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Best approximation and fixed-point theorems for discontinuous increasing maps in Banach lattices

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Abstract
In this paper, we extend and prove Ky Fan's Theorem for discontinuous increasing maps f in a Banach lattice X when f has no compact conditions. The main tools of analysis are the variational characterization of the generalized projection operator and order-theoretic fixed-point theory. Moreover, we establish a sequence $\{x_n\}$ which converges strongly to the unique best approximation point. As an application of our best approximation theorems, a fixed-point theorem for non-self maps is established and proved under some conditions. Our results generalize and improve many recent results obtained by many authors.

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Keywords: best approximation theorem; generalized projection operator; discontinuous increasing map; Banach lattice

1 Introduction

Ky Fan's approximation theorem (Theorem 2 in [1]) has attracted great attention worldwide over the last few decades. The normed space version of the theorem is as follows.

Theorem *Let K be a non-empty compact convex set in a normed linear space X. If f is a continuous map from K into X, then there exists a point u in K such that $\|u - f(u)\| = d(f(u), K)$. The point u in the theorem above is called a best approximation point of f in K.*

Ky Fan's Theorem is of great importance in nonlinear analysis, approximation theory, game theory and minimax theorems. In recent years, the theorem has been studied and generalized in various respects and applied in the analysis of many problems. Lin and Park [2], O'Regan and Shahzad [3] obtained a multivalued version of Ky Fan's result for condensing maps. Tan and Yuan [4] and Liu [5, 6] extended the theorem to the more general continuous 1-set-contractive maps under some stronger hypothesis. In the last decade, the study of random approximations and random fixed points have been a very active area of research in probabilistic functional analysis. Some results have already been achieved in this line such as those by Lin [7], Seghal and Singh [8], Seghal and Water [9], Liu [10, 11], Tan and Yuan [4], Beg and Shahzad [12]. Meanwhile, Lin [13] proved Fan's theorem for a continuous condensing map defined on a closed ball in a Banach space. Subsequently, Lin and Yen [14] proved that Ky Fan's Theorem is true for a semi-contractive map defined on a closed convex subset of a Hilbert space. Very recently, Liu [5] proved that Ky Fan's Theorem is true for the 1-set-contractive maps defined on a bounded closed convex subset in a Banach space when $\|\cdot\|$ is replaced by Minkowski's function. For more results, the reader is referred to Shahzad [15], Markin and Shahzad [16], Amini-Harandi [17], Roux and Singh [18], Liu [19, 20], O'Regan [21], and so on.

However, so far, Ky Fan's Theorem has not been well investigated for the cases where f is a discontinuous map and has no compact conditions. Partly motivated by this difficulty, Alber [22] introduced the notion of a generalized projection operator and noted that Π_C can be used instead of P_C in Banach space. Based on this concept, Li and Ok [23] proved that the metric projection operator is order-preserving in partially ordered Banach spaces. Motivated and inspired by the above mentioned work, in this paper, we obtain two best approximation theorems through the order-theoretic fixed-point theorems by using Π_C instead of P_C for reflexive, strictly convex and smooth Banach lattice. In the first best approximation theorem, we establish a sequence $\{x_n\}$ which converges strongly to the unique best approximation point; while in the second best approximation theorem, we obtain the existence of a minimum best approximation point and a maximum best approximation point in order intervals. As an application of our best approximation theorems, a fixed-point theorem for non-self maps is established under some conditions which do not need to require any continuous and compact conditions on f.

The rest of the paper is organized as follows. In Section 2, we review the definition of the generalized projection operator in Banach spaces and its basic properties, and also give some definitions in Banach lattice and some fundamental results as preliminaries for our theorems. In Section 3, we establish the properties of the generalized projection operator in Banach lattice under some assumptions. Then we combine these results with an order-theoretic fixed-point theorem to derive some best approximation theorems. Section 4 provides an application of these best approximation theorems to the fixed-point theory.

2 Preliminaries

2.1 The generalized projection operator

Let X be a real Banach space with the dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\| \|x\|, \|x\| = \|x^*\|\}, \tag{2.1}$$

for all $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X^* and X. It is well known that if X is reflexive, strictly convex and smooth, J is a surjective, injective, and single-valued map.

Let X be a reflexive, strictly convex and smooth Banach space and C a non-empty closed convex subset of X. Consider the Lyapunov functional defined by

$$W(x,y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2, \quad \forall x,y \in X. \tag{2.2}$$

Following Alber [22], the generalized projection operator $\Pi_C : X \to C$ is a map that assigns to an arbitrary point $x \in X$ the minimum point of the functional $W(x,y)$, that is, $\Pi_C(x) = \hat{x}$, where $\hat{x} \in C$ is the solution to the minimization problem

$$W(x,\hat{x}) = \inf_{y\in C} W(x,y). \tag{2.3}$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $W(x,y)$ and the strict monotonicity of the mapping J. It is obvious from the definition of the functional W that

$$\bigl(\|x\|-\|y\|\bigr)^2 \le W(x,y) \le \bigl(\|x\|+\|y\|\bigr)^2, \quad \forall x,y\in X. \tag{2.4}$$

If X is a Hilbert space, then $W(x,y)=\|x-y\|^2$ and $\Pi_C=P_C$.

If X is a reflexive, strictly convex, and smooth Banach space, then for $x,y\in X$, $W(x,y)=0$ if and only if $x=y$. It is sufficient to show that if $W(x,y)=0$ then $x=y$. From (2.4), we have $\|x\|=\|y\|$. This implies that $\langle Jx,y\rangle=\|y\|^2=\|Jx\|^2$. From the definition of J, one has $Jx=Jy$. Therefore, we have $x=y$; and for more details, the reader is referred to [24, 25].

As shown in [22], the generalized projection operator on a convex closed set C satisfies the following properties:

(i) The operator Π_C is fixed in each point $x\in C$, *i.e.*, $\hat{x}=x$.

(ii) The operator Π_C is d-accretive in X, *i.e.*,

$$\langle Jx-Jy,\hat{x}-\hat{y}\rangle \ge 0, \quad \forall x,y\in X. \tag{2.5}$$

(iii) The point $\Pi_C(x)=\hat{x}$ is a generalized projection of x on $C\subset X$ if and only if the following inequality is satisfied:

$$\langle Jx-J\hat{x},\hat{x}-y\rangle \ge 0, \quad \forall y\in C. \tag{2.6}$$

(iv) The operator Π_C gives the absolutely best approximation of $x\in X$ relative to the functional $W(x,y)$, *i.e.*,

$$W(\hat{x},y)=W(x,y)-W(x,\hat{x}), \quad \forall y\in C. \tag{2.7}$$

2.1.1 Banach lattices

Let $(X,\preceq)$ be a real partially ordered Banach space with the dual X^* and S be a subset of X. We say that an element x of X is an upper bound for S if $S\preceq x$, that is, $y\preceq x$ for each $y\in S$ (the notation $S\succeq x$ is similarly understood). We say that S is bounded from above if $S\preceq x$ for some $x\in X$, and bounded from below if $S\succeq x$ for some $x\in X$. In turn, S is said to be bounded if it is bounded both from above and below. The supremum of S is the minimum of the set of all upper bounds for S, and is denoted by $\bigvee_X S$ (the infimum of S is denoted as $\bigwedge_X S$). As is conventional, we denote $\bigvee_X\{x,y\}$ as $x\vee y$, and $\bigwedge_X\{x,y\}$ as $x\wedge y$, for any $x,y\in X$. If $x\vee y$ and $x\wedge y$ exist for every x and y in X, we say that $(X,\preceq)$ is a lattice. And if $\bigvee_X S$ and $\bigwedge_X S$ exist for every non-empty (bounded) $S\subseteq X$, we say that $(X,\preceq)$ is a (Dedekind) complete lattice. If Y is a non-empty subset of X which contains $x\wedge y$ and $x\vee y$ for every $x,y\in Y$, then Y is said to be a sublattice of X.

A normed lattice X is a vector lattice with a norm $\|\cdot\|$ such that the following condition is satisfied:

$$|x|\preceq|y| \quad \text{implies} \quad \|x\|\le\|y\|, \quad \text{for all } x,y\in X,$$

where $|x|$ is defined by $|x|=x\vee(-x)$ for each $x\in X$.

A Riesz space is a lattice $(X, \preceq)$ where X is a (real) linear space whose linear structure is compatible with the partial order $\preceq$ in the sense that for all $x, y \in X$, $x \preceq y$ implies $\alpha x + z \preceq \alpha y + z$ for every $z \in X$ and real number $\alpha > 0$. The positive cone of $(X, \preceq)$ is $X_+ = \{x \in X : x \succeq \theta\}$, which is a pointed convex cone in X. We will assume throughout the paper that the positive cones is closed.

Let $(X, \preceq)$ be a Banach lattice, that is, $(X, \preceq)$ is an ordered Riesz space with X being Banach space (if X is a Hilbert space here, $(X, \preceq)$ is referred to as a Hilbert lattice). The cone X_+ is said to be solid if X_+ has a non-empty interior *i.e.* $\operatorname{int} X_+ \neq \emptyset$. The cone X_+ is said to be normal if there is a number $K > 0$ such that for all $x, y \in X$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying this inequality is called the normal constant of X_+.

Definition 2.1 ([23]) Let $(X, \preceq)$ be a Banach lattice, a sublattice Y of X is said to be regular if $\|\cdot\|^2$ is submodular on Y with respect to $\preceq$, that is,

$$\|x \wedge y\|^2 + \|x \vee y\|^2 \leq \|x\|^2 + \|y\|^2, \quad \forall x, y \in Y. \tag{2.8}$$

Obviously, if $(X, \preceq)$ is itself regular, then every sublattice of X is regular. We know every Hilbert normed lattice is regular and the positive cones of many Banach lattices are regular. For example, if $p \geq 2$, every sublattice S of $R^{n,p}$ with $S \subseteq R^n_+$ is regular; if $p \geq 2$, every sublattice S of ℓ^p with $S \subseteq \ell^p_+$ is regular.

Definition 2.2 ([23]) For any lattices $(X, \preceq_X)$ and $(Y, \preceq_Y)$, we say that a map $F : X \to Y$ is order-preserving if $x \preceq_X y$ implies $F(x) \preceq_Y F(y)$.

2.1.2 Order-dual

Let $(X, \preceq)$ be a Banach lattice. The dual of $\preceq$ is the partial order $\preceq^*$ on X^* defined as follows:

$$\phi \preceq^* \varphi \quad \text{iff} \quad \langle \varphi - \phi, x\rangle \geq 0, \quad \forall x \in X_+. \tag{2.9}$$

It is well known that $(X^*, \preceq^*)$ is a Banach lattice, which is called the dual of $(X, \preceq)$. As usual, we denote the positive cone of $(X^*, \preceq^*)$ by X^*_+, and recall that $x \in X_+$ iff $\langle \varphi, x\rangle \geq 0$ for every $\varphi \in X^*_+$ (see Meyer-Nieberg [26], Proposition 1.4.2).

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $K \subseteq H$ be a closed convex cone. Recall that $K^* = \{x \in H : \langle x, y\rangle \geq 0, \forall y \in K\}$ is called the dual cone of K. The cone K is called subdual if $K \subseteq K^*$ and superdual if $K^* \subseteq K$. Suppose $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert lattice and K be its positive cone, for any $x, y \in H$, we denote the minimum (supremum) with respect to K as $x \wedge y$ ($x \vee y$) and the minimum (supremum) with respect to K^* as $x \wedge^* y$ ($x \vee^* y$).

The following fixed-point theorem is fundamental for the proof of the best approximation theorem.

Theorem 2.1 ([27]) *Let K be a normal and solid cone. Suppose that $f : K \to K$ is increasing and satisfies the following conditions*:

(i) *There exist $v \in \operatorname{int} K$ and $c > 0$, such that $\theta \prec f(v) \preceq v$, $f(\theta) \succeq cf(v)$.*
(ii) *For any $0 < a < b < 1$ and any bounded subset $B \subset K$, there exists $\eta(a, b, B) > 0$ such that*

$$f(tx) \succeq t(1 + \eta)f(x), \quad \forall x \in B, t \in [a, b]. \tag{2.10}$$

Then f has a unique fixed point x^* in K such that $\theta \prec x^* \preceq v$. Moreover, if $\forall x_0 \in K$ such that $x_n = f(x_{n-1})$ $(n = 1, 2, 3, \ldots)$, then $\|x_n - x^*\| \to 0$ for $n \to \infty$.

We denote $d_W(x, K) = \inf\{W(x, y) : y \in K\}$, where $x \in X$ and W is a Lyapunov functional in X.

3 Best approximation theorems

First we establish the following properties of the generalized projection operators.

Lemma 3.1 *Let $(X, \preceq)$ be a partially ordered space, and let X_+ be its positive cone, then*

$$\Pi_{X_+}(tx) = t\Pi_{X_+}(x), \quad \forall t > 0, x \in X.$$

Proof For every $t > 0$, $x \in X$, we take $y \in X_+$. It is obvious that $\frac{y}{t} \in X_+$, and so by equation (2.6) we have

$$\left\langle Jx - J\Pi_{X_+}(x), \Pi_{X_+}(x) - \frac{y}{t} \right\rangle \geq 0, \tag{3.1}$$

and further, by the positive homogeneity of J, we get

$$\langle Jtx - Jt\Pi_{X_+}(x), t\Pi_{X_+}(x) - y\rangle \geq 0, \quad \forall y \in X_+. \tag{3.2}$$

Using (2.6), we obtain $\Pi_{X_+}(tx) = t\Pi_{X_+}(x)$. □

Lemma 3.2 *For a reflexive, strictly convex, and smooth Banach lattice $(X, \preceq)$, the following statements are equivalent*:

(H_1) *The normalized duality mapping J is order-preserving*;
(H_2) $\forall x, y \in X$, $x \preceq y$ *implies* $\|Jx \wedge Jy\|^2 + \|Jx \vee Jy\|^2 \leq \|x\|^2 + \|y\|^2$.

Proof (H_1) $\Rightarrow$ (H_2) If J is order-preserving, for $x \preceq y$, we have $Jx \wedge Jy = Jx$, $Jx \vee Jy = Jy$. It is thus obvious that (H_2) holds.

(H_2) $\Rightarrow$ (H_1) Assume that J is not order-preserving, then there exist $x_0, y_0 \in X$, $x_0 \preceq y_0$ such that $Jx_0 \wedge Jy_0 \neq Jx_0$, $Jx_0 \vee Jy_0 \neq Jy_0$. Since X is a reflexive Banach lattice, J is surjective and X^* is a Banach lattice, which implies that there exist $z_1 \neq x_0$, $z_2 \neq y_0 \in X$, such that $Jx_0 \wedge Jy_0 = Jz_1$, $Jx_0 \vee Jy_0 = Jz_2$. Indeed we have

$$2\langle Jx_0 \wedge Jy_0, x_0\rangle \leq 2\|Jx_0 \wedge Jy_0\|\|x_0\| \leq \|Jx_0 \wedge Jy_0\|^2 + \|x_0\|^2. \tag{3.3}$$

Since X is strictly convex, which implies only in the case $z_1 = tx_0$, $t \geq 0$, the relation $2\langle Jx_0 \wedge Jy_0, x_0\rangle = 2\|Jx_0 \wedge Jy_0\|\|x_0\|$ holds. Moreover, only in the case $\|z_1\| = \|x_0\|$, the relation $2\|Jx_0 \wedge Jy_0\|\|x_0\| = \|Jx_0 \wedge Jy_0\|^2 + \|x_0\|^2$ holds. This obviously implies $2\langle Jx_0 \wedge Jy_0, x_0\rangle = \|Jx_0 \wedge Jy_0\|^2 + \|x_0\|^2$ if and only if $z_1 = x_0$. From the assumption, it is impossible that $z_1 = x_0$. Thus

$$2\langle Jx_0 \wedge Jy_0, x_0\rangle < \|Jx_0 \wedge Jy_0\|^2 + \|x_0\|^2. \tag{3.4}$$

In a similar way, we get

$$2\langle Jx_0 \vee Jy_0, y_0\rangle < \|Jx_0 \vee Jy_0\|^2 + \|y_0\|^2. \tag{3.5}$$

Adding equations (3.4) and (3.5), we have

$$\begin{aligned} &2\langle Jx_0 \wedge Jy_0, x_0\rangle + 2\langle Jx_0 \vee Jy_0, y_0\rangle \\ &\quad < \|Jx_0 \wedge Jy_0\|^2 + \|x_0\|^2 + \|Jx_0 \vee Jy_0\|^2 + \|y_0\|^2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &2\langle Jx_0 \wedge Jy_0 - Jx_0, x_0\rangle + 2\langle Jx_0 \vee Jy_0 - Jy_0, y_0\rangle \\ &\quad < \|Jx_0 \wedge Jy_0\|^2 - \|x_0\|^2 + \|Jx_0 \vee Jy_0\|^2 - \|y_0\|^2. \end{aligned} \tag{3.7}$$

Using $Jx_0 \wedge Jy_0 + Jx_0 \vee Jy_0 = Jx_0 + Jy_0$, we obtain

$$\begin{aligned} &2\langle Jx_0 \vee Jy_0 - Jy_0, -x_0\rangle + 2\langle Jx_0 \vee Jy_0 - Jy_0, y_0\rangle \\ &\quad < \|Jx_0 \wedge Jy_0\|^2 + \|Jx_0 \vee Jy_0\|^2 - \|x_0\|^2 - \|y_0\|^2, \end{aligned} \tag{3.8}$$

and thus

$$2\langle Jx_0 \vee Jy_0 - Jy_0, y_0 - x_0\rangle < \|Jx_0 \wedge Jy_0\|^2 + \|Jx_0 \vee Jy_0\|^2 - \|x_0\|^2 - \|y_0\|^2. \tag{3.9}$$

Since $Jx_0 \vee Jy_0 - Jy_0 \in X_+^*$ and $y_0 - x_0 \in X_+$, we have

$$\langle Jx_0 \vee Jy_0 - Jy_0, y_0 - x_0\rangle \geq 0.$$

This contradicts (H_2). So J is order-preserving and the assertion is proved. □

Lemma 3.3 *Let* $(X, \preceq)$ *be a reflexive, strictly convex, smooth Banach lattice and satisfy condition* (H_2) *and* C *a closed convex regular sublattice of* X. *Then,* Π_C *is increasing.*

Proof To derive a contradiction, assume that Π_C is not increasing. Then, there exist $x_0, y_0 \in X$, $x_0 \preceq y_0$ such that $\Pi_C x_0 \wedge \Pi_C y_0 \neq \Pi_C x_0$, $\Pi_C x_0 \vee \Pi_C y_0 \neq \Pi_C y_0$. Because C is a sublattice of X, we have $\Pi_C x_0 \wedge \Pi_C y_0 \in C$. It follows from the definition of Π_C that $W(x_0, \Pi_C x_0) < W(x_0, \Pi_C x_0 \wedge \Pi_C y_0)$, that is,

$$2\langle Jx_0, \Pi_C x_0 \wedge \Pi_C y_0 - \Pi_C x_0\rangle < \|\Pi_C x_0 \wedge \Pi_C y_0\|^2 - \|\Pi_C x_0\|^2. \tag{3.10}$$

On the other hand, as $\Pi_C x_0 \vee \Pi_C y_0 \in C$, we trivially have $W(y_0, \Pi_C y_0) < W(y_0, \Pi_C x_0 \vee \Pi_C y_0)$, that is,

$$2\langle Jy_0, \Pi_C x_0 \vee \Pi_C y_0 - \Pi_C y_0\rangle < \|\Pi_C x_0 \vee \Pi_C y_0\|^2 - \|\Pi_C y_0\|^2. \tag{3.11}$$

Using the fact that $x \vee y + x \wedge y = x + y$, we can write the inequality (3.10) as

$$2\langle -Jx_0, \Pi_C x_0 \vee \Pi_C y_0 - \Pi_C y_0\rangle < \|\Pi_C x_0 \wedge \Pi_C y_0\|^2 - \|\Pi_C x_0\|^2. \tag{3.12}$$

Combining equations (3.11) and (3.12) yields

$$\begin{aligned} &2\langle Jy_0 - Jx_0, \Pi_C x_0 \vee \Pi_C y_0 - \Pi_C y_0 \rangle \\ &\quad < \|\Pi_C x_0 \wedge \Pi_C y_0\|^2 + \|\Pi_C x_0 \vee \Pi_C y_0\|^2 - \|\Pi_C y_0\|^2 - \|\Pi_C x_0\|^2. \end{aligned} \tag{3.13}$$

Thus, by the regularity of C, we get

$$\langle Jy_0 - Jx_0, \Pi_C x_0 \vee \Pi_C y_0 - \Pi_C y_0 \rangle < 0.$$

By Lemma 3.2, $Jy_0 - Jx_0 \in X_+^*$, and so $\Pi_C x_0 \vee \Pi_C y_0 - \Pi_C y_0$ does not belong to X_+, which is a contradiction. This proves that Π_C is increasing. □

Lemma 3.4 *Let H be a Hilbert normed lattice with its positive cone K and C a closed convex sublattice of H. Then, P_C is increasing.*

Proof It is well known that J is an identity function in Hilbert space. $\forall x, y \in K$, we have $|x-y| = (x-y) \vee (y-x) \preceq x+y$. As H is a normed lattice, we get $\|x-y\| \leq \|x+y\|$, that is, $\|x-y\|^2 \leq \|x+y\|^2$. Furthermore, $\langle x, y\rangle \geq 0$. Thus, $x \in K^*$. Conclusion: K is subdual cone. Let $x, y \in H$ be such that $x \preceq y$, we have $y - x \in K \subseteq K^*$, that is, $x \preceq^* y$. Thus $x \vee^* y = y$ and $x \wedge^* y = x$. We have $\|x \vee^* y\|^2 + \|x \wedge^* y\|^2 = \|x\|^2 + \|y\|^2$. So (H_2) holds. From Lemma 3.2.1 in [23], we know that C is regular. By Lemma 3.3, P_C is increasing. □

From Theorem 2.1 and the properties of the generalized projection operator, we obtain the following best approximation theorems.

Theorem 3.1 *Let $(X, \preceq)$ be a reflexive, strictly convex, smooth Banach lattice satisfying condition (H_2), and X_+ be normal, solid and regular. Suppose that $f : X_+ \to X$ is increasing and satisfies the following conditions*:

(H_3) *There exist $v \in \operatorname{int} X_+$ and $c > 0$, such that $f(v) \preceq v$, $\Pi_{X_+}(f(v)) \neq \theta$ and $f(\theta) \succeq cf(v)$.*

(H_4) *For any $0 < a < b < 1$ and any bounded subset $B \subset X_+$, there exists $\eta(a,b,B) > 0$ such that*

$$f(tx) \succeq t(1+\eta) f(x), \quad \forall x \in B, t \in [a,b]. \tag{3.14}$$

Then f has a unique point x^* in X_+, satisfying $\theta \prec x^* \preceq v$, such that $W(f(x^*), x^*) = d_W(f(x^*), X_+)$. Moreover, if $x_0 \in X_+$ and $x_n = \Pi_{X_+}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$, then $\|x_n - x^*\| \to 0$ for $n \to \infty$.

Proof Define $F : X_+ \to X_+$ by $F(x) = \Pi_{X_+}(f(x))$. It is obvious that X_+ is a sublattice of X. By Lemma 3.3, it is easy to see that F is increasing. Since Π_{X_+} is increasing and $f(v) \preceq v$, $f(\theta) \succeq cf(v)$, we get

$$\theta \prec \Pi_{X_+}\big(f(v)\big) \preceq \Pi_{X_+}(v) = v. \tag{3.15}$$

Using Lemma 3.1, we have

$$\Pi_{X_+}\big(f(\theta)\big) \succeq \Pi_{X_+}\big(cf(v)\big) = c\Pi_{X_+}\big(f(v)\big). \tag{3.16}$$

From $f(tx) \succeq t(1+\eta)f(x)$, $\forall x \in B$, $t \in [a,b]$, we obtain

$$\Pi_{X_+}\big(f(tx)\big) \succeq \Pi_{X_+}\big(t(1+\eta)f(x)\big) = t(1+\eta)\Pi_{X_+}\big(f(x)\big). \tag{3.17}$$

Thus F satisfies all conditions of Theorem 2.1, and so F has a unique fixed point x^* in X_+, such that $\theta \prec x^* \preceq v$, and $\|x_n - x^*\| \to 0$ for $n \to \infty$. Now we consider $F(x^*) = x^*$, *i.e.* $\Pi_{X_+}(f(x^*)) = x^*$. By the definition of Π_{X_+}, we get

$$W\big(f\big(x^*\big),x^*\big) = \inf_{y\in X_+} W\big(f\big(x^*\big),y\big) = d_W\big(f\big(x^*\big),X_+\big).$$

The assertion is proved. □

Remark 3.1 In Theorem 3.1, f is a discontinuous map and has no compact conditions.

Corollary 3.1 *Let H be a Hilbert normed lattice and its positive cone K be solid. Suppose that $f: K \to H$ is increasing and satisfies* (H_3), (H_4) *in K. Then f has a unique point x^* in K, satisfying $\theta \prec x^* \preceq v$, such that $\|f(x^*) - x^*\| = d(f(x^*),K) = \inf_{y\in K}\|f(x^*) - y\|$. Moreover, if $x_0 \in K$ and $x_n = P_K(f(x_{n-1}))$ $(n = 1,2,3,\ldots)$, then $\|x_n - x^*\| \to 0$ for $n \to \infty$.*

Proof The assertion follows from the above Lemma 3.4 and Theorem 3.1. □

Let $(X,\preceq)$ be a Banach lattice. Given $u_0, v_0 \in X$ such that $u_0 \prec v_0$, we denote by $[u_0,v_0]$ the set:

$$[u_0,v_0] = \{z \in X : u_0 \preceq z \preceq v_0\}.$$

It is easy to see that $[u_0,v_0]$ is a sublattice of X.

Theorem 3.2 *Let $(X,\preceq)$ be a reflexive, strictly convex, smooth Banach and Dedekind complete lattice satisfying condition* (H_2). *Suppose that $[u_0,v_0]$ is regular and $f:[u_0,v_0] \to X$ is increasing. Then, f has a minimum best approximation point x_* and a maximum best approximation point x^* with respect to $W(x,y)$ in $[u_0,v_0]$, such that*

$$u_0 \preceq u_1 \preceq \cdots \preceq u_n \preceq \cdots \preceq x_* \preceq x^* \preceq \cdots \preceq v_n \preceq \cdots \preceq v_1 \preceq v_0, \tag{3.18}$$

where $u_n = \Pi_{[u_0,v_0]}(f(u_{n-1}))$, $v_n = \Pi_{[u_0,v_0]}(f(v_{n-1}))$ $(n = 1,2,3,\ldots)$.

Proof Define $F:[u_0,v_0] \to [u_0,v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. From Lemma 3.3, we see that F is increasing. It is easy to see that $u_0 \preceq F(u_0)$ and $F(v_0) \preceq v_0$. Thus, F satisfies all conditions of Theorem 2.1.2 in [28]. Then, F has a minimum fixed point x_* and a maximum fixed point x^* and satisfies (3.18). By the definition of $\Pi_{[u_0,v_0]}$, the assertion is proved. □

Remark 3.2 In Theorem 3.2, f is a discontinuous map and has no compact conditions.

Example 3.1 Let $(X,\preceq) = (\ell^2,\preceq)$. Here $\preceq$ stands for the coordinatewise ordering. Given $u_0, v_0 \in \ell^2$ such that $u_0 \prec v_0$. Then, by Theorem 3.2, every increasing $f:[u_0,v_0] \to \ell^2$ has a minimum best approximation point and a maximum best approximation point with respect to $W(x,y)$ in $[u_0,v_0]$.

Example 3.2 Let $(X, \preceq) = (L^2(\Omega), \preceq)$, the space of measurable functions which are 2nd power summable on Ω. Endow $L^2(\Omega)$ with the following norm and $\preceq$:

$$\|x\| = \left(\int_\Omega |x(t)|^2 \, d\mu \right)^{\frac{1}{2}},$$

$$L^2(\Omega)_+ = \{ x \in L^2(\Omega) : x(t) \geq 0, \forall \text{ a.e. } t \in \Omega \}.$$

It is easy to see that $(L^2(\Omega), \preceq)$ is a Hilbert normed lattice. Given $u_0, v_0 \in L^2(\Omega)$ such that $u_0 \prec v_0$; then, by Theorem 3.2, every increasing $f : [u_0, v_0] \to L^2(\Omega)$ has a minimum best approximation point and a maximum best approximation point with respect to $W(x, y)$ in $[u_0, v_0]$.

4 Fixed-point theorems

From the above best approximation theorems, we can obtain the following fixed-point theorems.

Theorem 4.1 *Suppose that all conditions in Theorem* 3.1 *are satisfied. Moreover, one of the following conditions holds*:

(H_5) $f(\theta) \succeq \theta$;
(H_6) $f(v) \succ \theta$.

Then, f has a unique fixed point x^* in X_+, which satisfies $\theta \prec x^* \preceq v$. Moreover, if $x_0 \in X_+$ and $x_n = \Pi_{X_+}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$, then $\|x_n - x^*\| \to 0$ for $n \to \infty$.

Proof It suffices to show that x^* is the fixed point of f. Indeed, if (H_5) holds, using $\theta \prec x^* \preceq v$, we get $f(\theta) \preceq f(x^*) \preceq f(v)$. Thus $f(x^*) = (f(x^*) - f(\theta)) + f(\theta) \in X_+$ and $W(f(x^*), x^*) = d_W(f(x^*), X_+) = 0$. Hence $f(x^*) = x^*$.

If (H_6) holds, using $f(\theta) \succeq cf(v)$, we get $f(\theta) \in X_+$. From (H_5), we have $f(x^*) = x^*$. The assertion is proved. □

If $f : [u_0, v_0] \to [u_0, v_0]$ is self-projective, then $\Pi_{[u_0,v_0]}(f(x)) = f(x)$, and Theorem 3.2 reduces to the following fixed-point theorem:

Corollary 4.1 *Let $(X, \preceq)$ be a Dedekind complete lattice. Suppose that $f : [u_0, v_0] \to X$ is increasing and satisfies*:

$$u_0 \preceq f(u_0), \qquad f(v_0) \preceq v_0.$$

Then, f has a minimum fixed point x_* and a maximum fixed point x^* in $[u_0, v_0]$. Moreover, if $u_n = f(u_{n-1})$ and $v_n = f(v_{n-1})$ $(n = 1, 2, 3, \ldots)$, then equation (3.18) *holds.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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