Kirk and Shahzad *Fixed Point Theory and Applications* 2014, **2014**:177
http://www.fixedpointtheoryandapplications.com/content/2014/1/177

Fixed Point Theory and Applications
a SpringerOpen Journal

CORRECTION **Open Access**

Correction: Generalized metrics and Caristi's theorem

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The assertion in [1] that Caristi's theorem holds in generalized metric spaces is based, among other things, on the false assertion that if $\{p_n\}$ is a sequence in a generalized metric space (X,d), and if $\{p_n\}$ satisfies $\sum_{i=1}^{\infty} d(p_i,p_{i+1}) < \infty$, then $\{p_n\}$ is a Cauchy sequence. In Example 1 below we give a counter-example to this assertion, and in Example 2 we show that, in fact, Caristi's theorem fails in such spaces. We apologize for any inconvenience.

For convenience we give the definition of a generalized metric space. The concept is due to Branciari [2].

Definition 1 Let X be a nonempty set and $d : X \times X \to [0,\infty)$ a mapping such that for all $x,y \in X$ and all distinct points $u,v \in X$, each distinct from x and y:

(i) $d(x,y) = 0 \Leftrightarrow x = y$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \le d(x,u) + d(u,v) + d(v,y)$ (quadrilateral inequality).

Then X is called a *generalized metric space*.

The following example is a modification of Example 1 of [3].

Example 1 Let $X := \mathbb{N}$, and define the function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by putting, for all $m,n \in \mathbb{N}$ with $m > n$:

$$d(n,n) := 0;$$

$$d(m,n) = d(n,m) := \frac{1}{2^n} \quad \text{if } m = n+1;$$

$$d(m,n) = d(n,m) := 1 \quad \text{if } m-n \text{ is even};$$

$$d(m,n) = d(n,m) := \sum_{i=n}^{m} d(i,i+1) \quad \text{if } m-n \text{ is odd}.$$

To see that (X,d) is a generalized metric space, suppose $m,n \in \mathbb{N}$ with $m > n$ and suppose $p,q \in \mathbb{N}$ are distinct with each distinct from m and n. Also we assume $q > p$. We now show that

$$d(n,m) \le d(n,p) + d(p,q) + d(q,m). \tag{Q}$$

If one of the three numbers $|n-p|$, $q-p$ or $|q-m|$ is even, then, since

$$d(n,m) \le 1,$$

clearly (Q) holds. If all three numbers are odd, then, since $m-n=(m-q)+(q-p)+(p-n)$, $m-n$ is odd and

$$d(n,m)=\sum_{i=n}^{m} d(i,i+1).$$

In this instance there are four cases to consider:

(i) $n<p<q<m$,
(ii) $p<n<q<m$,
(iii) $n<p<m<q$,
(iv) $p<n<m<q$.

If (i) holds then

$$\begin{aligned} d(n,m) &= \sum_{i=n}^{m} d(i,i+1) \\ &= \sum_{i=n}^{p} d(i,i+1) + \sum_{i=p}^{q} d(i,i+1) + \sum_{i=q}^{m} d(i,i+1) \\ &= d(n,p)+d(p,q)+d(q,m). \end{aligned}$$

In the other three cases

$$d(n,m) < d(n,p)+d(p,q)+d(q,m).$$

Therefore (X,d) is a generalized metric space. Now suppose $\{n_k\}$ is a Cauchy sequence in (X,d). Then if $n_i \neq n_k$ and $d(n_i,n_k)<1$, $|n_i-n_k|$ must be odd. However, if $\{n_k\}$ is infinite, $|n_i-n_k|$ cannot be odd for all sufficiently large i, k. (Suppose $n_i>n_j>n_k$. If n_i-n_j and n_j-n_k are odd, then n_i-n_k is even.) Thus any Cauchy sequence in (X,d) must eventually be constant. It follows that (X,d) is complete and that $\{n\}$ is not a Cauchy sequence in (X,d). However, $\sum_{i=1}^{\infty} d(i,i+1)<\infty$.

Theorem 2 of [1] asserts that the analog of Caristi's theorem holds in a complete generalized metric space (X,d). Thus a mapping $f: X\to X$ in such a space should always have a fixed point if there exists a lower semicontinuous function $\varphi: X\to \mathbb{R}^+$ such that

$$d\bigl(x,f(x)\bigr) \le \varphi(x)-\varphi\bigl(f(x)\bigr) \quad \text{for each } x\in X.$$

The following example shows this is not true in the space described in Example 1.

Example 2 Let (X,d) be the space of Example 1, let $f(n)=n+1$ for $n\in\mathbb{N}$, and define $\varphi:\mathbb{N}\to\mathbb{R}^+$ by setting $\varphi(n)=\frac{2}{n}$. Obviously f has no fixed points and, because the space is discrete, φ is continuous. On the other hand, f satisfies Caristi's condition:

$$\frac{1}{2^n}=d\bigl(n,f(n)\bigr)\le \varphi(n)-\varphi\bigl(f(n)\bigr)=\frac{2}{n}-\frac{2}{n+1}.$$

To see this, observe that

$$\frac{1}{2^n} \leq \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)}.$$

This is equivalent to the assertion that

$$2^{n+1} \geq n(n+1). \tag{C}$$

The proof is by induction. Clearly (C) holds if $n = 1$ or $n = 2$. Assume (C) holds for some $n \in \mathbb{N}$, $n \geq 2$. Then

$$\begin{aligned} 2^{n+2} &= 2\left(2^{n+1}\right) \\ &\geq 2n(n+1) \\ &= (n+n)(n+1) \\ &\geq (n+1)(n+2). \end{aligned}$$

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Received: 1 August 2014 Accepted: 1 August 2014 Published: 19 August 2014

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doi:10.1186/1687-1812-2014-177
Cite this article as: Kirk and Shahzad: **Correction: Generalized metrics and Caristi's theorem.** *Fixed Point Theory and Applications* 2014 **2014**:177.