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Some common fixed point and invariant approximation results for nonexpansive mappings in convex metric space

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Abstract
In this work, we introduce a new class of self-maps which satisfy the (E.A.) property with respect to some $q \in M$, where M is q-starshaped subset of a convex metric space and common fixed point results are established for this new class of self-maps. After that we obtain some invariant approximation results as an application. Our results represent a very strong variant of the several recent results existing in the literature. We also provide some illustrative examples in the support of proved results.
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1 Introduction

In 1976, Jungck [1] established some common fixed point results for a pair of commuting self-maps in the setting of complete metric space. The first ever attempt to relax the commutativity of mappings was initiated by Sessa [2] who introduced a class of noncommuting maps called 'namely' weak commutativity. Further, in order to enlarge the domain of noncommuting mappings, Jungck [3] in 1986 introduced the concept of 'compatible maps' as a generalization of weakly commuting maps.

Definition 1 Two self-maps I and T of a metric space (X, d) are called compatible if and only if

$$\lim_{n\to\infty} d(ITx_n, TIx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

In 2002, Aamri and Moutawakil [4] obtained the notion of (E.A.) property which enables us to study the existence of a common fixed points of self-maps satisfying nonexpansive or Lipschitz type condition in the setting of non-complete metric space.

Definition 2 Two self-maps I and T of a metric space (X, d) are said to satisfy the (E.A.) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

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On the other side, in 1970 Takahashi [5] introduced the notion of convexity into the metric space, studied properties of such spaces and proved several fixed point theorems for nonexpansive mappings. Afterward Guay *et al.* [6], Beg and Azam [7], Fu and Huang [8], Ding [9], Ćirić *et al.* [10], and many others have studied fixed point theorem in convex metric spaces. In the recent past, fixed point theorems have been extensively applied to best approximation theory. Meinardus [11] was the first who employed the Schauder's fixed point theorem to prove a result regarding invariant approximation. Later on, Brosowski [12] generalized the result of Meinardus under different settings. Further significant contribution to this area was made by a number of authors (see [13–35]). Many of them considered the pair of commuting or noncommuting mappings in the setting of normed or Banach spaces. In 1992, Beg *et al.* [36] proved some results on the existence of a common fixed point in the setting of a convex metric space and utilized the same to prove the best approximation results. After that, several authors studied common fixed point and invariant approximation results in the setting of convex metric space (see [36–40] and references therein).

In this work, we introduce a new class of self-maps which satisfy the (E.A.) property with respect to some $q \in M$, where M is q-starshaped subset of a convex metric space and establish some common fixed point results for this class of self-maps. After that we obtain some invariant approximation results as application. Our results represent a very strong variant of the several recent results existing in the literature.

2 Preliminaries

Firstly, we recall some useful definitions and auxiliary results that will be needed in the sequel. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

Definition 3 [5] Let (X,d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is called a convex structure on X if, for all $x, y \in X$ and $\lambda \in [0,1]$, we have

$$d\big(u, W(x,y,\lambda)\big) \leq \lambda d(u,x) + (1-\lambda)d(u,y) \tag{2.1}$$

for all $u \in X$.

A metric space (X,d) equipped with a convex structure is called a convex metric space.

Definition 4 A subset M of a convex metric space (X,d) is called a convex set if $W(x,y,\lambda) \in M$ for all $x, y \in M$ and $\lambda \in [0,1]$. The set M is said to be q-starshaped if there exists $q \in M$ such that $W(x,q,\lambda) \in M$ for all $x \in M$ and $\lambda \in [0,1]$. A set M is called starshaped if it is q-starshaped with respect to any $q \in M$.

Clearly, each convex set M is starshaped but the converse assertion is not true. Thus, the class of starshaped sets properly contains the class of convex sets.

Definition 5 A convex metric space (X,d) is said to satisfy the Property (I), if for all $x, y, z \in X$ and $\lambda \in [0,1]$,

$$d\big(W(x,z,\lambda), W(y,z,\lambda)\big) \leq \lambda d(x,y). \tag{2.2}$$

A normed linear space X and each of its convex subset are simple examples of convex metric spaces with W given by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ for all $x,y \in X$ and $0 \leq \lambda \leq 1$. Also, Property (I) is always satisfied in a normed linear space. There are many convex metric spaces which are not normed linear spaces (see [5, 6]). For further information on a convex metric space, refer to [5–10, 36–42].

Definition 6 Let (X,d) be a convex metric space and M a subset of X. A mapping $I : M \to M$ is said to be

(1) affine, if M is convex and $I(W(x,y,\lambda)) = W(Ix,Iy,\lambda)$ for all $x,y \in M$ and $\lambda \in [0,1]$;
(2) q-affine, if M is q-starshaped and $I(W(x,q,\lambda)) = W(Ix,q,\lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.

In [43] Pant define the concept of reciprocal continuity as follows.

Definition 7 Let (X,d) be a metric space and $I,T : X \to X$. Then the pair (I,T) is said to be reciprocally continuous if and only if

$$\lim_{n\to\infty} ITx_n = It \quad \text{and} \quad \lim_{n\to\infty} TIx_n = Tt$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to see that if I and T are continuous, then the pair (I,T) is reciprocally continuous but the converse is not true in general (see [[44], Example 2.3]). Moreover, in the setting of common fixed point theorems for compatible pairs of self-mappings satisfying some contractive conditions, continuity of one of the mappings implies their reciprocal continuity.

Definition 8 [45] A pair (I,T) of self-maps of a metric space (X,d) is said to be subcompatible if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X \text{ and } \lim_{n\to\infty} d(ITx_n, TIx_n) = 0.$$

Obviously, compatible maps which satisfy the (E.A.) property are subcompatible but the converse statement does not hold in general (see [40], Example 2.5).

Definition 9 Let (X,d) be a metric space, M a nonempty subset of X, and I and T be self-maps of M. A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ ($Ix = Tx = x$). The set of coincidence points of I and T is denoted by $C(I,T)$ and the set of fixed points of I and T is denoted by $F(I)$ and $F(T)$, respectively. The pair $\{I,T\}$ is called:

(1) Commuting if $ITx = TIx$ for all $x \in M$.
(2) Weakly compatible [46] if $ITx = TIx$ for all $x \in C(I,T)$.
(3) Banach operator pair [24] if the set $F(I)$ is T-invariant, *i.e.* $T(F(I)) \subseteq F(I)$.

For more details about these classes, one can refer [27, 47].

Definition 10 [19] Let M be a q-starshaped subset of convex metric space (X,d) such that $q \in F(I)$ and is both I- and T-invariant. Then the self-maps I and T are called R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq R\,\mathrm{dist}(Ix, [q, Tx])$, where $[q,x] = \{W(x,q,\lambda) : 0 \leq \lambda \leq 1\}$.

Clearly, R-subweakly commuting maps are compatible but the converse assertion is not necessarily true (see [31], Example 15).

For a nonempty subset M of a metric space (X,d) and $p \in X$, an element $y \in M$ is called a best approximation to p if $d(p,y) = \mathrm{dist}(p,M)$, where $\mathrm{dist}(p,M) = \inf\{d(p,z) : z \in M\}$. The set of all best approximations to p is denoted by $B_M(p)$.

3 Main results

We start to this section with following definition.

Definition 11 Let M be a q-starshaped subset of a convex metric space (X,d) and let $I, T : M \to M$ with $q \in F(I)$. The pair (I,T) is said to satisfy the (E.A.) property with respect to q if there exists a sequence $\{x_n\}$ in M such that for all $\lambda \in [0,1]$

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} T_\lambda x_n = t \quad \text{for some } t \in M, \tag{3.1}$$

where $T_\lambda x = W(Tx, q, \lambda)$.

Obviously, if the pair (I,T) satisfies the (E.A.) property with respect to q, then I and T satisfy the (E.A.) property but the converse assertion is not necessarily true. This can be seen by the following simple example.

Example 12 Let $X = \mathbb{R}^2$ be equipped with the metric $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$d\big((x_1,x_2),(y_1,y_2)\big) = |x_1 - y_1| + |x_2 - y_2|$$

and let $M = \{(x,y) : x \geq 1, y \geq 1\}$. Then (X,d) is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, $0 \leq \lambda \leq 1$, and M is q-starshaped with $q = (2,3)$. Define $I, T : M \to M$ by

$$I(x,y) = (x,y) \quad \text{and} \quad T(x,y) = (2x-1, 3y-2).$$

Firstly, we show I and T satisfy the (E.A.) property. Take a sequence $\{z_n\} = \{(x_n,y_n)\}$ in M such that $z_n \to (1,1)$, then $x_n \to 1$ and $y_n \to 1$. Thus

$$\lim_{n\to\infty} Iz_n = \lim_{n\to\infty} Tz_n = (1,1) \in M. \tag{3.2}$$

Now we will show that the pair (I,T) does not satisfy the (E.A.) property with respect to $q = (2,3)$. Take $\{z_n\} = \{(x_n,y_n)\}$ be any sequence in M, then $\lim_{n\to\infty} T_0 z_n = (2,3)$ and hence

$$\lim_{n\to\infty} Iz_n = \lim_{n\to\infty} T_0 z_n = (2,3) \quad \text{if and only if} \quad x_n \to 2, y_n \to 3.$$

But, for $x_n \to 2$, $y_n \to 3$, we get

$$\lim_{n\to\infty} Iz_n = (2,3) \neq (3,7) = \lim_{n\to\infty} T_1 z_n.$$

Thus, it is not possible to find a sequence $\{z_n\}$ in M such that for each $\lambda \in [0,1]$

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} T_\lambda x_n = t \in M.$$

So, the pair (I,T) does not satisfy the (E.A.) property with respect to $q = (2,3)$.

Remark 13 If M is convex subset of a convex metric space X and p is common fixed point of the self-maps I and T of M, then the pair (I,T) satisfies the (E.A.) property with respect to p but converse is not true in general. This can be seen by the following example.

Example 14 Let $X = \mathbb{R}$ be endowed with the usual metric and let $M = [0,1]$. Define $I,T : M \to M$ by

$$I(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{1}{4}, \\ \frac{1}{4} & \text{if } \frac{1}{4} \le x \le \frac{1}{2}, \\ \frac{x}{4} + \frac{1}{8} & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Clearly, X is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ and M is $\frac{1}{2}$-starshaped. Take $x_n = \frac{1}{4} - \frac{1}{2(n+1)}$ for each $n \ge 1$.Then for each $\lambda \in [0,1]$, we have

$$\lim_{n\to\infty} I(x_n) = \lim_{n\to\infty} T_\lambda x_n = \frac{1}{2} \in M.$$

Hence the pair (I,T) satisfies the (E.A.) property with respect to $q = \frac{1}{2}$, but I and T do not have a common fixed point.

The following lemma is a particular case of Theorem 4.1 of Chauhan and Pant [48].

Lemma 15 *Let I and T be self-maps of a metric space (X,d). If the pair (I,T) is subcompatible, reciprocally continuous and satisfy*

$$d(Tx,Ty) \le \lambda \max\{d(Ix,Iy), d(Ix,Tx), d(Iy,Ty), d(Ix,Ty), d(Iy,Tx)\} \tag{3.3}$$

for some $\lambda \in (0,1)$ and all $x,y \in X$. Then I and T have a unique common fixed point in X.

Now we prove our first result.

Theorem 16 *Let M be a nonempty q-starshaped subset of a convex metric space (X,d) with Property (I) and let I and T be continuous self-maps on M such that the pair (I,T) satisfies the (E.A.) property with respect to q. Assume that I is q-affine, $\mathrm{cl}(T(M))$ is compact. If I and T are compatible and satisfy the inequality*

$$\begin{aligned} d(Tx,Ty) \le \max\{ & d(Ix,Iy), \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \\ & \operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\} \end{aligned} \tag{3.4}$$

for all $x,y \in M$, then $M \cap F(T) \cap F(I) \ne \phi$.

Proof For each $n \in \mathbb{N}$, we define $T_n : M \to M$ by

$$T_n x = W(Tx, q, \lambda_n) \quad \text{for all } x \in M, \tag{3.5}$$

where λ_n is a sequence in $(0,1)$ such that $\lambda_n \to 1$.

Now, we have to show that for each $n \in \mathbb{N}$, the pair (T_n, I) is subcompatible. Since I and T satisfy the (E.A.)-property with respect to q, there exists a sequence $\{x_m\}$ in M such that for all $\lambda \in [0,1]$

$$\lim_{m\to\infty} Ix_m = \lim_{m\to\infty} T_\lambda x_m = t \in M, \tag{3.6}$$

where $T_\lambda x_m = W(Tx_m, q, \lambda)$.

Since $\lambda_n \in (0,1)$, in the light of (3.5) and (3.6), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \lim_{m\to\infty} T_n x_m &= \lim_{m\to\infty} W(Tx_m, q, \lambda_n) \\ &= \lim_{m\to\infty} T_{\lambda_n} x_m = t \in M. \end{aligned}$$

Thus, we have

$$\lim_{m\to\infty} Ix_m = \lim_{m\to\infty} T_n x_m = t \in M. \tag{3.7}$$

Now, using the fact that I is q-affine and Property (I) is satisfied, we get

$$\begin{aligned} d(T_n Ix_m, IT_n x_m) &= d\big(W(TIx_m, q, \lambda_n), I\big(W(Tx_m, q, \lambda_n)\big)\big) \\ &= d\big(W(TIx_m, q, \lambda_n), W(ITx_m, q, \lambda_n)\big) \\ &\le \lambda_n d(TIx_m, ITx_m). \end{aligned} \tag{3.8}$$

Since I and T are compatible, in view of (3.6), we have

$$\lim_{m\to\infty} d(ITx_m, TIx_m) = 0.$$

Now, letting $m \to \infty$ in (3.8), we get

$$\lim_{m\to\infty} d(IT_n x_m, T_n Ix_m) = 0. \tag{3.9}$$

Hence, on account of (3.7) and (3.9), it follows that the pair (T_n, I) is subcompatible for each $n \in \mathbb{N}$. Since I and T are continuous, for each $n \in \mathbb{N}$, the pair (T_n, I) is reciprocally continuous. Also, by (3.4),

$$\begin{aligned} &d(T_n x, T_n y) \\ &\quad = d\big(W(Tx, q, \lambda_n), W(Ty, q, \lambda_n)\big) \\ &\quad \le \lambda_n d(Tx, Ty), \quad \text{Property (I)} \\ &\quad \le \lambda_n \max\big\{d(Ix, Iy), \operatorname{dist}\big(Ix, [Tx, q]\big), \operatorname{dist}\big(Iy, [Ty, q]\big), \operatorname{dist}\big(Ix, [Ty, q]\big), \operatorname{dist}\big(Iy, [Tx, q]\big)\big\} \\ &\quad \le \lambda_n \max\big\{d(Ix, Iy), d(Ix, T_n x), d(Iy, T_n y), d(Ix, T_n y), d(Iy, T_n x)\big\} \end{aligned} \tag{3.10}$$

for each $x, y \in M$ and $0 < \lambda_n < 1$. By Lemma 15, for each $n \in \mathbb{N}$, there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$.

Now the compactness of $\mathrm{cl}(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z$ as $m \to \infty$. Further, it follows that

$$x_m = T_m x_m = W(Tx_m, q, \lambda_m) \to z \quad \text{as } m \to \infty.$$

By the continuity of I and T, we obtain $Iz = z = Tz$. Thus, $M \cap F(T) \cap F(I) \neq \phi$. □

Now we present a nontrivial example in support of Theorem 16.

Example 17 Let $X = \mathbb{R}$ endowed with the usual metric and let $M = [0,1]$. Define $I, T : M \to M$ by

$$I(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{3.11}$$

Then (X, d) is a convex metric space with the convex structure $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Firstly, we check the following:

(a) I is q affine with $q = \frac{1}{2}$.
(b) The pair (I, T) satisfies the (E.A.) property with respect to $q = \frac{1}{2}$.
(c) I and T are compatible.

Proof (a) Let $x \in [0, \frac{1}{2}]$. Then $W(x, \frac{1}{2}, \lambda) = \lambda x + (1 - \lambda)\frac{1}{2} \in [0, \frac{1}{2}]$ and hence

$$I\left(W\left(x, \frac{1}{2}, \lambda\right)\right) = \frac{1}{2} = \frac{\lambda}{2} + (1 - \lambda)\frac{1}{2} = W\left(Ix, \frac{1}{2}, \lambda\right).$$

Again, if $x \in [\frac{1}{2}, 1]$, then $W(x, \frac{1}{2}, \lambda) \in [\frac{1}{2}, 1]$, therefore we have

$$\begin{aligned} I\left(W\left(x, \frac{1}{2}, \lambda\right)\right) &= 1 - W\left(x, \frac{1}{2}, \lambda\right) \\ &= \frac{1 + \lambda}{2} - \lambda x \\ &= \lambda(1 - x) + (1 - \lambda)\frac{1}{2} \\ &= \lambda Ix + (1 - \lambda)\frac{1}{2} = W\left(Ix, \frac{1}{2}, \lambda\right). \end{aligned} \tag{3.12}$$

So, $I(W(x, \frac{1}{2}, \lambda)) = W(Ix, \frac{1}{2}, \lambda)$ for all $x \in M$ and hence I is q-affine with $q = \frac{1}{2}$. □

Proof (b) Clearly, $I(q) = q$ for $q = \frac{1}{2}$. Take $x_n = \frac{1}{2} - \frac{1}{n+1}$, $n \geq 1$, then for each n, $x_n \in [0, \frac{1}{2})$. So for each $\lambda \in [0, 1]$, we have

$$\lim_{n \to \infty} T_\lambda x_n = W\left(\frac{1}{2}, \frac{1}{2}, \lambda\right) = \frac{1}{2} = \lim_{n \to \infty} Ix_n.$$

Thus, the pair (I, T) satisfies the (E.A.) property with respect to $q = \frac{1}{2}$. □

Proof (c) If $\{x_n\}$ be a sequence in M such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in M$, then t lies in the closure of both $I(M) = [0, \frac{1}{2}]$ and $T(M) = [\frac{1}{2}, \frac{3}{4}]$, so $t = \frac{1}{2}$. Using the continuity of I and T we obtain

$$\lim_{n\to\infty} TIx_n = T\left(\frac{1}{2}\right) = \frac{1}{2} = I\left(\frac{1}{2}\right) = \lim_{n\to\infty} ITx_n.$$

Hence, I and T are compatible. □

Now we will show that the inequality (3.4) holds for each $x, y \in M$. If $x = y$, then $d(Tx, Ty) = 0$ and hence (3.4) obviously holds. Let $x, y \in M$ with $x \neq y$, then we have the following cases.

(1) If $x, y \in [0, \frac{1}{2}]$, then $d(Tx, Ty) = 0$ and so inequality (3.4) trivially holds.

(2) If $x, y \in [\frac{1}{2}, 1]$, then

$$d(Tx, Ty) = \frac{1}{2}|x - y| < |x - y| = d(Ix, Iy).$$

Thus, the inequality (3.4) holds.

(3) If $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$, then $Tx = \frac{1}{2}$, $Ty = \frac{y}{2} + \frac{1}{4}$, and $Ix = \frac{1}{2}$, $Iy = 1 - y$. Therefore

$$d(Tx, Ty) = \frac{1}{2}\left|y - \frac{1}{2}\right| < \left|y - \frac{1}{2}\right| = d(Ix, Iy).$$

Hence, the inequality (3.4) holds.

(3) If $x \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$, then, due to the symmetric property of metric d, the inequality (3.4) follows from case 3.

So, for each $x, y \in M$, the maps I and T satisfy the inequality (3.4). Also, $\mathrm{cl}(T(M)) = [\frac{1}{2}, \frac{3}{4}]$ is compact, I and T are continuous. Thus, from the above discussion we conclude that I and T satisfy all the hypotheses of Theorem 16 and consequently, $M \cap F(T) \cap F(I) \neq \phi$. Here $\frac{1}{2} \in M$ is a common fixed point of I and T.

Remark 18 Note that, in Example 17, $T(M) = [\frac{1}{2}, \frac{3}{4}] \not\subseteq [0, \frac{1}{2}] = I(M)$. Also, most of the common fixed point results in which the pair of maps is taken to be commuting, weakly commuting, R-subweakly commuting, compatible, and weakly compatible guarantee the existence of a common fixed point under the hypothesis $T(M) \subseteq I(M)$ (for example see [17–22, 25, 28–30, 36–40]). Thus, all these results are not applicable to finding the common fixed point of the maps I and T defined in Example 17.

Now, we present an example that will show if the condition 'The pair (I, T) satisfy the (E.A.) property with respect to q' of Theorem 16 fails to hold, then I and T may not have a common fixed point.

Example 19 Let $X = [0, \infty)$ be endowed with the usual metric and $M = [0, 2]$. Define $I, T : M \to M$ by

$$I(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ x & \text{if } \frac{1}{3} \leq x \leq 2, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{6} & \text{if } 0 \leq x \leq \frac{1}{3}, \\ \frac{x}{2} & \text{if } \frac{1}{3} \leq x \leq 2. \end{cases} \tag{3.13}$$

Then (X,d) is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ and M is q-starshaped with $q = \frac{1}{3}$. Clearly, I and T are continuous and $\mathrm{cl}(T(M)) = [\frac{1}{6},1]$ is compact. Using a routine calculation as is done in Example 17, it can easily be shown that I is q-affine with $q = \frac{1}{3}$ and also the maps I and T satisfy inequality (3.4) for each $x,y \in M$. Now we show that the map $I$and T do not satisfy the (E.A.) property. On the contrary, assume $\{x_n\}$ is a sequence in M such that

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in M. \tag{3.14}$$

Then t lies in the closure of both $I(M) = [\frac{1}{3},2]$ and $T(M) = [\frac{1}{6},1]$, so $t \in [\frac{1}{3},1]$. Further, employing the definition of maps I and T with 3.14, we have

$$t = \lim_{n\to\infty} Tx_n = \frac{1}{2}\lim_{n\to\infty} Ix_n = \frac{t}{2}.$$

This is not true for any $t \in [\frac{1}{3},1]$ and hence our assumption is wrong, so there does not exist any sequence $\{x_n\}$ in M such that 3.14 holds. Thus, the maps I and T do not satisfy the (E.A.) property and consequently the pair (I,T) does not satisfy (E.A.)-property with respect to $q = \frac{1}{3}$. Moreover, I and T are vacuously compatible and we observe I and T have no common fixed point.

Thus, if we relax the condition 'The pair (I,T) satisfy the (E.A.) property with respect to q' of Theorem 16, then I and T may not have a common fixed point.

The following corollaries immediately follow from Theorem 16.

Corollary 20 *Let M be a nonempty q-starshaped subset of a convex metric space (X,d) with Property (I) and let I and T be continuous self-maps on M such that the pair (I,T) satisfies the $(E.A.)$ property with respect to q. Assume that I is q-affine, $\mathrm{cl}(T(M))$ is compact. If I and T are compatible and satisfy the inequality*

$$\begin{aligned} d(Tx,Ty) \le \max\bigl\{ & d(Ix,Iy), \mathrm{dist}\bigl(Ix,[Tx,q]\bigr), \mathrm{dist}\bigl(Iy,[Ty,q]\bigr), \\ & 1/2\bigl[\mathrm{dist}\bigl(Ix,[Ty,q]\bigr) + \mathrm{dist}\bigl(Iy,[Tx,q]\bigr)\bigr]\bigr\} \end{aligned} \tag{3.15}$$

for all $x,y \in M$, then $M \cap F(T) \cap F(I) \neq \phi$.

Corollary 21 *Let M be a nonempty q-starshaped subset of a convex metric space (X,d) with Property (I) and let I and T be continuous self-maps on M such that the pair (I,T) satisfies the $(E.A.)$ property with respect to q. Assume that I is q-affine, $\mathrm{cl}(T(M))$ is compact. If I and T are R-subweakly commuting and satisfy the inequality*

$$\begin{aligned} d(Tx,Ty) \le \max\bigl\{ & d(Ix,Iy), \mathrm{dist}\bigl(Ix,[Tx,q]\bigr), \mathrm{dist}\bigl(Iy,[Ty,q]\bigr), \\ & \mathrm{dist}\bigl(Ix,[Ty,q]\bigr), \mathrm{dist}\bigl(Iy,[Tx,q]\bigr)\bigr\} \end{aligned} \tag{3.16}$$

for all $x,y \in M$, then $M \cap F(T) \cap F(I) \neq \phi$.

4 Invariant approximation

Now, we present some invariant approximation results as an application of Theorem 16.

Theorem 22 *Let I and T be self-maps of a convex metric space (X,d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that $B_M(p)$ is nonempty, q-starshaped with $I(B_M(p)) \subset B_M(p)$ and I is q-affine and continuous on $B_M(p)$. If the maps I and T are compatible, satisfy the $(E.A.)$ property with respect to q on $B_M(p)$, and also satisfy for all $x,y \in B_M(p) \cup \{p\}$*

$$d(Tx,Ty) \leq \begin{cases} d(Ix,Ip) & \text{if } y = p, \\ \max\{d(Ix,Iy), \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), & \\ \quad \operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\} & \text{if } y \in B_M(p), \end{cases} \tag{4.1}$$

then I and T have a common fixed point in $B_M(p)$, provided $\operatorname{cl}(T(B_M(p)))$ *is compact and T is continuous on $B_M(p)$.*

Proof Let $x \in B_M(p)$. Then for all $\lambda \in (0,1)$, we have

$$d\bigl(p, W(x,p,\lambda)\bigr) \leq \lambda d(p,x) + (1-\lambda)d(p,p) = \lambda d(p,x) < \operatorname{dist}(p,M).$$

Thus, it follows that $\{W(x,p,\lambda) : \lambda \in (0,1)\} \cap M = \phi$ and so $x \in \delta M \cap M$. As $T(\delta M \cap M) \subseteq M$, therefore $Tx \in M$. Since $Ix \in B_M(p)$ and $p \in F(I) \cap F(T)$, on account of (4.1), we have

$$d(Tx,p) = d(Tx,Tp) \leq d(Ix,Ip) = d(Ix,p) = \operatorname{dist}(p,M),$$

which shows that $Tx \in B_M(p)$, and in all I and T are self-maps on $B_M(p)$. In view of Theorem 16 there exists a $z \in B_M(p)$ such that z is a common fixed point of I and T. □

Example 23 Consider $X = [0,1]$ equipped with the usual metric, $M = (0,\frac{1}{2}]$ and define a mapping $W : X \times X \times [0,1] \to X$

$$W(x,y,\lambda) = \lambda x + (1-\lambda)y.$$

Then (X,d) is a convex metric space with Property (I). Define $I,T : X \to X$ by

$$I(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,\frac{2}{3}), \\ 0 & \text{if } x \in (\frac{2}{3},1], \\ \frac{2}{3} & \text{if } x = \frac{2}{3} \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,\frac{1}{2}] \cup (\frac{2}{3},1], \\ x & \text{if } x \in (\frac{1}{2},\frac{2}{3}]. \end{cases} \tag{4.2}$$

Clearly, $F(I) = \{\frac{1}{2},\frac{2}{3}\}$, $F(T) = [\frac{1}{2},\frac{2}{3}]$ and $T(\delta M \cap M) = T(\frac{1}{2}) = \frac{1}{2} \subseteq (0,\frac{1}{2}] = M$. Take $p = \frac{2}{3} \in F(I) \cap F(T) = \{\frac{1}{2},\frac{2}{3}\}$, then $B_M(p) = \{\frac{1}{2}\}$. Here, we observe that $B_M(p)$ is nonempty, $q = \frac{1}{2}$-starshaped with $I(B_M(p)) = \{\frac{1}{2}\} \subset B_M(p)$, and also I is q-affine and continuous on $B_M(p)$.

Further, I and T are commuting on $B_M(p)$ and hence compatible. Also, $\operatorname{cl}(T(B_M(p))) = \{\frac{1}{2}\}$ is compact, T is continuous on $B_M(p)$ and on account of Remark 13, the pair (I,T) satisfies the (E.A.) property with respect to $q = \frac{1}{2}$. Moreover, it can easily be checked that I and T satisfy inequality 4.1 for all $x,y \in B_M(p) \cup \{p\}$. Thus all the conditions of Theorem 22 are satisfied and consequently I and T have a common fixed point in $B_M(p)$. Here $x = \frac{1}{2}$ is such point.

Corollary 24 *Let I and T be self-maps of a convex metric space (X, d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that $B_M(p)$ is nonempty, q-starshaped with $I(B_M(p)) \subset B_M(p)$ and I is q-affine and continuous on $B_M(p)$. If the maps I and T are R-subweakly commuting, satisfy the $(E.A.)$ property with respect to q on $B_M(p)$ and also satisfy for all $x, y \in B_M(p) \cup \{p\}$*

$$d(Tx, Ty) \leq \begin{cases} d(Ix, Ip) & \text{if } y = p, \\ \max\{d(Ix, Iy), \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]), & \\ \quad \operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\} & \text{if } y \in B_M(p), \end{cases} \tag{4.3}$$

then I and T have a common fixed point in $B_M(p)$, provided $\operatorname{cl}(T(B_M(p)))$ *is compact and T is continuous on $B_M(p)$.*

We define $D = B_M(p) \cap C_M^I(p)$, where $C_M^I(p) = \{x \in M : Ix \in B_M(p)\}$.

Theorem 25 *Let I and T be self-maps of a convex metric space (X, d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that D is nonempty, q-starshaped with $I(D) \subset D$ and I is q-affine and nonexpansive on D. If the maps I and T are compatible, satisfy the $(E.A.)$ property with respect to q on D, and also satisfy for all $x, y \in D \cup \{p\}$*

$$d(Tx, Ty) \leq \begin{cases} d(Ix, Ip) & \text{if } y = p, \\ \max\{d(Ix, Iy), \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]), & \\ \quad \operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\} & \text{if } y \in D, \end{cases} \tag{4.4}$$

then I and T have a common fixed point in $B_M(p)$, provided $\operatorname{cl}(T(D))$ *is compact and T is continuous on D.*

Proof Let $x \in D$. Then $x \in B_M(p)$, and therefore, proceeding as in the proof of Theorem 22, we have $Tx \in B_M(p)$. Since I is nonexpansive and $p \in F(I) \cap F(T)$, it follows from (4.4) that

$$d(ITx, p) = d(ITx, Ip) \leq d(Tx, p) = d(Tx, Tp) \leq d(Ix, p) = \operatorname{dist}(p, M).$$

Thus $ITx \in B_M(p)$ and so $Tx \in C_M^I(p)$, which gives $Tx \in D$. Hence I and T are self-maps on D. Now, in the light of Theorem 16, there exists $z \in B_M(p)$ such that z is a common fixed point of I and T. □

Corollary 26 *Let I and T be self-maps of a convex metric space (X, d) with Property (I), $p \in F(I) \cap F(T)$, and M be a subset of X such that $T(\delta M \cap M) \subseteq M$, where δM denotes the boundary of M. Suppose that D is nonempty, q-starshaped with $I(D) \subset D$, and I is q-affine and nonexpansive on D. If the maps I and T are R-subweakly commuting, satisfy the $(E.A.)$*

property with respect to q on D, and also satisfy for all $x, y \in D \cup \{p\}$

$$d(Tx, Ty) \leq \begin{cases} d(Ix, Ip) & \text{if } y = p, \\ \max\{d(Ix, Iy), \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]), & \\ \quad \operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\} & \text{if } y \in D, \end{cases} \tag{4.5}$$

then I and T have a common fixed point in $B_M(p)$, *provided* $\operatorname{cl}(T(D))$ *is compact and T is continuous on D.*

Let $D_M^{R,I}(p) = B_M(p) \cap G_{R,I}^M(p)$, where $G_M^{R,I}(p) = \{x \in M : d(Ix, p) \leq (2R+1)\operatorname{dist}(p, M)\}$.

Theorem 27 *Let I and T be self-maps of a convex metric space* (X, d) *with Property* (I), $p \in F(I) \cap F(T)$, *and M be a subset of X such that* $T(\delta M \cap M) \subseteq M$, *where* δM *denotes the boundary of M. Suppose that* $D_M^{R,I}(p)$ *is nonempty, q-starshaped with* $I(D_M^{R,I}(p)) \subset D_M^{R,I}(p)$, *and I is q-affine and continuous on* $D_M^{R,I}(p)$. *If the maps I and T are R-subweakly commuting, satisfy the (E.A.) property with respect to q on* $D_M^{R,I}(p)$, *and also satisfy for all* $x, y \in D_M^{R,I}(p) \cup \{p\}$

$$d(Tx, Ty) \leq \begin{cases} d(Ix, Ip) & \text{if } y = p, \\ \max\{d(Ix, Iy), \operatorname{dist}(Ix, [Tx, q]), \operatorname{dist}(Iy, [Ty, q]), & \\ \quad \operatorname{dist}(Ix, [Ty, q]), \operatorname{dist}(Iy, [Tx, q])\} & \text{if } y \in B_M(p), \end{cases} \tag{4.6}$$

then I and T have a common fixed point in $B_M(p)$, *provided* $\operatorname{cl}(T(D_M^{R,I}(p)))$ *is compact and T is continuous on* $D_M^{R,I}(p)$.

Proof Let $x \in D_M^{R,I}(p)$, then using an argument similar to that in Theorem 22, we have $Tx \in B_M(p)$. Since I and T are R-subweakly commuting and $p \in F(I) \cap F(T)$, on account of (4.6) it follows that

$$\begin{aligned} d(ITx, p) &= d(ITx, Tp) \\ &\leq d(ITx, TIx) + d(TIx, Tp) \\ &\leq R \operatorname{dist}\bigl(Tx, [q, Ix]\bigr) + d\bigl(I^2x, Ip\bigr) \\ &\leq Rd(Tx, Ix) + d\bigl(I^2x, Ip\bigr) \\ &\leq R\bigl[d(Tx, Tp) + d(Ix, Tp)\bigr] + d\bigl(I^2x, Ip\bigr) \\ &\leq R\bigl[\operatorname{dist}(p, M) + \operatorname{dist}(p, M)\bigr] + \operatorname{dist}(p, M) \\ &= (2R+1)\operatorname{dist}(p, M). \end{aligned}$$

Thus $Tx \in G_{R,I}^M(p)$. Hence I and T are self-maps on $D_M^{R,I}(p)$. Regarding, Theorem 16, there exists $z \in B_M(p)$ such that z is a common fixed point of I and T. □

Remark 28

(1) The class of compatible and R-subweakly commuting are different from the class of Banach operator pairs and so our results are different from the results of [41].

(2) In the existing literature, common fixed point results are proved with the assumption $T(M) \subseteq I(M)$ but here we replace it with the assumption of the (E.A.) property with respect to some $q \in M$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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