), we have that $z_{n_i}\rightharpoonup w$ and $k_{n_i}\rightharpoonup w$. From (3.17), (3.19), (3.21), we have that $u_{n_i}\rightharpoonup w$, $\Lambda_{n_i}^m u_{n_i}\rightharpoonup w$, $z_{n_i}\rightharpoonup w$ and $k_{n_i}\rightharpoonup w$, where $m\in\{1,2,\ldots,N\}$. Since S is uniformly continuous, by (3.25) we get $\lim_{n\to\infty}\|z_n-S^mz_n\| = 0$ for any $m\ge 1$. Hence from Lemma 2.4 we obtain $w\in\operatorname{Fix}(S)$. In the meantime, utilizing Lemma 2.4, we deduce from $k_{n_i} \rightharpoonup w$, (3.32) and (3.34) that $w \in \mathrm{SGEP}(G)$ and $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ (due to Lemma 2.3). Utilizing similar arguments to those in the proof of [16, Theorem 3.1], we can derive $w \in \mathrm{GMEP}(\Theta, \varphi, A) \cap \bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i)$. Consequently, $w \in \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Next let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then we get $w' \in \Omega$. If $w \neq w'$, from the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - w\| &= \lim_{i\to\infty} \|x_{n_i} - w\| < \lim_{i\to\infty} \|x_{n_i} - w'\| \\ &= \lim_{n\to\infty} \|x_n - w'\| = \lim_{j\to\infty} \|x_{n_j} - w'\| \\ &< \lim_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.\end{aligned}$$

This attains a contradiction. So we have $w = w'$. Put $w_n = P_\Omega x_n$. Since $w \in \Omega$, we have $\langle x_n - w_n, w_n - w\rangle \geq 0$. By Lemma 2.7, we have that $\{w_n\}$ converges strongly to some $\tilde{w} \in \Omega$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - \tilde{w}, \tilde{w} - w\rangle \geq 0.$$

Therefore we obtain $w = \tilde{w} = \lim_{n\to\infty} P_\Omega x_n$. This completes the proof. □

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let Θ, Θ_1, Θ_2 be three bifunctions from $C \times C$ to $\mathbf{R}$ satisfying* (H1)-(H4) *and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex functional. Let $R_i : C \to 2^H$ be a maximal monotone mapping, and let $A, A_k : H \to H$ and $B_i : C \to H$ be ζ-inverse strongly monotone, ζ_k-inverse strongly monotone and η_i-inverse strongly monotone, respectively, for $k = 1, 2$ and $i = 1, 2$. Let $S : C \to C$ be a uniformly continuous asymptotically k-strict pseudocontractive mapping in the intermediate sense for some $0 \leq k < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{c_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings on H and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let V be a $\bar{\gamma}$-strongly positive bounded linear operator and $f : H \to H$ be an l-Lipschitzian mapping with $\gamma l < (1 + \mu)\bar{\gamma}$. Let W_n be the W-mapping defined by* (1.4). *Assume that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \mathrm{I}(B_2, R_2) \cap \mathrm{I}(B_1, R_1) \cap \mathrm{Fix}(S)$ is nonempty, where G is defined as in Proposition* 1.1. *Let $\{r_n\}$ be a sequence in $[0, 2\zeta]$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $0 < k + \epsilon \leq \delta_n \leq d < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2)J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n(I + \mu V)]W_n G k_n, \quad \forall n \geq 1. \end{cases} \tag{3.35}$$

Assume that the following conditions are satisfied:

(i) *$K : H \to \mathbf{R}$ is strongly convex with constant $\sigma > 0$ and its derivative K' is Lipschitz continuous with constant $\nu > 0$ such that the function $x \mapsto \langle y - x, K'(x)\rangle$ is weakly upper semicontinuous for each $y \in H$*;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta$;

(iv) $\nu_k \in (0, 2\zeta_k)$ *and* $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ *for* $k = 1, 2$ *and* $i = 1, 2$.

If $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, then* $\{x_n\}$ *converges weakly to* $w = \lim_{n\to\infty} P_\Omega x_n$.

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* Θ, Θ_1, Θ_2 *be three bifunctions from* $C \times C$ *to* $\mathbf{R}$ *satisfying* (H1)-(H4) *and* $\varphi : C \to \mathbf{R}$ *be a lower semicontinuous and convex functional. Let* $R : C \to 2^H$ *be a maximal monotone mapping, and let* $A, A_k : H \to H$ *and* $B : C \to H$ *be* ζ*-inverse strongly monotone,* ζ_k*-inverse strongly monotone and* η*-inverse strongly monotone, respectively, for* $k = 1, 2$. *Let* $S : C \to C$ *be a uniformly continuous asymptotically* k*-strict pseudocontractive mapping in the intermediate sense for some* $0 \le k < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$ *such that* $\sum_{n=1}^\infty \gamma_n < \infty$ *and* $\{c_n\} \subset [0, \infty)$ *such that* $\sum_{n=1}^\infty c_n < \infty$. *Let* V *be a* $\bar{\gamma}$*-strongly positive bounded linear operator and* $f : H \to H$ *be an* l*-Lipschitzian mapping with* $\gamma l < (1 + \mu)\bar{\gamma}$. *Assume that* $\Omega := \mathrm{GMEP}(\Theta, \varphi, A) \cap \mathrm{SGEP}(G) \cap \mathrm{I}(B, R) \cap \mathrm{Fix}(S)$ *is nonempty, where* G *is defined as in Proposition* 1.1. *Let* $\{r_n\}$ *be a sequence in* $[0, 2\zeta]$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0, 1)$ *such that* $\sum_{n=1}^\infty \alpha_n < \infty$ *and* $0 < k + \epsilon \le \delta_n \le d < 1$. *Pick any* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by the following algorithm*:

$$\begin{cases} u_n = S_{r_n}^{(\Theta,\varphi)}(I - r_n A)x_n, \\ z_n = J_{R,\rho_n}(I - \rho_n B)u_n, \\ k_n = \delta_n z_n + (1 - \delta_n)S^n z_n, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n k_n + [(1 - \beta_n)I - \alpha_n(I + \mu V)]Gk_n, \quad \forall n \ge 1. \end{cases} \tag{3.36}$$

Assume that the following conditions are satisfied:

(i) $K : H \to \mathbf{R}$ *is strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that the function* $x \mapsto \langle y - x, K'(x)\rangle$ *is weakly upper semicontinuous for each* $y \in H$;

(ii) *for each* $x \in H$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \notin D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), z_x - y\rangle < 0;$$

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\zeta$;

(iv) $\nu_k \in (0, 2\zeta_k)$ *and* $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$ *for* $k = 1, 2$.

If $S_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, then* $\{x_n\}$ *converges weakly to* $w = \lim_{n\to\infty} P_\Omega x_n$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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