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Some new iterative algorithms with errors for common solutions of two finite families of accretive mappings in a Banach space

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Abstract

The purpose of this paper is to prove some new theorems of strong convergence to common solutions for two finite families of accretive mappings in a real uniformly smooth and uniformly convex Banach space by means of some new iterative algorithms with errors, which extend the corresponding works by some authors. As applications, the theorems of strong convergence to common fixed points of two finite families of pseudo-contractive mappings are presented.

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Keywords: accretive mapping; pseudo-contractive mapping; common zeros; uniformly smooth (or convex) Banach space

1 Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

We use J to denote the normalized duality mapping from E to 2^{E^*}, which is defined by

$$Jx := \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

It is well known that J is single-valued if E^* is strictly convex. Moreover, $J(cx) = cJx$, for $\forall x \in E$ and $c \in R^1$. We call J weakly sequentially continuous if each $\{x_n\} \subset E$ which converges weakly to x implies that $\{Jx_n\}$ converges in the sense of weak* to Jx.

Let C be a nonempty, closed, and convex subset of E and Q be a mapping of E onto C. Then Q is said to be sunny [1] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [1] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for $\forall x, y \in C$. We use $F(T)$ to denote the fixed point set of T, that is, $F(T) := \{x \in C : Tx = x\}$. A mapping $T : E \supset D(T) \to R(T) \subset E$ is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$.

A subset C of E is said to be a sunny nonexpansive retract of E [2] if there exists a sunny nonexpansive retraction of E onto C and it is called a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C. If E is reduced to a Hilbert space H, then the metric projection P_C is a sunny nonexpansive retraction from H to any closed and convex subset C of H. But this is not true in a general Banach space. We note that if E is smooth and Q is a retraction of C onto $F(T)$, then Q is sunny and nonexpansive if and only if for $\forall x \in C$, $z \in F(T)$, $\langle Qx - x, J(Qx - z)\rangle \leq 0$ [3].

A mapping $T : C \to C$ is called pseudo-contractive [2] if there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2$ holds for all $x, y \in C$.

Interest in pseudo-contractive mappings stems mainly from their firm connection with the important class of nonlinear accretive mappings. A mapping $A : D(A) \subset E \to E$ is said to be accretive if $\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$, for $\forall x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, which is called the resolvent of A. We also know that for an accretive mapping A, $N(A) = F(J_r^A)$, where $N(A) = \{x \in D(A) : Ax = 0\}$. An accretive mapping A is said to be m-accretive if $R(I + \lambda A) = E$, for $\forall \lambda > 0$.

It is well known that if A is an accretive mapping, then the solutions of the problem $0 \in Ax$ correspond to the equilibrium points of some evolution equations. Hence, the problem of finding a solution $x \in E$ with $0 \in Ax$ has been studied by many researchers (see [4–12] and the references contained therein).

One classical method for studying the problem $0 \in Ax$, where A is an m-accretive mapping, is the following so-called proximal method (*cf.* [4]), presented in a Hilbert space:

$$x_0 \in H, \quad x_{n+1} \approx J_{r_n}^A x_n, \quad n \geq 0, \tag{1.1}$$

where $J_{r_n}^A := (I + r_n A)^{-1}$. It was shown that the sequence generated by (1.1) converges weakly or strongly to a zero point of A under some conditions.

On the other hand, one explicit iterative process was first introduced, in 1967, by Halpern [13] in the frame of Hilbert spaces:

$$u \in C, \quad x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset [0, 1]$ and $T : C \to C$ is a nonexpansive mapping. It was proved that under some conditions, the sequence $\{x_n\}$ produced by (1.2) converges strongly to a point in $F(T)$.

In 2007, Qin and Su [6] presented the following iterative algorithm:

$$\begin{aligned} &x_1 \in C, \\ &y_n = \beta_n x_n + (1 - \beta_n)J_{r_n}^A x_n, \\ &x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n. \end{aligned} \tag{1.3}$$

They showed that $\{x_n\}$ generated by (1.3) converges strongly to a point in $N(A)$.

Motivated by iterative algorithms (1.1) and (1.2), Zegeye and Shahzad extended their discussion to the case of finite m-accretive mappings. They presented in [14] the following iterative algorithm:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)S_r x_n, \quad n \geq 0, \tag{1.4}$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$ and $\sum_{i=0}^{l} a_i = 1$. If $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$, they proved that $\{x_n\}$ generated by (1.4) converges strongly to the common point in $N(A_i)$ $(i = 1, 2, \ldots, l)$ under some conditions.

The work in [14] was then extended to the following one presented by Hu and Liu in [15]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \vartheta_n S_{r_n} x_n, \quad n \geq 0, \tag{1.5}$$

where $S_{r_n} = a_0 I + a_1 J_{r_n}^{A_1} + a_2 J_{r_n}^{A_2} + \cdots + a_l J_{r_n}^{A_l}$ with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $\sum_{i=0}^{l} a_i = 1$. We have $\{\alpha_n\}, \{\beta_n\}, \{\vartheta_n\} \subset (0,1)$ and $\alpha_n + \beta_n + \vartheta_n = 1$. If $\bigcap_{i=1}^{l} N(A_i) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to the common point in $N(A_i)$ $(i = 1, 2, \ldots, l)$ under some conditions.

In 2009, Yao *et al.* presented the following iterative algorithm in the frame of Hilbert space in [16]:

$$\begin{aligned} &x_1 \in C, \\ &y_n = P_C\big[(1 - \alpha_n)x_n\big], \\ &x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad n \geq 1. \end{aligned} \tag{1.6}$$

Here $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim_{n\to\infty} \alpha_n = 0$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ constructed by (1.6) converges strongly to a point in $F(T)$.

The following lemma is commonly used in proving the convergence of the iterative algorithms in a Banach space.

Lemma 1.1 ([17]) *Let E be a real uniformly smooth Banach space, then there exists a non-decreasing continuous function $\beta : [0, +\infty) \to [0, +\infty)$ with $\lim_{t\to 0^+} \beta(t) = 0$ and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$, such that for all $x, y \in E$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + \max\big\{\|x\|, 1\big\}\|y\|\beta\big(\|y\|\big).$$

Motivated by the work in [14] and [16], and after imposing an additional condition on the function β in Lemma 1.1 that

$$\beta(t) \leq \frac{t}{\max\{1, 2r_1\}}, \tag{1.7}$$

where $r_1 > 0$ is a constant satisfying some conditions, Shehu and Ezeora presented the following result.

Theorem 1.1 ([2]) *Let E be a real uniformly smooth and uniformly convex Banach space, and let C be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_C is the sunny nonexpansive retraction of E onto C. Supposed the duality mapping $J : E \to E^*$ is weakly sequentially continuous. For each $i = 1, 2, \ldots, N$, let $A_i : C \to E$ be an m-accretive* *mapping such that* $\bigcap_{i=1}^{N} N(A_i) \neq \emptyset$. *Let* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfy* (a) *and* (b). *Let* $\{x_n\}$ *be generated iteratively by*

$$\begin{aligned} &x_1 \in C, \\ &y_n = Q_C\big[(1-\alpha_n)x_n\big], \\ &x_{n+1} = (1-\beta_n)x_n + \beta_n S_N y_n, \quad n \geq 1. \end{aligned} \tag{1.8}$$

Here $S_N := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_N J_{A_N}$ *with* $J_{A_i} = (I + A_i)^{-1}$, *for* $i = 1, 2, \ldots, N$. $0 < a_k < 1$, *for* $k = 0, 1, 2, \ldots, N$, *and* $\sum_{k=0}^{N} a_k = 1$. *Then* $\{x_n\}$ *converges strongly to the common point in* $N(A_i)$, *where* $i = 1, 2, \ldots, N$.

How do we show the convergence of the iterative sequence $\{x_n\}$ in (1.8) if β loses the additional condition (1.7)? How about the convergence of $\{x_n\}$ if different A_i has different coefficient in (1.8)?

To answer these questions, Wei and Tan presented the following iterative scheme in [18]:

$$\begin{aligned} &x_1 \in C, \\ &u_n = Q_C\big[(1-\alpha_n)(x_n + e_n)\big], \\ &v_n = (1-\beta_n)x_n + \beta_n S_n u_n, \\ &x_{n+1} = \gamma_n x_n + (1-\gamma_n) S_n v_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where $\{e_n\} \subset E$ is the error sequence and $\{A_i\}_{i=1}^{N}$ is a finite family of m-accretive mappings. $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \cdots + a_N J_{r_{n,N}}^{A_N}$, $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$, for $i = 1, 2, \ldots, N$, $\sum_{k=0}^{N} a_k = 1$, $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$. Some strong convergence theorems are obtained.

In this paper, our main purpose is to extend the discussion of (1.9) from one family of m-accretive mappings $\{A_i\}_{i=1}^{N}$ to that of two families of m-accretive mappings $\{A_i\}_{i=1}^{N}$ and $\{B_j\}_{j=1}^{M}$. We shall first present and study the following three-step iterative algorithm (A) with errors $\{e_n\} \subset E$:

$$\begin{aligned} &x_1 \in C, \\ &u_n = Q_C\big[(1-\alpha_n)(x_n + e_n)\big], \\ &v_n = (1-\beta_n)x_n + \beta_n S_n u_n, \\ &x_{n+1} = \gamma_n x_n + (1-\gamma_n) W_n S_n v_n, \quad n \geq 1, \end{aligned} \tag{A}$$

where $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \cdots + a_N J_{r_{n,N}}^{A_N}$, and $W_n := b_0 I + b_1 J_{s_{n,1}}^{B_1} + b_2 J_{s_{n,2}}^{B_2} + \cdots + b_M J_{s_{n,M}}^{B_M}$. For $i = 1, 2, \ldots, N$, $J_{r_{n,i}}^{A_i} = (I + r_{n,i} A_i)^{-1}$. For $j = 1, 2, \ldots, M$, $J_{s_{n,j}}^{B_j} = (I + s_{n,j} B_j)^{-1}$. $a_0, a_1, \ldots, a_N$ and $b_0, b_1, \ldots, b_M$ are real numbers in $(0,1)$ and $\sum_{i=0}^{N} a_i = 1$, $\sum_{j=0}^{M} b_j = 1$. $r_{n,i} > 0$, for $i = 1, 2, \ldots, N$, and $s_{n,j} > 0$, for $j = 1, 2, \ldots, M$ and $n \geq 1$.

Later, we introduce and study the following one:

$$\begin{aligned} &x_1 \in C, \\ &u_n = Q_C\big[(1-\alpha_n)(x_n + e_n)\big], \end{aligned} \tag{B}$$

$$v_n = (1-\beta_n)x_n + \beta_n S_n u_n,$$
$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)U_n S_n v_n, \quad n \geq 1,$$

where $U_n := c_0 I + c_1 J_{t_{n,1}}^{B_1} + c_2 J_{t_{n,2}}^{B_2} J_{t_{n,1}}^{B_1} + \cdots + c_M J_{t_{n,M}}^{B_M} J_{t_{n,M-1}}^{B_{M-1}} \cdots J_{t_{n,1}}^{B_1}$, $c_0, c_1, \ldots, c_M$ are real numbers in $(0,1)$, $\sum_{j=0}^{M} c_j = 1$, and $J_{t_{n,j}}^{B_j} = (I + t_{n,j}B_j)^{-1}$ and $t_{n,j} > 0$, for $j = 1, 2, \ldots, M$ and $n \geq 1$.

More details will be presented in Section 3. Some strong convergence theorems are obtained, which can be regarded as the extension of the work done in [2, 6, 14, 15, 18], *etc.* As a consequence, some new iterative algorithms are constructed to converge strongly to the common fixed point of two finite families of pseudo-contractive mappings from C to E.

2 Preliminaries

Now, we list some results we need in sequel.

Lemma 2.1 ([19]) *Let E be a real uniformly convex Banach space and let C be a nonempty, closed, and convex subset of E and $T : C \to C$ is a nonexpansive mapping such that $F(T) \neq \emptyset$, then $I - T$ is demiclosed at zero.*

Lemma 2.2 ([15]) *Let E be a strictly convex Banach space which has a uniformly Gâteaux differential norm, and let C be a nonempty, closed, and convex subset of E. Let $\{A_i\}_{i=1}^{N}$ be a finite family of accretive mappings with $\bigcap_{i=1}^{N} N(A_i) \neq \emptyset$, satisfying the following range conditions*:

$$\overline{D(A_i)} \subseteq C \subset \bigcap_{r>0} R(I + rA_i), \quad i = 1, 2, \ldots, N.$$

Let $a_0, a_1, \ldots, a_N$ be real numbers in $(0,1)$ such that $\sum_{i=0}^{N} a_i = 1$ and $S_{r_n} = a_0 I + a_1 J_{r_n}^{A_1} + a_2 J_{r_n}^{A_2} + \cdots + a_N J_{r_n}^{A_N}$, where $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $r_n > 0$, then S_{r_n} is nonexpansive and $F(S_{r_n}) = \bigcap_{i=1}^{N} N(A_i)$.

Lemma 2.3 ([12]) *In a real Banach space E, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 2.4 ([20]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - c_n)a_n + b_n c_n, \quad \forall n \geq 1,$$

where $\{c_n\} \subset (0,1)$ such that (i) $c_n \to 0$ *and* $\sum_{n=1}^{\infty} c_n = +\infty$, (ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n c_n| < +\infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([21]) *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E such that $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$, for $n \geq 1$. Suppose $\{\beta_n\} \subset (0,1)$ satisfying $0 < \liminf_{n\to+\infty} \beta_n \leq \limsup_{n\to+\infty} \beta_n < 1$. If $\limsup_{n\to+\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to+\infty} \|y_n - x_n\| = 0$.*

Lemma 2.6 ([22]) *Let E be a Banach space and let A be an m-accretive mapping. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

3 Main results

Lemma 3.1 ([2]) *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, and Q_C be the sunny nonexpansive retraction of E onto C. Let $T : C \to C$ be nonexpansive with $F(T) \neq \emptyset$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous. If for each $t \in (0,1)$, define $T_t : C \to C$ by*

$$T_t x := TQ_C\big[(1-t)x\big]. \tag{3.1}$$

Then T_t is a contraction and has a fixed point z_t, which satisfies $\|z_t - Tz_t\| \to 0$, as $t \to 0$.

Lemma 3.2 ([2]) *Under the assumptions of Lemma 3.1, suppose further that β in Lemma 1.1 satisfies (1.7), where $r_1 > 0$ is a sufficiently large constant such that $z_t \in C \bigcap \{z \in E : \|z - x^*\| \le r_1\}$, x^* is in $F(T)$ and $t \in (0,1)$, then $\lim_{t\to 0} z_t = z_0 \in F(T)$.*

Remark 3.1 Lemma 1.1 with additional condition (1.7) is employed as a key tool to prove Lemma 3.2. In the following lemma, we shall show that Lemma 2.3 can be used instead of Lemma 1.1, which simplifies the proof and weakens the assumption.

Lemma 3.3 *Only under the assumptions of Lemma 3.1, the result of Lemma 3.2 is true, which ensures that the assumption is weaker than that in Lemma 3.2.*

Proof To show that $\lim_{t\to 0} z_t = z_0 \in F(T)$, it suffices to show that for any sequence $\{t_n\}$ such that $t_n \to 0$, we have $\lim_{n\to\infty} z_{t_n} = z_0 \in F(T)$.

In fact, Lemma 3.1 implies that $z_t \in F(T)$ such that $z_t = TQ_C[(1-t)z_t]$, $t \in (0,1)$. By using Lemma 2.3, we have for $\forall p \in F(T)$,

$$\begin{aligned}\|z_t - p\|^2 &= \big\|TQ_C\big[(1-t)z_t\big] - TQ_C p\big\|^2 \\ &\le \|z_t - p - tz_t\|^2 \\ &\le \|z_t - p\|^2 - 2t\|z_t - p - tz_t\|^2 - 2t\big\langle p + tz_t, J(z_t - p - tz_t)\big\rangle.\end{aligned}$$

This implies that

$$\|z_t - p - tz_t\|^2 \le \big\langle p, J(p + tz_t - z_t)\big\rangle + t\big\langle z_t, J(p + tz_t - z_t)\big\rangle. \tag{3.2}$$

In particular,

$$\|z_{t_n} - p - t_n z_{t_n}\|^2 \le \big\langle p, J(p + t_n z_{t_n} - z_{t_n})\big\rangle + t_n\big\langle z_{t_n}, J(p + t_n z_{t_n} - z_{t_n})\big\rangle. \tag{3.3}$$

Since $\forall p \in F(T)$,

$$\begin{aligned} \|z_t - p\| &= \big\|TQ_C\big[(1-t)z_t\big] - TQ_C p\big\| \\ &\leq \big\|Q_C\big[(1-t)z_t\big] - Q_C p\big\| \\ &\leq \big\|(1-t)z_t - p\big\| = \big\|(1-t)(z_t - p) - tp\big\| \\ &\leq (1-t)\|z_t - p\| + t\|p\|, \end{aligned}$$

$\{z_t\}$ is bounded.

Without loss of generality, we can assume that $\{z_{t_n}\}$ converges weakly to z_0. Using Lemma 3.1 and Lemma 2.1, we have $z_0 \in F(T)$.

Substituting z_0 for p in (3.3), we obtain

$$\|z_{t_n} - z_0 - t_n z_{t_n}\|^2 \leq \big\langle z_0, J(z_0 + t_n z_{t_n} - z_{t_n})\big\rangle + t_n\big\langle z_{t_n}, J(z_0 + t_n z_{t_n} - z_{t_n})\big\rangle. \tag{3.4}$$

Then from (3.4) and the weak convergence of J, we have $z_{t_n} - z_0 - t_n z_{t_n} \to 0$, as $n \to \infty$.

Then from $\|z_{t_n} - z_0\| \leq \|z_{t_n} - z_0 - t_n z_{t_n}\| + t_n\|z_{t_n}\|$, we see that $z_{t_n} \to z_0$, as $n \to \infty$.

Suppose there exists another sequence $z_{t_m} \rightharpoonup x_0$, as $t_m \to 0$ and $m \to \infty$. Then from Lemma 3.1 that $\|z_{t_m} - Tz_{t_m}\| \to 0$ and $I - T$ is demi-closed at zero, we have $x_0 \in F(T)$. Moreover, repeating the above proof, we have $z_{t_m} \to x_0$, as $m \to \infty$. Next, we want to show that $z_0 = x_0$.

Using (3.2), we have

$$\|z_{t_m} - z_0 - t_m z_{t_m}\|^2 \leq \big\langle z_0, J(z_0 + t_m z_{t_m} - z_{t_m})\big\rangle + t_m\big\langle z_{t_m}, J(z_0 + t_m z_{t_m} - z_{t_m})\big\rangle. \tag{3.5}$$

By letting $m \to \infty$, (3.5) implies that

$$\|x_0 - z_0\|^2 \leq \big\langle z_0, J(z_0 - x_0)\big\rangle. \tag{3.6}$$

Interchanging x_0 and z_0 in (3.6), we obtain

$$\|z_0 - x_0\|^2 \leq \big\langle x_0, J(x_0 - z_0)\big\rangle. \tag{3.7}$$

Then (3.6) and (3.7) ensure

$$2\|x_0 - z_0\|^2 \leq \|x_0 - z_0\|^2, \tag{3.8}$$

which implies that $x_0 = z_0$.

Therefore, $\lim_{t\to 0} z_t = z_0 \in F(T)$.

This completes the proof. □

Lemma 3.4 *Let E be a strictly convex Banach space and let C be a nonempty, closed, and convex subset of E. Let $A_i : C \to E$ $(i = 1, 2, \ldots, N)$ be a finite family of m-accretive mappings such that $\bigcap_{i=1}^N N(A_i) \neq \emptyset$.*

Let $a_0, a_1, \ldots, a_N$ be real numbers in $(0,1)$ such that $\sum_{i=0}^N a_i = 1$ and $S_n = a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \cdots + a_N J_{r_{n,N}}^{A_N}$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ and $r_{n,i} > 0$, for $i = 1, 2, \ldots, N$, and $n \geq 1$, then $S_n : C \to C$ is nonexpansive and $F(S_n) = \bigcap_{i=1}^N N(A_i)$, for $n \geq 1$.

Proof The proof is from [18]. For later use, we present the proof in the following.

It is easy to check that $S_n : C \to C$ is nonexpansive and $\bigcap_{i=1}^N N(A_i) \subset F(S_n)$.

On the other hand, for $\forall p \in F(S_n)$, then $p = S_n p = a_0 p + a_1 J_{r_{n,1}}^{A_1} p + a_2 J_{r_{n,2}}^{A_2} p + \cdots + a_N J_{r_{n,N}}^{A_N} p$.

For $\forall q \in \bigcap_{i=1}^N N(A_i) \subset F(S_n)$, we have

$$\begin{aligned}\|p-q\| &\le a_0\|p-q\| + a_1\left\|J_{r_{n,1}}^{A_1}p - q\right\| + \cdots + a_N\left\|J_{r_{n,N}}^{A_N}p - q\right\| \\ &\le (a_0 + a_1 + \cdots + a_{N-1})\|p-q\| + a_N\left\|J_{r_{n,N}}^{A_N}p - q\right\| \\ &= (1-a_N)\|p-q\| + a_N\left\|J_{r_{n,N}}^{A_N}p - q\right\| \\ &\le \|p-q\|.\end{aligned}$$

Therefore, $\|p-q\| = (1-a_N)\|p-q\| + a_N\|J_{r_{n,N}}^{A_N}p - q\|$, which implies that $\|p-q\| = \|J_{r_{n,N}}^{A_N}p - q\|$. Similarly, $\|p-q\| = \|J_{r_{n,1}}^{A_1}p - q\| = \cdots = \|J_{r_{n,N}}^{A_N}p - q\|$.

Then $\|p-q\| = \|\frac{a_1}{\sum_{i=1}^N a_i}(J_{r_{n,1}}^{A_1}p - q) + \frac{a_2}{\sum_{i=1}^N a_i}(J_{r_{n,2}}^{A_2}p - q) + \cdots + \frac{a_N}{\sum_{i=1}^N a_i}(J_{r_{n,N}}^{A_N}p - q)\|$, which implies from the strict convexity of E that $p - q = J_{r_{n,1}}^{A_1}p - q = J_{r_{n,2}}^{A_2}p - q = \cdots = J_{r_{n,N}}^{A_N}p - q$.

Therefore, $J_{r_{n,i}}^{A_i}p = p$, for $i = 1, 2, \ldots, N$. We have $p \in \bigcap_{i=1}^N N(A_i)$, which completes the proof. □

Similar to Lemma 3.4, we have the following lemma.

Lemma 3.5 *Let E and C be the same as those in Lemma* 3.4. *Let $\{B_j\}_{j=1}^M$ be a finite family of m-accretive mappings such that $\bigcap_{j=1}^M N(B_j) \neq \emptyset$.*

Let $b_0, b_1, \ldots, b_M$ be real numbers in $(0,1)$ *such that $\sum_{j=0}^M b_j = 1$ and $W_n = b_0 I + b_1 J_{s_{n,1}}^{B_1} + b_2 J_{s_{n,2}}^{B_2} + \cdots + b_M J_{s_{n,M}}^{B_M}$, where $J_{s,j}^{B_j} = (I + s_{n,j}B_j)^{-1}$ and $s_{n,j} > 0$, for $j = 1, 2, \ldots, M$, then $W_n : C \to C$ is nonexpansive and $F(W_n) = \bigcap_{j=1}^M N(B_j)$, for $n \ge 1$.*

Lemma 3.6 *Let E, C, S_n, and W_n be the same as those in Lemmas* 3.4 *and* 3.5. *Suppose $D := (\bigcap_{i=1}^N N(A_i)) \cap (\bigcap_{j=1}^M N(B_j)) \neq \emptyset$. Then $W_n S_n, S_n W_n : C \to C$ are nonexpansive and $F(W_n S_n) = F(S_n W_n) = D$.*

Proof From Lemmas 3.4 and 3.5, we can easily check that $W_n S_n, S_n W_n : C \to C$ are nonexpansive and $F(S_n) \cap F(W_n) = D$. So, it suffices to show that $F(S_n) \cap F(W_n) \supset F(W_n S_n)$ since $F(S_n) \cap F(W_n) \subset F(W_n S_n)$ is trivial.

For $\forall p \in F(W_n S_n)$, then $p = W_n S_n p$.

For $\forall q \in F(S_n) \cap F(W_n) \subset F(W_n S_n)$, then $q = W_n S_n q$. Now,

$$\|p-q\| \le \|S_n p - S_n q\| \le a_0\|p-q\| + a_1\left\|J_{r_{n,1}}^{A_1}p - q\right\| + \cdots + a_N\left\|J_{r_{n,N}}^{A_N}p - q\right\|.$$

Then repeating the discussion in Lemma 3.4, we know that $p \in F(S_n)$. Then $p = W_n S_n p = W_n p$, thus $p \in F(W_n)$, which completes the proof. □

Theorem 3.1 *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_C is the sunny nonexpansive retraction of E onto C. Let $A_i, B_j : C \to E$ be m-accretive mappings, where $i = 1, 2, \ldots N$, $j = 1, 2, \ldots, M$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous and $D := (\bigcap_{i=1}^N N(A_i)) \cap (\bigcap_{j=1}^M N(B_j)) \neq \emptyset$. Let $\{x_n\}$ be generated* *by the iterative algorithm* (A), *where* $S_n := a_0 I + a_1 J_{r_{n,1}}^{A_1} + a_2 J_{r_{n,2}}^{A_2} + \cdots + a_N J_{r_{n,N}}^{A_N}$, *and* $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$, *for* $i = 1, 2, \ldots, N$, $0 < a_k < 1$, *for* $k = 0, 1, 2, \ldots, N$, $\sum_{k=0}^{N} a_k = 1$. $W_n = b_0 I + b_1 J_{s_{n,1}}^{B_1} + b_2 J_{s_{n,2}}^{B_2} + \cdots + b_M J_{s_{n,M}}^{B_M}$, *where* $J_{s_{n,j}}^{B_j} = (I + s_{n,j}B_j)^{-1}$, *for* $j = 1, 2, \ldots, M$, $0 < b_k < 1$, *for* $k = 0, 1, 2, \ldots, M$, $\sum_{k=0}^{M} b_k = 1$. *Suppose* $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$, *and* $\{r_{n,i}\}, \{s_{n,j}\} \subset (0, +\infty)$ *satisfy the following conditions*:

(i) $\alpha_n \to 0$, $\beta_n \to 0$, *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = +\infty$;

(iii) $0 < \liminf_{n\to+\infty} \gamma_n \le \limsup_{n\to+\infty} \gamma_n < 1$;

(iv) $\sum_{n=1}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \ge \varepsilon > 0$, *for* $n \ge 1$ *and* $i = 1, 2, \ldots, N$;

(v) $\sum_{n=1}^{\infty} |s_{n+1,j} - s_{n,j}| < +\infty$ *and* $s_{n,j} \ge \varepsilon > 0$, *for* $n \ge 1$ *and* $j = 1, 2, \ldots, M$;

(vi) $\frac{\|e_n\|}{\alpha_n} \to 0$, *as* $n \to +\infty$, *and* $\sum_{n=1}^{\infty} \|e_n\| < +\infty$.

Then $\{x_n\}$ *converges strongly to a point* $p_0 \in D$.

Proof We shall split the proof into five steps:

Step 1. $\{x_n\}$, $\{u_n\}$, $\{S_n u_n\}$, $\{v_n\}$, and $\{S_n x_n\}$ are all bounded.

We shall first show that $\forall p \in D$,

$$\|x_{n+1} - p\| \le M_1 + \sum_{i=1}^{n} \|e_i\|, \tag{3.9}$$

where $M_1 = \max\{\|x_1 - p\|, \|p\|\}$.

By using the induction method, we see that for $n = 1$, $\forall p \in D$,

$$\begin{aligned}
\|x_2 - p\| &\le \gamma_1 \|x_1 - p\| + (1 - \gamma_1) \|W_1 S_1 v_1 - p\| \\
&\le \gamma_1 \|x_1 - p\| + (1 - \gamma_1) \|v_1 - p\| \\
&\le \gamma_1 \|x_1 - p\| + (1 - \gamma_1)(1 - \beta_1) \|x_1 - p\| + \beta_1 (1 - \gamma_1) \|u_1 - p\| \\
&\le \gamma_1 \|x_1 - p\| + (1 - \gamma_1)(1 - \beta_1) \|x_1 - p\| + \beta_1 (1 - \gamma_1) \big\|(1 - \alpha_1)(x_1 + e_1) - p\big\| \\
&\le \big[1 - \alpha_1 \beta_1 (1 - \gamma_1)\big] \|x_1 - p\| + \alpha_1 \beta_1 (1 - \gamma_1) \|p\| + (1 - \alpha_1) \beta_1 (1 - \gamma_1) \|e_1\| \\
&\le M_1 + \|e_1\|.
\end{aligned}$$

Suppose that (3.9) is true for $n = k$. Then, for $n = k + 1$,

$$\begin{aligned}
\|x_{k+2} - p\| &\le \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \|v_{k+1} - p\| \\
&\le \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \big[(1 - \beta_{k+1}) \|x_{k+1} - p\| + \beta_{k+1} \|u_{k+1} - p\|\big] \\
&\le \gamma_{k+1} \|x_{k+1} - p\| + (1 - \gamma_{k+1}) \big[(1 - \beta_{k+1}) \|x_{k+1} - p\| \\
&\quad + \beta_{k+1} \big\|(1 - \alpha_{k+1})(x_{k+1} + e_{k+1}) - p\big\|\big] \\
&\le \big[1 - \alpha_{k+1} \beta_{k+1} (1 - \gamma_{k+1})\big] \|x_{k+1} - p\| + \alpha_{k+1} \beta_{k+1} (1 - \gamma_{k+1}) \|p\| \\
&\quad + \beta_{k+1} (1 - \alpha_{k+1})(1 - \gamma_{k+1}) \|e_{k+1}\| \\
&\le M_1 + \big[1 - \alpha_{k+1} \beta_{k+1} (1 - \gamma_{k+1})\big] \sum_{i=1}^{k} \|e_i\| + (1 - \alpha_{k+1}) \beta_{k+1} (1 - \gamma_{k+1}) \|e_{k+1}\| \\
&\le M_1 + \sum_{i=1}^{k+1} \|e_i\|.
\end{aligned}$$

Thus (3.9) is true for all $n \in N$. Since $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, (3.9) ensures that $\{x_n\}$ is bounded.

For $\forall p \in D$, from $\|u_n - p\| \leq \|(1-\alpha_n)(x_n + e_n) - p\| \leq \|x_n\| + \|e_n\| + \|p\|$, we see that $\{u_n\}$ is bounded.

Since $\|S_n u_n\| \leq \|S_n u_n - S_n p\| + \|p\| \leq \|u_n - p\| + \|p\|$, $\{S_n u_n\}$ is bounded. Since both $\{S_n u_n\}$ and $\{x_n\}$ are bounded, $\{v_n\}$ is bounded. Similarly, $\{S_n x_n\}$, $\{S_n v_n\}$, $\{J_{r_{n,i}}^{A_i} u_n\}$, $\{J_{r_{n,i}}^{A_i} v_n\}$, and $\{J_{s_{n,j}}^{B_j} S_n v_n\}$ are all bounded, for $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, M$.

Then we set $M_2 = \sup\{\|u_n\|, \|J_{r_{n,i}}^{A_i} u_n\|, \|v_n\|, \|J_{r_{n,i}}^{A_i} v_n\|, \|S_n u_n\|, \|S_n v_n\|, \|x_n\|, \|J_{s_{n,j}}^{B_j} S_n v_n\| : n \geq 1, i = 1, 2, \ldots, N; j = 1, 2, \ldots, M\}$.

Step 2. $\lim_{n\to\infty} \|x_n - W_n S_n v_n\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In fact,

$$\begin{aligned} &\|W_{n+1} S_{n+1} v_{n+1} - W_n S_n v_n\| \\ &\quad \leq b_0 \|S_{n+1} v_{n+1} - S_n v_n\| + \sum_{j=1}^{M} b_j \bigl\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - J_{s_{n,j}}^{B_j} S_n v_n\bigr\|. \end{aligned} \tag{3.10}$$

Next, we discuss $\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - J_{s_{n,j}}^{B_j} S_n v_n\|$.

If $s_{n,j} \leq s_{n+1,j}$, then, using Lemma 2.6,

$$\begin{aligned} &\bigl\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - J_{s_{n,j}}^{B_j} S_n v_n\bigr\| \\ &\quad = \left\|J_{s_{n,j}}^{B_j}\left(\frac{s_{n,j}}{s_{n+1,j}} S_{n+1} v_{n+1} + \left(1 - \frac{s_{n,j}}{s_{n+1,j}}\right) J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1}\right) - J_{s_{n,j}}^{B_j} S_n v_n\right\| \\ &\quad \leq \left\|\frac{s_{n,j}}{s_{n+1,j}} S_{n+1} v_{n+1} + \left(1 - \frac{s_{n,j}}{s_{n+1,j}}\right) J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - S_n v_n\right\| \\ &\quad \leq \frac{s_{n,j}}{s_{n+1,j}} \|S_{n+1} v_{n+1} - S_n v_n\| + \left(1 - \frac{s_{n,j}}{s_{n+1,j}}\right) \bigl\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - S_n v_n\bigr\| \\ &\quad \leq \|S_{n+1} v_{n+1} - S_n v_n\| + 2M_2 \frac{s_{n+1,j} - s_{n,j}}{\varepsilon}. \end{aligned} \tag{3.11}$$

If $s_{n+1,j} \leq s_{n,j}$, then imitating the proof of (3.11), we have

$$\bigl\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - J_{s_{n,j}}^{B_j} S_n v_n\bigr\| \leq \|S_{n+1} v_{n+1} - S_n v_n\| + 2M_2 \frac{s_{n,j} - s_{n+1,j}}{\varepsilon}. \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\begin{aligned} &\bigl\|J_{s_{n+1,j}}^{B_j} S_{n+1} v_{n+1} - J_{s_{n,j}}^{B_j} S_n v_n\bigr\| \\ &\quad \leq \|S_{n+1} v_{n+1} - S_n v_n\| + 2M_2 \frac{|s_{n,j} - s_{n+1,j}|}{\varepsilon}. \end{aligned} \tag{3.13}$$

Putting (3.13) into (3.10), we have

$$\|W_{n+1} S_{n+1} v_{n+1} - W_n S_n v_n\| \leq \|S_{n+1} v_{n+1} - S_n v_n\| + \frac{2M_2}{\varepsilon} \sum_{j=1}^{M} |s_{n,j} - s_{n+1,j}|. \tag{3.14}$$

Similarly, we have

$$\|S_{n+1}u_{n+1} - S_n u_n\| \leq \|u_{n+1} - u_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| \tag{3.15}$$

and

$$\|S_{n+1}v_{n+1} - S_n v_n\| \leq \|v_{n+1} - v_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}|. \tag{3.16}$$

Therefore,

$$\begin{aligned}
&\|W_{n+1}S_{n+1}v_{n+1} - W_n S_n v_n\| \\
&\quad\leq \|v_{n+1} - v_n\| + \frac{2M_2}{\varepsilon}\sum_{j=1}^{M}|s_{n,j} - s_{n+1,j}| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| \\
&\quad\leq \|x_{n+1} - x_n\| + \beta_n\|x_n\| + \beta_{n+1}\|x_{n+1}\| + |\beta_{n+1} - \beta_n|\|S_{n+1}u_{n+1}\| + \beta_n\|S_{n+1}u_{n+1} - S_n u_n\| \\
&\qquad + \frac{2M_2}{\varepsilon}\sum_{j=1}^{M}|s_{n,j} - s_{n+1,j}| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| \\
&\quad\leq \|x_{n+1} - x_n\| + \beta_n\|x_n\| + \beta_{n+1}\|x_{n+1}\| + |\beta_{n+1} - \beta_n|\|S_{n+1}u_{n+1}\| + \beta_n\|u_{n+1} - u_n\| \\
&\qquad + \frac{4M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + \frac{2M_2}{\varepsilon}\sum_{j=1}^{M}|s_{n,j} - s_{n+1,j}| \\
&\quad\leq \|x_{n+1} - x_n\| + \beta_n\|x_n\| + \beta_{n+1}\|x_{n+1}\| + |\beta_{n+1} - \beta_n|\|S_{n+1}u_{n+1}\| \\
&\qquad + \beta_n\big\|(1 - \alpha_{n+1})(x_{n+1} + e_{n+1}) - (1 - \alpha_n)(x_n + e_n)\big\| \\
&\qquad + \frac{4M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + \frac{2M_2}{\varepsilon}\sum_{j=1}^{M}|s_{n,j} - s_{n+1,j}| \\
&\quad\leq (1 + \beta_n)\|x_{n+1} - x_n\| + (\beta_n + \alpha_n\beta_n)\|x_n\| + (\beta_{n+1} + \alpha_{n+1}\beta_n)\|x_{n+1}\| \\
&\qquad + |\beta_{n+1} - \beta_n|\|S_{n+1}u_{n+1}\| + \beta_n\|e_{n+1} - e_n\| + \beta_n\|\alpha_{n+1}e_{n+1} - \alpha_n e_n\| \\
&\qquad + \frac{4M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + \frac{2M_2}{\varepsilon}\sum_{j=1}^{M}|s_{n,j} - s_{n+1,j}|.
\end{aligned} \tag{3.17}$$

Thus $\limsup_{n\to+\infty}(\|W_{n+1}S_{n+1}v_{n+1} - W_n S_n v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Using Lemma 2.5, we have from (3.17) $\lim_{n\to\infty}\|x_n - W_n S_n v_n\| = 0$ and then $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \gamma_n)\|W_n S_n v_n - x_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - W_n S_n x_n\| = 0$.

In fact,

$$\begin{aligned}
\|x_n - W_n S_n x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - W_n S_n x_n\| \\
&\leq \|x_{n+1} - x_n\| + \big\|\gamma_n x_n + (1 - \gamma_n)W_n S_n v_n - W_n S_n x_n\big\| \\
&\leq \|x_{n+1} - x_n\| + \gamma_n\|x_n - W_n S_n v_n\| + \|W_n S_n v_n - W_n S_n x_n\|
\end{aligned}$$

$$\begin{aligned}
&\le \|x_{n+1} - x_n\| + \gamma_n \|x_n - W_n S_n v_n\| + \beta_n \|S_n u_n - x_n\| \\
&\le \|x_{n+1} - x_n\| + \gamma_n \|x_n - W_n S_n v_n\| + 2\beta_n M_2. \qquad (3.18)
\end{aligned}$$

Then (3.18) and step 2 imply that $\|x_n - W_n S_n x_n\| \to 0$, as $n \to +\infty$, since $\beta_n \to 0$.

Step 4. $\limsup_{n\to+\infty} \langle p_0, J(p_0 - x_n)\rangle \le 0$, where p_0 is an element in D.

From Lemma 3.6, we know that $W_n S_n : C \to C$ is nonexpansive and $F(W_n S_n) = D$. Then Lemma 3.1 and Lemma 3.3 imply that there exists $z_t \in C$ such that $z_t = W_n S_n Q_C[(1-t)z_t]$ for $t \in (0,1)$. Moreover, $z_t \to p_0 \in D$, as $t \to 0$.

Since $\|z_t - p_0\| \le \|(1-t)z_t - p_0\| \le (1-t)\|z_t - p_0\| + \|p_0\|$, $\{z_t\}$ is bounded. Let $M_3 = \sup\{\|z_t - x_n\| : n \ge 1, t > 0\}$. Then from step 1, we know that M_3 is a positive constant. Using Lemma 2.3, we have

$$\begin{aligned}
\|z_t - x_n\|^2 &= \|z_t - W_n S_n x_n + W_n S_n x_n - x_n\|^2 \\
&\le \|z_t - W_n S_n x_n\|^2 + 2\langle W_n S_n x_n - x_n, J(z_t - x_n)\rangle \\
&\le \|z_t - W_n S_n x_n\|^2 + 2\|W_n S_n x_n - x_n\|\|z_t - x_n\| \\
&\le \big\|(1-t)z_t - x_n\big\|^2 + 2\|W_n S_n x_n - x_n\|\|z_t - x_n\| \\
&\le \|z_t - x_n\|^2 - 2t\big\langle z_t, J\big[(1-t)z_t - x_n\big]\big\rangle + 2M_3\|W_n S_n x_n - x_n\|.
\end{aligned}$$

So $\langle z_t, J[(1-t)z_t - x_n]\rangle \le \frac{M_3}{t}\|W_n S_n x_n - x_n\|$, which implies that $\lim_{t\to 0}\limsup_{n\to+\infty}\langle z_t, J[(1-t)z_t - x_n]\rangle \le 0$ in view of step 3.

Since $\{x_n\}$ is bounded and J is uniformly continuous on each bounded subset of E, $\langle p_0, J(p_0 - x_n) - J[(1-t)z_t - x_n]\rangle \to 0$, as $t \to 0$.

Moreover, noticing the fact that

$$\begin{aligned}
\big\langle p_0, J(p_0 - x_n)\big\rangle &= \big\langle p_0, J(p_0 - x_n) - J\big[(1-t)z_t - x_n\big]\big\rangle \\
&\quad + \big\langle p_0 - z_t, J\big[(1-t)z_t - x_n\big]\big\rangle + \big\langle z_t, J\big[(1-t)z_t - x_n\big]\big\rangle,
\end{aligned}$$

we have $\limsup_{n\to+\infty}\langle p_0, J(p_0 - x_n)\rangle \le 0$.

Since $\langle p_0, J[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n]\rangle = \langle p_0, J[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n] - J(p_0 - x_n)\rangle + \langle p_0, J(p_0 - x_n)\rangle$, and J is uniformly continuous on each bounded subset of E,

$$\limsup_{n\to+\infty}\big\langle p_0, J\big[p_0 - x_n - (1-\alpha_n)e_n + \alpha_n x_n\big]\big\rangle \le 0. \qquad (3.19)$$

Step 5. $x_n \to p_0$, as $n \to +\infty$, where $p_0 \in D$ is the same as in step 4.

Let $M_4 = \sup\{\|(1-\alpha_n)(x_n + e_n) - p_0\| : n \ge 1\}$. By using Lemma 2.3 again, we have

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad \le \gamma_n\|x_n - p_0\|^2 + (1-\gamma_n)\|v_n - p_0\|^2 \\
&\quad \le \gamma_n\|x_n - p_0\|^2 + (1-\gamma_n)(1-\beta_n)\|x_n - p_0\|^2 + (1-\gamma_n)\beta_n\|u_n - p_0\|^2 \\
&\quad = (1 - \beta_n + \beta_n\gamma_n)\|x_n - p_0\|^2 + (1-\gamma_n)\beta_n\|u_n - p_0\|^2 \\
&\quad \le (1 - \beta_n + \beta_n\gamma_n)\|x_n - p_0\|^2 + (1-\gamma_n)\beta_n\big\|(1-\alpha_n)(x_n + e_n) - p_0\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\beta_n+\beta_n\gamma_n)\|x_n-p_0\|^2+(1-\gamma_n)\beta_n(1-\alpha_n)\|x_n-p_0\|^2\\
&\quad+2(1-\gamma_n)\beta_n(1-\alpha_n)\bigl\langle e_n,J\bigl[(1-\alpha_n)(x_n+e_n)-p_0\bigr]\bigr\rangle\\
&\quad+2\alpha_n\beta_n(1-\gamma_n)\bigl\langle p_0,J\bigl[p_0-x_n-(1-\alpha_n)e_n+\alpha_n x_n\bigr]\bigr\rangle\\
&\le \bigl[1-\alpha_n\beta_n(1-\gamma_n)\bigr]\|x_n-p_0\|^2+2(1-\gamma_n)(1-\alpha_n)\beta_n M_4\|e_n\|\\
&\quad+2\alpha_n\beta_n(1-\gamma_n)\bigl\langle p_0,J\bigl[p_0-x_n-(1-\alpha_n)e_n+\alpha_n x_n\bigr]\bigr\rangle.
\end{aligned}
\tag{3.20}
$$

Let $c_n = (1-\gamma_n)\alpha_n\beta_n$, then (3.20) reduces to $\|x_{n+1}-p_0\|^2 \le (1-c_n)\|x_n-p_0\|^2 + 2c_n\{\langle p_0,J[p_0-x_n-(1-\alpha_n)e_n+\alpha_n x_n]\rangle + (1-\alpha_n)M_4\frac{\|e_n\|}{\alpha_n}\}$.

From (3.19), (3.20), and the assumptions, by using Lemma 2.4, we know that $x_n \to p_0$, as $n \to +\infty$.

This completes the proof. □

If in Theorem 3.1, $C = E$, then we have the following theorem.

Theorem 3.2 *Let E and D be the same as those in Theorem* 3.1. *Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous. Let $A_i : E \to E$ $(i = 1,2,\dots,N)$ and $B_j : E \to E$ $(j = 1,2,\dots,M)$ be two finite families of m-accretive mappings. Let $\{e_n\} \subset E$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$, and $\{r_{n,i}\},\{s_{n,j}\} \subset (0,+\infty)$ satisfy the some conditions presented in Theorem* 3.1.

Let $\{x_n\}$ be generated by the following scheme:

$$
\begin{aligned}
&x_1 \in E,\\
&u_n = (1-\alpha_n)(x_n+e_n),\\
&v_n = (1-\beta_n)x_n+\beta_n S_n u_n,\\
&x_{n+1} = \gamma_n x_n+(1-\gamma_n)W_n S_n v_n, \quad n \ge 1.
\end{aligned}
\tag{C}
$$

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$, where S_n and W_n are the same as those in Theorem 3.1.

Lemma 3.7 *Let E, C and $\{B_j\}_{j=1}^M$ be the same as those in Lemma* 3.5. $\bigcap_{j=1}^M N(B_j) \neq \emptyset$.

Let $c_0,c_1,\dots,c_M$ be real numbers in $(0,1)$ such that $\sum_{j=0}^M c_j = 1$ and $U_n = c_0 I + c_1 J_{t_{n,1}}^{B_1} + c_2 J_{t_{n,2}}^{B_2} J_{t_{n,1}}^{B_1} + \cdots + c_M J_{t_{n,M}}^{B_M} J_{t_{n,M-1}}^{B_{M-1}} \cdots J_{t_{n,1}}^{B_1}$, where $J_{t_{n,j}}^{B_j} = (I+t_{n,j}B_j)^{-1}$ and $t_{n,j} > 0$, for $j = 1,2,\dots,M$, and $n \ge 1$, then $U_n : C \to C$ is nonexpansive and $F(U_n) = \bigcap_{j=1}^M N(B_j)$, for $n \ge 1$.

Proof It is easy to check that $U_n : C \to C$ is nonexpansive and $\bigcap_{j=1}^M N(B_j) \subset F(U_n)$.

On the other hand, for $\forall p \in F(U_n)$, then $p = U_n p = c_0 p + c_1 J_{t_{n,1}}^{B_1} p + c_2 J_{t_{n,2}}^{B_2} J_{t_{n,1}}^{B_1} p + \cdots + c_M J_{t_{n,M}}^{B_M} J_{t_{n,M-1}}^{B_{M-1}} \cdots J_{t_{n,1}}^{B_1} p$.

For $\forall q \in \bigcap_{j=1}^M N(B_j) \subset F(U_n)$, then

$$
\begin{aligned}
\|p-q\| &\le c_0\|p-q\| + c_1\bigl\|J_{t_{n,1}}^{B_1}p-q\bigr\| + \cdots + c_M\bigl\|J_{t_{n,M}}^{B_M} J_{t_{n,M-1}}^{B_{M-1}} \cdots J_{t_{n,1}}^{B_1}p-q\bigr\|\\
&\le (c_0+c_2+\cdots+c_M)\|p-q\| + c_1\bigl\|J_{t_{n,1}}^{B_1}p-q\bigr\|\\
&= (1-c_1)\|p-q\| + c_1\bigl\|J_{t_{n,1}}^{B_1}p-q\bigr\|\\
&\le \|p-q\|.
\end{aligned}
$$

Therefore, $\|p-q\| = (1-c_1)\|p-q\| + c_1\|J^{B_1}_{t_{n,1}}p-q\|$, which implies that $\|p-q\| = \|J^{B_1}_{t_{n,1}}p-q\|$. Similarly, $\|p-q\| = \|J^{B_1}_{t_{n,1}}p-q\| = \|J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}p-q\| = \cdots = \|J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}p-q\|$.

Then $\|p-q\| = \|\frac{c_1}{\sum_{j=1}^M c_j}(J^{B_1}_{t_{n,1}}p-q) + \frac{c_2}{\sum_{j=1}^M c_j}(J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}p-q) + \cdots + \frac{c_M}{\sum_{j=1}^M c_j}(J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}p-q)\|$, which implies from the strict convexity of E that $p-q = J^{B_1}_{t_{n,1}}p-q = J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}p-q = \cdots = J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}p-q$.

Therefore, $J^{B_1}_{t_{n,1}}p = p$, and then we can easily see that $J^{B_j}_{t_{n,j}}p = p$, for $j = 2,\ldots,M$. Thus $p \in \bigcap_{j=1}^M N(B_j)$, which completes the proof. □

Lemma 3.8 *Let E and C be the same as those in Lemma* 3.4. *Let S_n and U_n be the same as those in Lemmas* 3.4 *and* 3.7, *respectively. Suppose $D := (\bigcap_{i=1}^N N(A_i)) \cap (\bigcap_{j=1}^M N(B_j)) \neq \emptyset$. Then $S_nU_n, U_nS_n : C \to C$ are nonexpansive and $F(U_nS_n) = F(S_nU_n) = D$.*

Proof From Lemmas 3.4 and 3.7, we can easily check that $U_nS_n, S_nU_n : C \to C$ are nonexpansive and $F(S_n) \cap F(U_n) = D$. So, it suffices to show that $F(S_n) \cap F(U_n) \supset F(U_nS_n)$ since $F(S_n) \cap F(U_n) \subset F(U_nS_n)$ is trivial.

For $\forall p \in F(U_nS_n)$, then $p = U_nS_np$.

For $\forall q \in F(S_n) \cap F(U_n) \subset F(U_nS_n)$, then $q = U_nS_nq$. Now,

$$\|p-q\| = \|U_nS_np-q\| \le \|S_np-S_nq\| \le \|p-q\|.$$

Then repeating the discussion in Lemma 3.4, we know that $p \in F(S_n)$. Then $p = U_nS_np = U_np$, thus $p \in F(U_n)$, which completes the proof. □

Theorem 3.3 *Let E, C, Q_C, S_n, and D be the same as those in Theorem* 3.1. *Let $A_i, B_j : C \to E$ be m-accretive mappings, for $i = 1,2,\ldots N$, and $j = 1,2,\ldots,M$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous and $D \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative algorithm* (B), *where $U_n := c_0I + c_1J^{B_1}_{t_{n,1}} + c_2J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}} + \cdots + c_MJ^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}$, and $J^{B_j}_{t_{n,j}} = (I + t_{n,j}B_j)^{-1}$, for $j = 1,2,\ldots,M$, $0 < c_k < 1$, for $k = 0,1,2,\ldots,M$, and $\sum_{k=0}^M c_k = 1$. Suppose $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in* $(0,1)$ *and $\{r_{n,i}\}, \{t_{n,j}\} \subset (0,+\infty)$ satisfy the following conditions*:

(i) $\alpha_n \to 0$, $\beta_n \to 0$, *as* $n \to \infty$;

(ii) $\sum_{n=1}^\infty \alpha_n\beta_n = +\infty$;

(iii) $0 < \liminf_{n\to+\infty} \gamma_n \le \limsup_{n\to+\infty} \gamma_n < 1$;

(iv) $\sum_{n=1}^\infty |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \ge \varepsilon > 0$, *for* $n \ge 1$ *and* $i = 1,2,\ldots,N$;

(v) $\sum_{n=1}^\infty |t_{n+1,j} - t_{n,j}| < +\infty$ *and* $t_{n,j} \ge \varepsilon > 0$, *for* $n \ge 1$ *and* $j = 1,2,\ldots,M$;

(vi) $\frac{\|e_n\|}{\alpha_n} \to 0$, *as* $n \to +\infty$, *and* $\sum_{n=1}^\infty \|e_n\| < +\infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$.

Proof We shall split the proof into five steps:

Step 1. $\{x_n\}$, $\{u_n\}$, $\{S_nu_n\}$, $\{v_n\}$, $\{S_nv_n\}$ and $\{S_nx_n\}$ are all bounded.

Similar to the proof of step 1 in Theorem 3.1, we can get the result of step 1.

Then $\{J^{B_1}_{t_{n,1}}S_nv_n\}, \{J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}S_nv_n\}, \ldots, \{J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}S_nv_n\}$ are all bounded.

Step 2. $\lim_{n\to\infty} \|x_n - U_nS_nv_n\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In fact,

$$\begin{aligned}
&\|U_{n+1}S_{n+1}v_{n+1} - U_nS_nv_n\| \\
&\quad \le c_0\|S_{n+1}v_{n+1} - S_nv_n\| + c_1\big\|J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_1}_{t_{n,1}}S_nv_n\big\| \\
&\qquad + c_2\big\|J^{B_2}_{t_{n+1,2}}J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}S_nv_n\big\| + \cdots \\
&\qquad + c_M\big\|J^{B_M}_{t_{n+1,M}}J^{B_{M-1}}_{t_{n+1,M-1}}\cdots J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}S_nv_n\big\|. \qquad (3.21)
\end{aligned}$$

Similar to (3.13), we know that

$$\big\|J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_1}_{t_{n,1}}S_nv_n\big\| \le \|S_{n+1}v_{n+1} - S_nv_n\| + 2M_5\frac{|t_{n+1,1} - t_{n,1}|}{\varepsilon}, \qquad (3.22)$$

where $M_5 = \sup\{\|S_nv_n\|, \|J^{B_1}_{t_{n,1}}S_nv_n\|, \|J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}S_nv_n\|, \ldots, \|J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}S_nv_n\| : n \ge 1\}$.

Repeating (3.22), we have

$$\begin{aligned}
&\big\|J^{B_2}_{t_{n+1,2}}J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}S_nv_n\big\| \\
&\quad \le \big\|J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_1}_{t_{n,1}}S_nv_n\big\| + \frac{2M_5}{\varepsilon}|t_{n+1,2} - t_{n,2}|. \qquad (3.23)
\end{aligned}$$

Then (3.22) and (3.23) imply that

$$\begin{aligned}
&\big\|J^{B_2}_{t_{n+1,2}}J^{B_1}_{t_{n+1,1}}S_{n+1}v_{n+1} - J^{B_2}_{t_{n,2}}J^{B_1}_{t_{n,1}}S_nv_n\big\| \\
&\quad \le \|S_{n+1}v_{n+1} - S_nv_n\| + \frac{2M_5}{\varepsilon}\big(|t_{n+1,2} - t_{n,2}| + |t_{n+1,1} - t_{n,1}|\big). \qquad (3.24)
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\big\|J^{B_M}_{t_{n+1,M}}J^{B_{M-1}}_{t_{n+1,M-1}}\cdots J^{B_1}_{t_{n,1}}S_{n+1}v_{n+1} - J^{B_M}_{t_{n,M}}J^{B_{M-1}}_{t_{n,M-1}}\cdots J^{B_1}_{t_{n,1}}S_nv_n\big\| \\
&\quad \le \|S_{n+1}v_{n+1} - S_nv_n\| \\
&\qquad + \frac{2M_5}{\varepsilon}\big(|t_{n+1,M} - t_{n,M}| + \cdots + |t_{n+1,2} - t_{n,2}| + |t_{n+1,1} - t_{n,1}|\big). \qquad (3.25)
\end{aligned}$$

Going back to (3.21), we have

$$\begin{aligned}
&\|U_{n+1}S_{n+1}v_{n+1} - U_nS_nv_n\| \\
&\quad \le \|S_{n+1}v_{n+1} - S_nv_n\| \\
&\qquad + \frac{2M_5}{\varepsilon}\left(\sum_{j=1}^{M} c_j|t_{n,1} - t_{n+1,1}| + \sum_{j=2}^{M} c_j|t_{n,2} - t_{n+1,2}| + \cdots + c_M|t_{n,M} - t_{n+1,M}|\right). \qquad (3.26)
\end{aligned}$$

Therefore, similar to (3.17), we have

$$\begin{aligned}
&\|U_{n+1}S_{n+1}v_{n+1} - U_nS_nv_n\| \\
&\quad \le \|v_{n+1} - v_n\| + \frac{2M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2M_5}{\varepsilon}\left(\sum_{j=1}^{M} c_j|t_{n,1}-t_{n+1,1}|+\sum_{j=2}^{M} c_j|t_{n,2}-t_{n+1,2}|+\cdots+c_M|t_{n,M}-t_{n+1,M}|\right) \\
\le{} & (1+\beta_n)\|x_{n+1}-x_n\|+(\beta_n+\alpha_n\beta_n)\|x_n\|+(\beta_{n+1}+\alpha_{n+1}\beta_n)\|x_{n+1}\| \\
&+|\beta_{n+1}-\beta_n|\|S_{n+1}u_{n+1}\|+\beta_n\|e_{n+1}-e_n\|+\beta_n\|\alpha_{n+1}e_{n+1}-\alpha_n e_n\| \\
&+\frac{4M_2}{\varepsilon}\sum_{i=1}^{N}|r_{n,i}-r_{n+1,i}| \\
&+\frac{2M_5}{\varepsilon}\left(\sum_{j=1}^{M} c_j|t_{n,1}-t_{n+1,1}|+\sum_{j=2}^{M} c_j|t_{n,2}-t_{n+1,2}|+\cdots+c_M|t_{n,M}-t_{n+1,M}|\right).
\end{aligned}
\tag{3.27}
$$

Thus $\limsup_{n\to+\infty}(\|U_{n+1}S_{n+1}v_{n+1}-U_nS_nv_n\|-\|x_{n+1}-x_n\|)\le 0$. Using Lemma 2.5, we have from (3.27) $\lim_{n\to\infty}\|x_n-U_nS_nv_n\|=0$ and then $\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\gamma_n)\|U_nS_nv_n-x_n\|=0$.

Similar to Theorem 3.1, we have

Step 3. $\lim_{n\to\infty}\|x_n-U_nS_nx_n\|=0$.

Step 4. $\limsup_{n\to+\infty}\langle p_0,J(p_0-x_n)\rangle\le 0$, where p_0 is an element in D.

From Lemma 3.8, we know that $U_nS_n: C\to C$ is nonexpansive and $F(U_nS_n)=D$. Then Lemma 3.1 and Lemma 3.3 imply that there exists $z_t\in C$ such that $z_t=U_nS_nQ_C[(1-t)z_t]$ for $t\in(0,1)$. Moreover, $z_t\to p_0\in D$, as $t\to 0$. Then copy step 4 in Theorem 3.1, the result follows.

Step 5. $x_n\to p_0\in D$, which is the same as that in step 4.

Copy step 5 in Theorem 3.1, the result follows.

This completes the proof. □

If in Theorem 3.3, $C=E$, then we have the following theorem.

Theorem 3.4 *Let E and D be the same as those in Theorem* 3.3. *Suppose that the duality mapping $J: E\to E^*$ is weakly sequentially continuous. Let $A_i: E\to E$ ($i=1,2,\ldots,N$) and $B_j: E\to E$ ($j=1,2,\ldots,M$) be two finite families of m-accretive mappings. Let $\{e_n\}\subset E$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}\subset(0,1)$ and $\{r_{n,i}\},\{t_{n,j}\}\subset(0,+\infty)$ satisfy the some conditions presented in Theorem* 3.3.

Let $\{x_n\}$ be generated by the following scheme:

$$
\begin{aligned}
&x_1\in E, \\
&u_n=(1-\alpha_n)(x_n+e_n), \\
&v_n=(1-\beta_n)x_n+\beta_nS_nu_n, \\
&x_{n+1}=\gamma_nx_n+(1-\gamma_n)U_nS_nv_n, \quad n\ge 1.
\end{aligned}
\tag{D}
$$

Then $\{x_n\}$ converges strongly to a point $p_0\in D$, where S_n and U_n are the same as those in Theorem 3.3.

Next, we apply Theorems 3.1 and 3.3 to the cases of finite pseudo-contractive mappings.

Theorem 3.5 *Let E be a real uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny nonexpansive retract of E, where Q_C is the sunny* *nonexpansive retraction of E onto C. Let $T_i^{(1)}, T_j^{(2)} : C \to E$ be pseudo-contractive mappings such that $(I - T_i^{(1)})$ and $(I - T_j^{(2)})$ are m-accretive, where $i = 1, 2, \ldots N$, $j = 1, 2, \ldots, M$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous and $D := (\bigcap_{i=1}^N F(T_i^{(1)})) \cap (\bigcap_{j=1}^M F(T_j^{(2)})) \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative algorithm* (A), *where $S_n := a_0 I + a_1 J_{r_{n,1}}^{I-T_1^{(1)}} + a_2 J_{r_{n,2}}^{I-T_2^{(1)}} + \cdots + a_N J_{r_{n,N}}^{I-T_N^{(1)}}$, and $J_{r_{n,i}}^{I-T_i^{(1)}} = [I + r_{n,i}(I - T_i^{(1)})]^{-1}$, for $i = 1, 2, \ldots, N$, $0 < a_k < 1$, for $k = 0, 1, 2, \ldots, N$, $\sum_{k=0}^N a_k = 1$. $W_n = b_0 I + b_1 J_{s_{n,1}}^{I-T_1^{(2)}} + b_2 J_{s_{n,2}}^{I-T_2^{(2)}} + \cdots + b_M J_{s_{n,M}}^{I-T_M^{(2)}}$, where $J_{s_{n,j}}^{I-T_j^{(2)}} = [I + s_{n,j}(I - T_j^{(2)})]^{-1}$, for $j = 1, 2, \ldots, M$, $0 < b_k < 1$, for $k = 0, 1, 2, \ldots, M$, $\sum_{k=0}^M b_k = 1$. Suppose $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in* $(0,1)$ *and $\{r_{n,i}\}, \{s_{n,j}\} \subset (0, +\infty)$ satisfying the following conditions*:

(i) $\alpha_n \to 0$, $\beta_n \to 0$, *as* $n \to \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n \beta_n = +\infty$;
(iii) $0 < \liminf_{n\to+\infty} \gamma_n \leq \limsup_{n\to+\infty} \gamma_n < 1$;
(iv) $\sum_{n=1}^\infty |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \geq \varepsilon > 0$, *for* $n \geq 1$ *and* $i = 1, 2, \ldots, N$;
(v) $\sum_{n=1}^\infty |s_{n+1,j} - s_{n,j}| < +\infty$ *and* $s_{n,j} \geq \varepsilon > 0$, *for* $n \geq 1$ *and* $j = 1, 2, \ldots, M$;
(vi) $\frac{\|e_n\|}{\alpha_n} \to 0$, *as* $n \to +\infty$, *and* $\sum_{n=1}^\infty \|e_n\| < +\infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$.

Proof Let $A_i = (I - T_i^{(1)})$ and $B_j = (I - T_j^{(2)})$, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Then the result follows from Theorem 3.1. □

Similarly, from Theorem 3.3, we have the following result.

Theorem 3.6 *Let E, C, Q_C and D be the same as those in Theorem* 3.5. *Let $T_i^{(1)}, T_j^{(2)} : C \to E$ be pseudo-contractive mappings such that $(I - T_i^{(1)})$ and $(I - T_j^{(2)})$ are m-accretive mappings, where $i = 1, 2, \ldots N$, $j = 1, 2, \ldots, M$. Suppose that the duality mapping $J : E \to E^*$ is weakly sequentially continuous and $D \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative algorithm* (B), *where S_n is the same as that in Theorem* 3.5 *and $U_n = c_0 I + c_1 J_{t_{n,1}}^{I-T_1^{(2)}} + c_2 J_{t_{n,2}}^{I-T_2^{(2)}} J_{t_{n,1}}^{I-T_1^{(2)}} + \cdots + c_M J_{t_{n,M}}^{I-T_M^{(2)}} J_{t_{n,M-1}}^{I-T_{M-1}^{(2)}} \cdots J_{t_{n,1}}^{I-T_1^{(2)}}$, where $J_{t_{n,j}}^{I-T_j^{(2)}} = [I + t_{n,j}(I - T_j^{(2)})]^{-1}$, for $j = 1, 2, \ldots, M$, $0 < c_k < 1$, for $k = 0, 1, 2, \ldots, M$, $\sum_{k=0}^M c_k = 1$. Suppose $\{e_n\} \subset E$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in* $(0,1)$ *and $\{r_{n,i}\}, \{t_{n,j}\} \subset (0, +\infty)$ satisfying the following conditions*:

(i) $\alpha_n \to 0$, $\beta_n \to 0$, *as* $n \to \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n \beta_n = +\infty$;
(iii) $0 < \liminf_{n\to+\infty} \gamma_n \leq \limsup_{n\to+\infty} \gamma_n < 1$;
(iv) $\sum_{n=1}^\infty |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $r_{n,i} \geq \varepsilon > 0$, *for* $n \geq 1$ *and* $i = 1, 2, \ldots, N$;
(v) $\sum_{n=1}^\infty |t_{n+1,j} - t_{n,j}| < +\infty$ *and* $s_{n,j} \geq \varepsilon > 0$, *for* $n \geq 1$ *and* $j = 1, 2, \ldots, M$;
(vi) $\frac{\|e_n\|}{\alpha_n} \to 0$, *as* $n \to +\infty$, *and* $\sum_{n=1}^\infty \|e_n\| < +\infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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