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Nonlinear quasi-contractions in non-normal cone metric spaces

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Abstract

In the paper, we prove a new fixed point theorem of nonlinear quasi-contractions in non-normal cone metric spaces, which partially improve the recent results of Arandelović and Kečkić's and of Li and Jiang since some of the essential conditions therein are removed. A suitable example is presented to show the usability of our theorem. It is worth mentioning that the results in this paper could not be derived from the corresponding results in the setting of metric spaces by using a scalarization function or a Minkowski functional.

MSC: 06A07; 47H10

Keywords: nonlinear quasi-contraction; non-normal cone metric space

1 Introduction

In 2007, Huang and Zhang [1] introduced the concept of cone metric spaces, as a generalization of metric spaces, and gave the version of the Banach contraction principle and other basic theorems in the setting of cone metric spaces. Later on, by omitting the assumption of normality of the cone, Rezapour and Hamlbarani [2] improved the relevant results of [1], and presented a number of examples to support the existence of non-normal cones, which shows that such generalizations are meaningful. Since then, many authors have been interested in the study of fixed point results in non-normal cone metric spaces; see [3–17]. In the preceding references except [3–5, 16, 17], the involving contractions are always assumed to be restricted with a constant.

There are some references concerned with the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings in cone metric spaces; see [7–10]. Actually, it has been shown that each cone metric space (X,d) is equivalent to a usual metric space (X,d_e), where the real-valued metric function d_e is defined by a nonlinear scalarization function [8] or by a Minkowski functional [9]. Besides, it has been pointed out in [18] that many fixed point generalizations obtained in cone metric spaces are not real generalizations, and the authors should take care in obtaining real fixed point generalizations in cone metric spaces.

In 1974, Ćirić [11] introduced Ćirić's quasi-contractions in metric spaces as one of the most general classes of contractive-type mappings, and proved the well-known theorem that every Ćirić's quasi-contraction T has a unique fixed point, which was then generalized to cone metric spaces by [13–15]. There were many works concerned with the fixed point results of contractions or quasi-contractions restricted with nonlinear comparison functions, we refer the readers to [19–25]. Recently, Arandelović and Kečkić [16] considered nonlinear quasi-contractions in cone metric spaces, and by using the nonlinear scalarization method of Du [8], they obtained several fixed point theorems of nonlinear quasi-contractions and quasi-contractions restricted with linear contractive bounded mappings in cone metric spaces over locally convex Hausdorff topological vector spaces with the assumption that $(I-A)(\mathrm{int}P) \subset \mathrm{int}P$. Very recently, Li and Jiang [17] removed the contractive condition of linear bounded mappings appearing in [16], and they proved a fixed point result of quasi-contractions restricted with linear bounded mappings in non-normal cone metric spaces at the expense of

$$u_n \overset{w}{\to} \theta \quad \Rightarrow \quad Au_n \overset{w}{\to} \theta, \quad \forall\{u_n\} \subset P. \tag{H}$$

In this paper, we first show that every nondecreasing mapping $A : P \to P$ satisfies the condition (H) provided that it is continuous at θ and $A\theta = \theta$ (see Lemma 3), and consequently, the condition (H) in [17] is superfluous and could be omitted; see Remark 1. Then by using Lemma 3, we prove a new fixed point theorems of nonlinear quasi-contractions in non-normal cone metric spaces, which improved the relevant results of [16, 17] since the conditions $(I-A)(\mathrm{int}P) \subset \mathrm{int}P$ and (H) are removed. In addition, a suitable example is presented to show the usability of our theorem.

It is worth mentioning that the results in this paper could not be derived from the corresponding results in the setting of metric spaces by the methods of [8, 9] and also cannot be obtained by any existing fixed point results in cone metric spaces. Hence the results in this paper are real generalizations.

2 Preliminaries

Let $(E, \|\cdot\|)$ be a normed vector space. A cone of E is a nonempty closed subset P of E such that $ax + by \in P$ for each $x, y \in P$ and each $a, b \geq 0$, and $P \cap (-P) = \{\theta\}$, where θ is the zero element of E. A cone P of E determines a partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for each $x, y \in X$. In this case E is called an ordered normed vector space.

A cone P of a normed vector space E is solid if $\mathrm{int}P \neq \emptyset$, where $\mathrm{int}P$ is the interior of P. For each $x, y \in E$ with $y - x \in \mathrm{int}P$, we write $x \ll y$. Let P be a solid cone of a normed vector space E. A sequence $\{u_n\}$ of E weakly converges [5] to $u \in E$ (denote $u_n \overset{w}{\to} u$) if for each $\epsilon \in \mathrm{int}P$, there exists a positive integer n_0 such that $u - \epsilon \ll u_n \ll u + \epsilon$ for all $n \geq n_0$.

A cone P of E is normal if the unit ball is order-convex, which is equivalent to the condition that there is some positive number N such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leq N\|y\|$, and the minimal N is called a normal constant of P. Another equivalent condition is that

$$\inf\bigl\{\|x+y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\bigr\} > 0.$$

Then it is not hard to conclude that P is non-normal if and only if there exists a sequence $\{u_n\}, \{v_n\} \subset P$ such that

$$u_n + v_n \overset{\|\cdot\|}{\to} \theta \quad \not\Rightarrow \quad u_n \overset{\|\cdot\|}{\to} \theta,$$

which implies that the sandwich theorem does not hold in the case that P is non-normal. However, in the sense of weak convergence, the sandwich theorem still holds even if P is non-normal, and we have the following lemma.

Lemma 1 (see [6, 17]) *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $\{u_n\}, \{v_n\}, \{z_n\} \subset E$. If*

$$u_n \preceq z_n \preceq v_n, \quad \forall n,$$

and there exists some $z \in E$ such that $u_n \xrightarrow{w} z$ and $v_n \xrightarrow{w} z$, then $z_n \xrightarrow{w} z$.

Lemma 2 (see [5]) *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$. Then for each sequence $\{u_n\} \subset E$, $u_n \xrightarrow{\|\cdot\|} u$ implies $u_n \xrightarrow{w} u$.*

Lemma 3 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $A : P \to P$ a nondecreasing mapping. If A is continuous at θ and $A\theta = \theta$, then it satisfies* (H).

Proof Let $\{u_n\}$ be a sequence of P such that $u_n \xrightarrow{w} \theta$. It suffices to show $Au_n \xrightarrow{w} \theta$.

Fix $\epsilon \in \operatorname{int}P$. It is clear that $\frac{\epsilon}{m} \in \operatorname{int}P$ for each m. From $u_n \xrightarrow{w} u$ we find that, for each m, there exists n_m such that $u_n \ll \frac{\epsilon}{m}$ for each $n \geq n_m$. Since A is nondecreasing, $Au_n \preceq A(\frac{\epsilon}{m})$ for each $n \geq n_m$. Note that $\frac{\epsilon}{m} \xrightarrow{\|\cdot\|} \theta$ $(m \to \infty)$, then $A(\frac{\epsilon}{m}) \xrightarrow{\|\cdot\|} \theta$ $(m \to \infty)$ since A is continuous at θ and $A\theta = \theta$. Hence by Lemma 2, $A(\frac{\epsilon}{m}) \xrightarrow{w} \theta$ $(m \to \infty)$, which implies that, for each $c \in \operatorname{int}P$, there exists m_0 such that $A(\frac{\epsilon}{m}) \ll c$ for each $m \geq m_0$. Therefore we have $Au_n \ll c$ for each $n \geq u_{m_0}$, *i.e.*, $Au_n \xrightarrow{w} \theta$ $(n \to \infty)$. The proof is complete. □

Remark 1 Every linear bounded mapping $A : P \to P$ is certainly nondecreasing and continuous at θ, and hence it satisfies the condition (H) by Lemma 3. Therefore in Theorem 1 of [17], the condition (H) is superfluous and could be omitted.

Let X be a nonempty set and P be a cone of a topological vector space E. A cone metric on X is a mapping $d : X \times X \to P$ such that, for each $x, y, z \in X$,

(d1) $d(x,y) = \theta \Longleftrightarrow x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \preceq d(x,z) + d(z,y)$.

The pair (X,d) is called a cone metric space over P. A cone metric d on X over a solid cone P generates a topology τ_d on X which has a base of the family of open d-balls $\{B_d(x,\epsilon) : x \in X, \theta \ll \epsilon\}$, where $B_d(x,\epsilon) = \{y \in X : d(x,y) \ll \epsilon\}$ for each $x \in X$ and each $\epsilon \in \operatorname{int}P$.

Let (X,d) be a cone metric space over a solid cone P of a normed vector space E. A sequence $\{x_n\}$ of X converges [1, 5] to $x \in X$ (denote by $x_n \xrightarrow{\tau_d} x$) if $d(x_n,x) \xrightarrow{w} \theta$. A sequence $\{x_n\}$ of X is Cauchy [1, 5], if $d(x_n,x_m) \xrightarrow{w} \theta$. The cone metric space (X,d) is complete [1, 5], if each Cauchy sequence $\{x_n\}$ of X converges to a point $x \in X$.

3 Main results

Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$. A mapping $T : X \to X$ is called a quasi-contraction, if there exists a mapping $A : P \to P$ such that

$$d(Tx, Ty) \preceq Au, \quad \forall x, y \in X, \tag{1}$$

where $u \in \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. In particular when A is a linear bounded mapping, T is reduced to the one considered in [17].

Some slight modifications of the proof of [17, Theorem 1] yield the following result.

Theorem 1 *Let (X,d) be a complete cone metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$ and $T : X \to X$ a quasi-contraction. Assume that $A : P \to P$ is a nondecreasing and subadditive (i.e., $A(u+v) \preceq Au + Av$ for each $u, v \in P$) mapping with $A\theta = \theta$ such that*

$$\sum_{i=0}^{\infty} \|A^i u\| < \infty, \quad \forall u \in P. \tag{2}$$

If A and B are continuous at θ, where $Bu = \sum_{i=0}^{\infty} A^i u$ for each $u \in P$. Then T has a unique fixed point $x^* \in X$, and for each $x_0 \in X$, the Picard iterative sequence $\{x_n\}$ converges to x^*, where $x_n = T^n x_0$ for each n.

Remark 2 In particular when $A : P \to P$ is a linear bounded mapping with the spectra radius $r(A) < 1$, then (2) is naturally satisfied and B is continuous on P since $B = (I - A)^{-1}$ and $(I - A)^{-1} : P \to P$ is a linear bounded mapping, where $(I - A)^{-1}$ is the inverse of $I - A$.

The following example shows that there exists some nonlinear mapping $A : P \to P$ such that (2) is satisfied and B is continuous at θ.

Example 1 Let $E = C^1_{\mathbb{R}}[0,1]$ be endowed with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and $P = \{u \in E : u(t) \geq 0, \forall t \in [0,1]\}$ which is a non-normal cone [26]. Let $(Au)(t) = a \int_0^t u^2 \, ds$ for each $u \in P$ and each $t \in [0,1]$, where $a > 0$.

For each $u \in P$, we have $(A^n u)(t) \leq \frac{(at)^n}{n!} \|u\|_\infty^{\frac{1}{2}} \leq \frac{a^n}{n!} \|u\|^{\frac{1}{2}}$ for each $t \in [0,1]$ and each $n \geq 1$, and so $\|A^n u\|_\infty \leq \frac{a^n}{n!} \|u\|^{\frac{1}{2}}$ for each $n \geq 1$. Note that $(A^n u)'(t) = a(A^{n-1}u)^{\frac{1}{2}}(t)$ for each $u \in P$ and each $t \in [0,1]$, then $\|(A^n u)'\|_\infty \leq \frac{a^{\frac{n+1}{2}}}{\sqrt{(n-1)!}} \|u\|^{\frac{1}{4}}$ for each $u \in P$ and $n \geq 2$. Thus for each $u \in P$, we have

$$\|A^n u\| = \|A^n u\|_\infty + \|(A^n u)'\|_\infty \leq \frac{a^n}{n!} \|u\|^{\frac{1}{2}} + \frac{a^{\frac{n+1}{2}}}{\sqrt{(n-1)!}} \|u\|^{\frac{1}{4}}, \quad \forall n \geq 2,$$

and so

$$\sum_{i=0}^{\infty} \|A^i u\| \leq \|u\| + 2a\|u\|^{\frac{1}{2}} + \left(\sum_{i=2}^{\infty} \frac{a^i}{i!}\right) \|u\|^{\frac{1}{2}} + \left(\sum_{i=2}^{\infty} \frac{a^{\frac{i+2}{2}}}{\sqrt{i!}}\right) \|u\|^{\frac{1}{4}},$$

which implies that (2) is satisfied since the series $\sum_{i=2}^{\infty} \frac{a^i}{i!}$ and $\sum_{i=2}^{\infty} \frac{a^{\frac{i+2}{2}}}{\sqrt{i!}}$ are convergent.

Note that $B\theta = \theta$, then for each $u \in P$ we have

$$\begin{aligned} \|Bu - B\theta\| = \|Bu\| &\leq \sum_{i=0}^{\infty} \|A^i u\| \\ &\leq \|u\| + 2a\|u\|^{\frac{1}{2}} + \left(\sum_{i=2}^{\infty} \frac{a^i}{i!}\right) \|u\|^{\frac{1}{2}} + \left(\sum_{i=2}^{\infty} \frac{a^{\frac{i+2}{2}}}{\sqrt{i!}}\right) \|u\|^{\frac{1}{4}}, \end{aligned}$$

which implies that B is continuous at θ.

Proof of Theorem 1 It follows from (2) that the mapping B is well defined. Clearly, $B(P) \subset P$ and $B\theta = \theta$ since $A(P) \subset (P)$ and $A\theta = \theta$. By (2), we get

$$\lim_{n\to\infty} \|A^n u\| = 0, \quad \forall u \in P. \tag{3}$$

Since A and B are commutative,

$$BA = AB, \qquad B(I - A) = (I - A)B = B - AB = I. \tag{4}$$

We claim that, for all $n \geq 1$,

$$d(x_i, x_j) \preceq BAd(x_0, x_1), \quad \forall 1 \leq i, j \leq n. \tag{5}$$

In the following we shall show this claim by induction.

If $n = 1$, then $i = j = 1$, and so the claim is trivial.

Assume that (5) holds for n. To prove (5) holds for $n + 1$, it suffices to show

$$d(x_{i_0}, x_{n+1}) \preceq BAd(x_0, x_1), \quad \forall 1 \leq i_0 \leq n. \tag{6}$$

By (1),

$$d(x_{i_0}, x_{n+1}) \preceq Au, \tag{7}$$

where

$$u \in \bigl\{d(x_{i_0-1}, x_n), d(x_{i_0-1}, x_{i_0}), d(x_n, x_{n+1}), d(x_{i_0-1}, x_{n+1}), d(x_n, x_{i_0})\bigr\}.$$

Consider the case that $i_0 = 1$.

If $u = d(x_0, x_n)$, then by the triangle inequality, the nondecreasing property of A, (5), and (7),

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq Ad(x_0, x_n) \preceq A\bigl[d(x_0, x_1) + d(x_1, x_n)\bigr] \\ &\preceq A\bigl[d(x_0, x_1) + BAd(x_0, x_1)\bigr] = A(I + BA)d(x_0, x_1) \\ &= A\left(I + \sum_{i=1}^{\infty} A^i\right) d(x_0, x_1) = BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_0, x_1)$, then by (7) and $A(P) \subset P$,

$$d(x_{i_0}, x_{n+1}) \preceq Ad(x_0, x_1) \preceq \left(\sum_{i=1}^{\infty} A^i\right) d(x_0, x_1) = BAd(x_0, x_1),$$

i.e., (6) holds.

If $u = d(x_0, x_{n+1})$, then by the triangle inequality, the nondecreasing property and subadditivity of A, (7), and $A(P) \subset P$,

$$d(x_{i_0}, x_{n+1}) \preceq Ad(x_0, x_{n+1}) \preceq A\bigl[d(x_0, x_1) + d(x_{i_0}, x_{n+1})\bigr] \preceq Ad(x_0, x_1) + Ad(x_{i_0}, x_{n+1}),$$

which implies that

$$(I - A)d(x_{i_0}, x_{n+1}) \preceq Ad(x_0, x_1).$$

Act on the above inequality with B, then by (4) and $B(P) \subset P$,

$$d(x_{i_0}, x_{n+1}) \preceq BAd(x_0, x_1),$$

i.e., (6) holds.

If $u = d(x_n, x_{i_0})$, then by (5), (7), and $A(P) \subset P$,

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq Ad(x_{i_0}, x_n) \preceq A^2Bd(x_0, x_1) \\ &= \left(\sum_{i=2}^{\infty} A^i\right) d(x_0, x_1) \preceq \left(\sum_{i=1}^{\infty} A^i\right) d(x_0, x_1) \\ &= BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_n, x_{n+1})$, we set $i_1 = n - 1$, and then by (7),

$$d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}). \tag{8}$$

Consider the case that $2 \leq i_0 \leq n$.

If $u = d(x_{i_0-1}, x_n)$, or $u = d(x_{i_0-1}, x_{i_0})$, or $d(x_n, x_{i_0})$, then by (5), (7), and $A(P) \subset P$,

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq Au \preceq A^2Bd(x_0, x_1) \\ &= \left(\sum_{i=2}^{\infty} A^i\right) d(x_0, x_1) \preceq \left(\sum_{i=1}^{\infty} A^i\right) d(x_0, x_1) \\ &= BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_n, x_{n+1})$, or $u = d(x_{i_0-1}, x_{n+1})$, we set $i_1 = n$, or $i_1 = i_0 - 1 \geq 1$, respectively, and then (8) follows.

From the above discussions of both cases, we have the result that either (6) holds, and so the proof of our claim is complete, or there exists $i_1 \in \{1, 2, \ldots, n\}$ such that (8) holds. For the latter situation, continue in a similar way, and we will have the result that either

$$d(x_{i_1}, x_{n+1}) \preceq ABd(x_0, x_1),$$

which together with (8) forces

$$d(x_{i_0}, x_{n+1}) \preceq A^2Bd(x_0, x_1) \preceq ABd(x_0, x_1),$$

i.e., (6) holds, and so the proof of our claim is complete, or there exists $i_2 \in \{1, 2, \ldots, n\}$ such that

$$d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}).$$

If the above procedure ends by the kth step with $k \leq n-1$, that is, there exist $k+1$ integers $i_0, i_1, \ldots, i_k \in \{1,2,\ldots,n\}$ such that

$$\begin{aligned}
&d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}),\\
&d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}), \qquad \ldots,\\
&d(x_{i_{k-1}}, x_{n+1}) \preceq Ad(x_{i_k}, x_{n+1}),\\
&d(x_{i_k}, x_{n+1}) \preceq BAd(x_0, x_1),
\end{aligned}$$

then by $A(P) \subset P$,

$$\begin{aligned}
d(x_{i_0}, x_{n+1}) &\preceq A^{k+1}Bd(x_0, x_1) = \left(\sum_{i=k+1}^{\infty} A^i\right) d(x_0, x_1)\\
&\preceq \left(\sum_{i=1}^{\infty} A^i\right) d(x_0, x_1) = BAd(x_0, x_1),
\end{aligned}$$

i.e. (6) holds, and so the proof of our claim is complete.

If the above procedure continues more than n steps, then there exist $n+1$ integers $i_0, i_1, i_n \in \{1,2,\ldots,n\}$ such that

$$\begin{aligned}
&d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}),\\
&d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}), \qquad \ldots,\\
&d(x_{i_{n-1}}, x_{n+1}) \preceq Ad(x_{i_n}, x_{n+1}).
\end{aligned} \tag{9}$$

It is clear that $i_0, i_1, i_n \in \{1,2,\ldots,n\}$ implies there exist two integers $k, l \in \{0,1,2,\ldots,n\}$ with $k < l$ such that $i_k = i_l$, then by the nondecreasing property of A and (9),

$$d(x_{i_k}, x_{n+1}) \preceq A^{l-k} d(x_{i_l}, x_{n+1}) = A^{l-k} d(x_{i_k}, x_{n+1}),$$

and so

$$\left(I - A^{l-k}\right) d(x_{i_k}, x_{n+1}) \preceq \theta. \tag{10}$$

Set $B_1 u = \sum_{i=0}^{\infty} A^{i(l-k)} u$ for each $u \in P$. By (2), $B_1 : P \to P$ is well defined. Clearly, $B_1\theta = \theta$ and

$$B_1\left(I - A^{l-k}\right) = \left(I - A^{l-k}\right)B_1 = B_1 - A^{l-k}B_1 = I. \tag{11}$$

Act on (10) with B_1, then by (11), $B_1(P) \subset P$ and $B_1\theta = \theta$ we get $d(x_{i_k}, x_{n+1}) = \theta$, and hence (6) holds by (9). The proof of our claim is complete.

For each $1 < m < n$ and each $x_0 \in X$, set

$$C(x_0, m, n) = \left\{d\left(T^i x_0, T^j x_0\right) : m \leq i, j \leq n\right\}.$$

From (1), it follows that, for each $u \in C(x_0, m, n)$, there exists some $v \in C(x_0, m-1, n)$ such that $u \preceq Av$. Consequently for all $1 < m < n$, there exist $u_i \in C(x_0, m-i, n)$ $(i = 1, 2, \dots, m-1)$ such that

$$d(x_m, x_n) \preceq Au_1 \preceq A^2 u_2 \preceq \cdots \preceq A^{m-1} u_{m-1}, \tag{12}$$

since A is nondecreasing. Note that $u_{m-1} \in C(x_0, 1, n)$, then by (5),

$$u_{m-1} \preceq BAd(x_0, x_1),$$

and so by (12),

$$d(x_m, x_n) \preceq BA^m d(x_0, x_1), \quad \forall 1 < m < n. \tag{13}$$

It follows from (3) that $A^m d(x_0, x_1) \overset{\|\cdot\|}{\to} \theta$ $(m \to \infty)$, and hence $BA^m d(x_0, x_1) \overset{\|\cdot\|}{\to} \theta$ $(m \to \infty)$ since B is continuous at θ. This together with Lemma 2 implies that $BA^m d(x_0, x_1) \overset{w}{\to} \theta$ $(m \to \infty)$. Moreover, by (13) and Lemma 1, we get

$$d(x_m, x_n) \overset{w}{\to} \theta \quad (n > m \to \infty), \tag{14}$$

i.e., $\{x_n\}$ is a Cauchy sequence of X. Therefore by the completeness of X, there exists some $x^* \in X$ such that $x_n \overset{\tau_d}{\to} x^*$ $(n \to \infty)$, *i.e.*,

$$d\bigl(x_n, x^*\bigr) \overset{w}{\to} \theta \quad (n \to \infty). \tag{15}$$

By (1),

$$d\bigl(Tx^*, x^*\bigr) \preceq d\bigl(x_{n+1}, Tx^*\bigr) + d\bigl(x_{n+1}, x^*\bigr) \preceq Au + d\bigl(x_{n+1}, x^*\bigr), \quad \forall n, \tag{16}$$

where $u \in \{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1})\}$.

If $u = d(x_n, x^*)$, or $u = d(x_n, x_{n+1})$, or $u = d(x^*, x_{n+1})$, then by (14), (15), (16), Lemma 1, and Lemma 3, we get $d(Tx^*, x^*) = \theta$ since A is continuous at θ.

If $u = d(x^*, Tx^*)$, then by (16),

$$(I - A)d\bigl(x^*, Tx^*\bigr) \preceq d\bigl(x_{n+1}, x^*\bigr), \quad \forall n,$$

and hence by (15), for each $\epsilon \in \operatorname{int}P$, there exists n_0 such that, for each $n \geq n_0$,

$$(I - A)d\bigl(x^*, Tx^*\bigr) \preceq d\bigl(x_{n+1}, x^*\bigr) \ll \epsilon, \tag{17}$$

which implies that

$$(I - A)d\bigl(x^*, Tx^*\bigr) \preceq \theta. \tag{18}$$

Act on (18) with B, then by $B(P) \subset P$ and $B\theta = \theta$ we get $d(Tx^*, x^*) = \theta$.

If $u = d(x_n, Tx^*)$, then by the triangle inequality, the nondecreasing property, and subadditivity of A and (16), we have

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(x_{n+1}, x^*) + Ad(x_n, Tx^*) \\ &\preceq d(x_{n+1}, x^*) + A\bigl[d(x_n, x^*) + d(x^*, Tx^*)\bigr] \\ &\preceq d(x_{n+1}, x^*) + Ad(x_n, x^*) + Ad(x^*, Tx^*), \quad \forall n, \end{aligned}$$

and so

$$(I - A)d(x^*, Tx^*) \preceq d(x_{n+1}, x^*) + Ad(x_n, x^*), \quad \forall n.$$

Thus it follows from (15) and Lemma 3 that (17) holds for each $\epsilon \in \operatorname{int}P$ since A is continuous at θ. Consequently, we get (18). Act on (18) with B, then by $B(P) \subset P$ and $B\theta = \theta$ we get $d(Tx^*, x^*) = \theta$. This shows that x^* is a fixed point of T.

If x is another fixed point of T, then by (1),

$$d(x, x^*) = d(Tx, Tx^*) \preceq Au,$$

where $u \in \{d(x, x^*), d(x, Tx), d(x^*, Tx^*), d(x, Tx^*), d(x^*, Tx)\}$. If $u = d(x, Tx)$, or $u = d(x^*, Tx^*)$, then $u = \theta$, and hence $d(x, x^*) = \theta$. If $u = d(x, x^*)$, or $u = d(x, Tx^*)$ or $u = d(x^*, Tx)$, then we must have $u = d(x, x^*)$, and hence $(I - A)d(x, x^*) \preceq \theta$. Act on it with B, then by $B(P) \subset P$ and $B\theta = \theta$ we get $d(x, x^*) = \theta$. This shows x^* is the unique fixed point of T. The proof is complete. □

The following example shows the usability of Theorem 1.

Example 2 Let E and P be the same ones as those in Example 1 and $X = P$. Define a mapping $d : X \times X \to P$ by

$$d(x, y) = \begin{cases} \theta, & x = y, \\ x + y, & x \neq y. \end{cases}$$

Clearly, (X, d) is a complete cone metric space.

Let $(Tx)(t) = \int_0^t x^{\frac{1}{2}}(s)\,ds$ and $(Ax)(t) = \sqrt{2}(Tx)(t)$ for each $x \in X$ and each $t \in [0,1]$.

Clearly, $A : P \to P$ is a nondecreasing mapping with $A\theta = \theta$, and A is continuous at θ. From Example 1 we know that (2) is satisfied and B is continuous at θ. For each $u, v \in P$, we have $(A(u + v))(t) = \sqrt{2}\int_0^t (u(s) + v(s))^{\frac{1}{2}}\,ds \leq \sqrt{2}\int_0^t (u(s)^{\frac{1}{2}} + v(s)^{\frac{1}{2}})\,ds = (Au)(t) + (Av)(t)$ for each $t \in [0,1]$, and so $A(u + v) \preceq Au + Av$ for each $u, v \in P$, *i.e.*, A is subadditive.

Note that $(Tx)(t) + (Ty)(t) \int_0^t (x^{\frac{1}{2}}(s) + y^{\frac{1}{2}}(s))\,ds = \int_0^t (x(s) + y(s) + 2x^{\frac{1}{2}}(s)y^{\frac{1}{2}}(s))^{\frac{1}{2}}\,ds \leq \sqrt{2}\int_0^t (x(s) + y(s))^{\frac{1}{2}}\,ds = \sqrt{2}(T(x + y))(t)$ for each $t \in [0,1]$ and each $x, y \in X$, *i.e.*, $Tx + Ty \preceq \sqrt{2}T(x + y)$ for each $x, y \in P$, then

$$d(Tx, Ty) = \begin{cases} \theta = Ad(x, y), & x = y, \\ Tx + Ty \preceq \sqrt{2}T(x + y) = Ad(x, y), & x \neq y, \end{cases}$$

i.e., (1) is satisfied with $u = d(x, y)$.

Hence all the assumptions of Theorem 1 are satisfied, and so T has a unique fixed point. In fact, θ is the unique fixed point of T.

Remark 3

(i) Since in Example 2 the underlying mapping A is nonlinear, we can conclude that any of the theorems in [12–15, 17] cannot cope with Example 2.

(ii) Let $u_0(t) = \cos^2 t$ for each $t \in [0,1]$ in Example 2. Clearly, $u_0 \in \operatorname{int}P$ and $(Au_0)(t) = \sqrt{2}\int_0^t \cos s\,ds = \sqrt{2}\sin t$ for each $t \in [0,1]$. Take $t_0 = \frac{\pi}{4}$, we have $(Au_0)(t_0) = 1 > \frac{1}{2} = u_0(t_0)$, and so $Au_0 \not\preceq u_0$, *i.e.*, $(I - A)u_0 \notin P$. Note that it is necessarily assumed that $(I - A)(\operatorname{int}P) \subset \operatorname{int}P$ in [16], then Theorem 2 of [16] is not applicable.

In what follows, we shall show that the subadditivity of A assumed in Theorem 1 could be removed in the case that (1) is satisfied for $u = d(x,y)$.

Theorem 2 *Let (X,d) be a complete cone metric space over a solid cone P of a normed vector space $(E,\|\cdot\|)$ and $T : X \to X$. Assume that*

$$d(Tx,Ty) \preceq Ad(x,y), \quad \forall x,y \in X, \tag{19}$$

where $A : P \to P$ is a nondecreasing mapping with $A\theta = \theta$ such that (2) *is satisfied. If A and B are continuous at θ, where $Bu = \sum_{i=0}^{\infty} A^i u$ for each $u \in P$. Then T has a unique fixed point $x^* \in X$, and for each $x_0 \in X$, the Picard iterative sequence $\{x_n\}$ converges to x^*.*

Proof By the nondecreasing property of A and (19), we have

$$d(x_n,x_{n+1}) \preceq Ad(x_{n-1},x_n) \preceq A^2 d(x_{n-2},x_{n-1}) \preceq \cdots \preceq A^n d(x_0,x_1), \quad \forall n,$$

and so, by the triangle inequality,

$$\begin{aligned} d(x_n,x_m) &\preceq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \preceq \sum_{i=n}^{m-1} A^i d(x_0,x_1) \\ &= A^n\left(\sum_{i=n}^{m-1} A^i\right) d(x_0,x_1) = BA^n d(x_0,x_1), \quad \forall m > n. \end{aligned} \tag{20}$$

Since B is continuous at θ, it follows from (3) that $BA^n d(x_0,x_1) \xrightarrow{\|\cdot\|} \theta$ $(n \to \infty)$, which together with Lemma 2 implies that $BA^n d(x_0,x_1) \xrightarrow{w} \theta$ $(n \to \infty)$. Moreover, by (20) and Lemma 1, we get

$$d(x_m,x_n) \xrightarrow{w} \theta \quad (m > n \to \infty),$$

i.e., $\{x_n\}$ is a Cauchy sequence of X. Therefore by the completeness of X, there exists some $x^* \in X$ such that (15) is satisfied. By the triangle inequality and (19), we get

$$d\left(x^*,Tx^*\right) \preceq d\left(x^*,x_{n+1}\right) + d\left(Tx_n,Tx^*\right) \preceq d\left(x^*,x_{n+1}\right) + Ad\left(x_n,x^*\right), \quad \forall n,$$

which together with (15), Lemma 1, and Lemma 3 implies that $p(x^*, Tx^*) = \theta$ since A is continuous at θ. Hence x^* is a fixed point of T. Let x be another fixed point of T, then by (19),

$$d(x, x^*) = d(Tx, Tx^*) \preceq Ad(x, x^*),$$

and so $(I - A)d(x, x^*) \preceq \theta$. Act on it with B, then by $B(P) \subset P$ and $B\theta = \theta$ we get $d(x, x^*) = \theta$. This shows x^* is the unique fixed point of T. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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