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Strong convergence of hybrid Halpern iteration for Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in reflexive Banach spaces with application

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Abstract

In this paper, Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the framework of reflexive Banach spaces are established. Under suitable limit conditions, by using the shrinking projection method introduced by Takahashi, Kubota and Takeuchi, some strong convergence theorems for hybrid Halpern's iteration for a countable family of Bregman totally quasi-asymptotically nonexpansive multi-valued mappings are proved. We apply our main results to solve classical equilibrium problems in the framework of reflexive Banach spaces. The main result presented in the paper improves and extends the corresponding result in the work by Chang (Appl. Math. Comput. 2013, doi:10.1016/j.amc.2013.11.074; Appl. Math. Comput. 228:38-48, 2014), Suthep (Comput. Math. Appl., 64:489-499, 2012), Yi Li (Fixed Point Theory Appl. 2013:197, 2013), Reich and Sabach (Nonlinear Anal. 73:122-135, 2010), Nilsrakoo and Saejung (Appl. Math. Comput. 217(14):6577-6586, 2011), Qin *et al*. (Appl. Math. Lett. 22:1051-1055, 2009), Wang *et al*. (J. Comput. Appl. Math. 235:2364-2371, 2011), Su *et al*. (Nonlinear Anal. 73:3890-3906, 2010) and others.

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1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let D be a nonempty and closed subset of a real Banach space X. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty, closed, and bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\right\}$$

for all $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{\|x - y\|, y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive, if

$$H(Tx, Ty) \leq \|x - y\|, \quad \forall x, y \in D.$$

An element $p \in D$ is called *a fixed point* of the multi-valued mapping $T : D \to N(D)$ if $p \in T(p)$. The set of fixed points of T is denoted by $F(T)$.

In recent years, several types of iterative schemes have been constructed and proposed in order to get strong convergence results for finding fixed points of nonexpansive mappings in various settings. One classical and effective iteration process is defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad x_1, u \in D,$$

where $\alpha_n \in (0, 1)$. This method was introduced in 1967 by Halpern [10] and is often called Halpern's iteration. In fact, he proved, in a real Hilbert space, the strong convergence of $\{x_n\}$ to a fixed point of the nonexpansive mapping T, where $\alpha_n = n^{-a}$, $a \in (0, 1)$.

Because of the simple construction, Halpern's iteration is widely used to approximate fixed points of nonexpansive mappings and other classes of nonlinear mappings by mathematicians in different styles [3–42]. In particular, some strong convergence theorems for resolvents of accretive operators in Banach spaces were proved by Reich [11], and he also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces in [12]. In 2012, Halpern's iteration for Bregman strongly nonexpansive mappings in reflexive Banach spaces was introduced and a strong convergence theorem for Bregman strongly nonexpansive mappings by Halpern's iteration in the framework of reflexive Banach spaces was proved. Recently, a strong convergence theorem for Bregman strongly multi-valued nonexpansive mappings as regards Halpern's iteration in the framework of reflexive Banach spaces was proved by Chang [1, 2], Suthep [3] and Li [4].

The purpose of our work is to introduce a modified Halpern iteration for a countable family of Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the framework of reflexive Banach spaces, and to prove strong convergence theorems for these iterations under suitable limit conditions by using the shrinking projection method. We use our results to solve equilibrium problems in the framework of reflexive Banach spaces. The main results presented in the paper improve and extend the corresponding results in the work by Chang [1, 2], Suthep [3], Li [4], and others.

2 Preliminaries

In this section, we recall some basic definitions and results which will be used in the following.

Let X be a real reflexive Banach space with a norm $\|\cdot\|$, and let X^* be the dual space of X. Let $f : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous, and convex function. We denote by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ the domain of f.

Let $x \in \operatorname{int}\operatorname{dom} f$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in X\}. \tag{2.1}$$

The Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}.$$

We know that the Young-Fenchel inequality holds, that is,

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall x \in X, x^* \in X^*.$$

Furthermore, equality holds if $x^* \in \partial f(x)$ (see [13]). The set $\operatorname{lev}_{\leq}^{f}(r) := \{x \in X : f(x) \leq r\}$ for some $r \in \mathbb{R}$ is called a sublevel of f.

A function f on X is called coercive [14], if the sublevel sets of f are bounded, or equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

A function f on X is said to be *strongly coercive* [15], if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in X$, the right-hand derivative of f at x in the direction y is defined by

$$f^{\circ}(x,y) = \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x, if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^{\circ}(x,y)$ coincides with $\nabla f(x)$, the value of the gradient $\nabla f(x)$ of f at x. The function f is said to be *Gâteaux differentiable*, if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be *Fréchet differentiable* at x, if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset D of X, if the limit is attained uniformly, for $x \in D$ and $\|y\| = 1$. It is well known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\operatorname{int}\operatorname{dom} f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak*, continuous (resp. continuous) on $\operatorname{int}\operatorname{dom} f$ (see [16, 17]).

Definition 2.1 (*cf.* [18]) The function f is said to be

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.1 (*cf.* [19]) Let X be a reflexive Banach space. Then we have

(a) f is essentially smooth if and only if f^* is essentially strictly convex;
(b) $(\partial f)^{-1} = \partial f^*$;
(c) f is Legendre if and only if f^* is Legendre;
(d) If f is Legendre, then ∂f is a bijection which satisfies $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$.

Examples of Legendre functions can be found in [30]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(0 < p < +\infty)$ when X is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of X, *i.e.*, $\nabla f = J_p$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces. In this paper, we always assume that f is Legendre.

The following crucial lemma was proved by Reich and Sabach [20].

Lemma 2.1 (*cf.* [20]) *If $f : X \to R$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Let $f : X \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the Bregman distance with respect to f.

Recall that the Bregman projection [21] of $x \in \operatorname{int}\operatorname{dom} f$ onto a nonempty, closed, and convex set $D \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_D^f(x) \in D$ (for convenience, here we use $P_D^f(x)$ for $\operatorname{proj}_D^f(x)$) satisfying

$$D_f\left(\operatorname{proj}_D^f(x), x\right) = \inf\left\{D_f(y,x) : y \in D\right\}.$$

The modulus of the total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x,t) : [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x,t) := \inf\left\{D_f(y,x) : y \in \operatorname{dom} f, \|y - x\| = t\right\}.$$

The function f is called totally convex at x, if $\nu_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex, if it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$, and it is said to be totally convex on bounded sets, if $\nu_f(B,t) > 0$, for any nonempty bounded subset B of and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B,t) = \inf\left\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\right\}.$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (*cf.* [22]).

Recall that the function f is said to be sequentially consistent [22], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that the first sequence is bounded, the following implication holds:

$$\lim_{n \to +\infty} D_f(x_n, y_n) = 0 \quad \Rightarrow \quad \lim_{n \to +\infty} \|x_n - y_n\| = 0.$$

Recall that the function f is called sequentially consistent, if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom} f$, respectively, and $\{x_n\}$ is bounded, $D_f(y_n, x) \to 0$, then $\|y_n - x\| \to 0$.

The following crucial lemma was proved by Butnariu and Iusem [23].

Lemma 2.2 (*cf.* [23]) *If $x \in \operatorname{int}\operatorname{dom} f$, then the following statements are equivalent*:

(i) *The function f is totally convex at x.*

(ii) *For any sequence $\{y_n\} \subset \operatorname{dom} f$, $D_f(y_n, x) \to 0$, then $\|y_n - x\| \to 0$.*

Definition 2.2 (*cf.* [24]) Let D be a convex subset of $\operatorname{int}\operatorname{dom} f$ and let T be a multi-valued mapping of D. A point $p \in D$ is called an asymptotic fixed point of T if D contains a sequence $\{x_n\}$ which converges weakly to p such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$).

We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

Definition 2.3 A multi-valued mapping $T : D \to N(D)$ with a nonempty fixed point set is said to be:

(i) Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$, if

$$D_f(p, z) \le D_f(p, x), \quad \forall x \in D, p \in \hat{F}(T), z \in T(x)$$

and if, whenever $\{x_n\} \subset D$ is bounded, $p \in \hat{F}(T)$, and $\lim_{n\to\infty}[D_f(p, x_n) - D_f(p, z_n)] = 0$, then $\lim_{n\to\infty} D_f(x_n, z_n) = 0$, where $z_n \in Tx_n$.

(ii) Bregman firmly nonexpansive if

$$\begin{aligned} &\langle \nabla f(x^*) - \nabla f(y^*), x^* - y^* \rangle \\ &\quad \le \langle \nabla f(x) - \nabla f(y), x^* - y^* \rangle, \quad \forall x, y \in D, x^* \in Tx, y^* \in Ty. \end{aligned}$$

(iii) Bregman quasi-asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, +\infty)$, $k_n \to 1$ (as $n \to \infty$), if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$D_f(p, z) \le k_n D_f(p, x), \quad p \in F(T), \forall z \in T^n x, x \in D.$$

(iv) Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$, if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$D_f(p, z) \le D_f(p, x) + \nu_n \zeta\big(D_f(p, x)\big) + \mu_n, \quad p \in F(T), \forall z \in T^n x, x \in D.$$

(v) Closed, if for any sequence $\{x_n\} \subset D$ with $x_n \to x \in N(D)$ and $d(Tx_n, , y) \to 0$ $(y \in D)$, then $y \in Tx$.

Remark 2.2 (*cf.* [1]) From these definitions, it is obvious that if $\hat{F}(T) = F(T) \neq \emptyset$, then a Bregman strongly nonexpansive multi-valued mapping is a Bregman relatively nonexpansive mapping; a Bregman relatively nonexpansive multi-valued mapping is a Bregman quasi-nonexpansive multi-valued mapping; a Bregman quasi-nonexpansive multi-valued mapping is a Bregman quasi-asymptotically nonexpansive multi-valued mapping; a Bregman quasi-asymptotically nonexpansive multi-valued mapping must be a Bregman totally quasi-asymptotically nonexpansive multi-valued mapping. However, converses of these statements are not true.

In particular, the existence and approximation of Bregman firmly nonexpansive single value mappings was studied in [24]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X, then $F(T) = \hat{F}(T)$ and $F(T)$ is closed and convex (*cf.* [24]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \hat{F}(T)$. The class of single-valued Bregman totally quasi-asymptotically nonexpansive mappings was introduced first in [1]. For a wealth of results concerning this class of mappings (for example, see [1], Examples 2.11-2.15 and the references therein).

Remark 2.3 Let X be a uniformly smooth and uniformly convex Banach space, and D is nonempty, closed, and convex subset. An operator $T : C \to N(D)$ is called a strongly relatively nonexpansive multi-valued mapping on X, if $\hat{F}(T) \neq \Phi$ and

$$\phi(p, z) \leq \phi(p, x), \quad p \in \hat{F}(T), z \in Tx,$$

and, if whenever $\{x_n\} \subset D$ is bounded, $p \in \hat{F}(T)$, and $\lim_{n\to\infty}[\phi(p, x_n) - \phi(p, z_n)] = 0$, then $\lim_{n\to\infty} \phi(x_n, z_n) = 0$, where $z_n \in Tx_n$ and $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$.

Now, we give an example of Bregman totally quasi-asymptotically nonexpansive multi-valued mapping.

Example 2.1 (see [1], Example 2.11) Let D be a unit ball in a real Hilbert space l^2, $f(x) = \|x\|^2$. Since $\nabla f(y) = 2y$, the Bregman distance with respect to f

$$D_f(x, y) = \|x\|^2 - \|y\|^2 - 2\langle y, x - y\rangle = \|x - y\|^2, \quad \forall x, y \in D. \tag{2.2}$$

Let $T : D \to N(D)$ be a multi-valued mapping defined by

$$T : (x_1, x_2, \ldots) \to \left(0, x_1^2, a_2x_2, a_3x_3, \ldots\right) \in l^2, \quad \forall (x_1, x_2, \ldots) \in D$$

where any $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved in Goebal and Kirk [25] that

(i) $\|Tx - Ty\| \leq 2\|x - y\|$, $\forall x, y \in D$;

(ii) $\|T^n x - T^n y\| \leq 2\prod_{j=2}^{n} a_j \|x - y\|$, $\forall x, y \in D$, $n \geq 2$.

Let $\sqrt{k_1} = 2$, $\sqrt{k_n} = 2\prod_{j=2}^{n} a_j$, $n \geq 2$, then $\lim_{n\to\infty} k_n = 1$. Letting $\nu_n = k_n - 1$ ($n \geq 2$), $\zeta(t) = t$ ($t \geq 0$), and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \to 0$, then from (i) and (ii) we have

$$\left\|T^n x - T^n y\right\|^2 \leq \|x - y\|^2 + \nu_n \zeta\left(\|x - y\|^2\right) + \mu_n, \quad \forall x, y \in D.$$

Since D is a unit ball in a real Hilbert space l^2, it follows from (2.2) that $D_f(x, y) = \|x - y\|^2$, $\forall x, y \in D$. Above inequality can be written as

$$D_f\left(T^n x, T^n y\right) \leq D_f(x, y) + \nu_n \zeta\left(D_f(x, y)\right) + \mu_n, \quad \forall x, y \in D.$$

Again since $0 \in D$ and $0 \in F(T)$, this implies that $F(T) \neq \Phi$. From the above inequality we get

$$D_f(p, T^n x) \leq D_f(p,x) + \nu_n \zeta(D_f(p,x)) + \mu_n, \quad \forall p \in F(T), x \in D.$$

This shows that the mapping T defined as above is a Bregman total quasi-asymptotically nonexpansive multi-valued mapping.

Let D be a nonempty, closed, and convex subset of X. Let $f: X \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and $x \in X$. It is well known from [22] that $z = P_D^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \quad \forall y \in D. \tag{2.3}$$

We also know the following characterization:

$$D_f(y, P_D^f(x)) + D_f(P_D^f(x), x) \leq D_f(y,x), \quad \forall x, y \in D. \tag{2.4}$$

Let $f: X \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [31] and [32], we make use of the function $V_f: X \times X^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) + f^*(x^*) - \langle x, x^* \rangle, \quad \forall x \in X, x^* \in X^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in X$ and $x^* \in X^*$. Moreover, by the subdifferential inequality (see [26], Proposition 1(iii), p.1047),

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad \forall x \in X, x^*, y^* \in X^*.$$

In addition, if $f: X \to (-\infty, +\infty]$ is a proper and lower semi-continuous function, then $f^*: X^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function (see [33]). Hence V_f is convex in the second variable (see [26], Proposition 1(i), p.1047). Thus,

$$\begin{aligned} &D_f(z, \nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))) \\ &\quad \leq tD_f(z,x) + (1-t)D_f(z,y), \quad \forall t \in (0,1), \forall x, y \in X. \end{aligned} \tag{2.5}$$

The properties of the Bregman projection and the relative projection operators were studied in [22] and [27].

In 2013, Yi Li and Jin-hua Zhu proved the following result, respectively.

Let X be a real reflexive Banach space and let $f: X \to (-\infty, +\infty]$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of X. Let D be a nonempty, closed, and convex subset of $\operatorname{int}\operatorname{dom} f$. $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

(1) (see [28]) Let $T : D \to N(D)$ be a Bregman strongly nonexpansive mapping on X such that $F(T) = \hat{F}(T) \neq \emptyset$. Suppose that $u \in X$ and define the sequence $\{x_n\}$ by

$$x_1 \in D, \quad x_{n+1} = \nabla f^*\big(\alpha_n \nabla f(u) + (1-\alpha_n)\big(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(Tx_n)\big)\big), \quad n \geq 1.$$

Then $\{x_n\}$ strongly converges to $P^f_{F(T)}(u)$.

(2) (see [4]) Let $T : D \to N(D)$ be a Bregman strongly nonexpansive multi-valued mapping on X such that $F(T) = \hat{F}(T) \neq \emptyset$. Suppose that $u \in X$ and define the sequence $\{x_n\}$ by

$$x_1 \in D, \quad x_{n+1} = \nabla f^*\big(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)\big), \quad z_n \in Tx_n, n \geq 1.$$

Then $\{x_n\}$ strongly converges to $P^f_{F(T)}(u)$.

In 2014, Chang SS proved the following result.

Theorem 2.1 ([1]) *Let X be a real uniformly smooth, uniformly convex, and reflexive Banach space, D be a nonempty, closed, and convex subset of X. Let $f : D \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of D and let $\{T_i\} : D \to D$ be a family of closed and uniformly Bregman total quasi-asymptotically nonexpansive mappings with sequence $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and let there be a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that, for each $i \geq 1$, $\{T_i\}$ is uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let x_n be a sequence generated by*

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n) f(T_i^n x_n)], \quad i \geq 1, \\ D_{n+1} = \{z \in D_n : \sup_{i\geq 1} D_f(z, y_{n,i}) \leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = P^f_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases}$$

where $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(D_f(p, x_n)) + \mu_n$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $P^f_{D_{n+1}}$ is the Bregman projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $P^f_{\mathcal{F}} x_1$.

Definition 2.4

(1) A countable family of multi-valued mappings $\{T_i : D \to N(D)\}_{i=1}^{\infty}$ is said to be uniformly Bregman totally quasi-asymptotically nonexpansive, if $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$, such that

$$D_f(p, z_{n,i}) \leq D_f(p, x) + \nu_n \zeta\big(D_f(p, x)\big) + \mu_n, \quad p \in F(T), \forall z_{n,i} \in T_i^n x, x \in D. \tag{2.6}$$

(2) A countable family of multi-valued mappings $\{T_i : D \to N(D)\}_{i=1}^{\infty}$ is said to be uniformly Bregman quasi-asymptotically nonexpansive, if $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exist nonnegative real sequences $\{k_n\} \in [1, +\infty)$, $k_n \to 1$ (as $n \to \infty$), such that

$$D_f(p, z_{n,i}) \leq D_f(p, x) + \nu_n \zeta\big(D_f(p, x)\big) + \mu_n, \quad p \in F(T), \forall z_{n,i} \in T_i^n x, x \in D. \tag{2.7}$$

(3) A multi-valued mapping $T: D \to N(D)$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$H\big(T^n x, T^n y\big) \le L\|x - y\|, \quad \forall x, y \in D. \tag{2.8}$$

Now, we improve the above results, and the following main results are obtained.

3 Main results

To prove our main result, we first give the following propositions.

The proof of the following result in the case of single-valued Bregman totally quasi-asymptotically nonexpansive mappings was done in ([1], Lemma 2.6, and [24], Lemma 15.5). In the multi-valued case the proof is identical and therefore we will omit the exact details. The interesting reader will consult [1, 24].

Proposition 3.1 *Let $f: X \to (-\infty, +\infty]$ be a Legendre function and let D be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$*. Let $T: D \to N(D)$ be a Bregman totally quasi-asymptotically nonexpansive multi-valued mapping with respect to f. Then $F(T)$ is closed and convex.*

Theorem 3.1 *Let X be a real uniformly smooth, uniformly convex, and reflexive Banach space, D be a nonempty, closed, and convex subset of X. Let $f: D \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of D and let $\{T_i\}: D \to N(D)$ be a family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings with sequence $\{\nu_n\}, \{\mu_n\}, \nu_n, \mu_n \to 0$ (as $n \to \infty$), and let there be a strictly increasing continuous function $\zeta: R^+ \to R^+$ with $\zeta(0) = 0$ such that, for each $i \ge 1$, $\{T_i\}$ are uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$.

Let x_n be a sequence generated by

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n)(\beta_n \nabla f(x_n) \\ \qquad + (1-\beta_n)\nabla f(w_{n,i})], \quad w_{n,i} \in T_i^n x_n, i \ge 1, \\ D_{n+1} = \{z \in D_n : \sup_{i\ge 1} D_f(z, y_{n,i}) \le \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = P^f_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{3.1}$$

where $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(D_f(p, x_n)) + \mu_n$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $P^f_{D_{n+1}}$ is the Bregman projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $P^f_{\mathcal{F}} x_1$.

Proof (I) First, we prove that $\mathcal{F}$ and D_n are closed and convex subsets in D.

In fact, by Proposition 3.1 for each $i \ge 1$, $F(T_i)$ is closed and convex in D. Therefore $\mathcal{F}$ is a closed and convex subset in D. We use the assumption that $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \ge 1$. In view of the definition of $D_f(\cdot,\cdot)$, we have

$$\begin{aligned}D_{n+1} &= \Big\{z \in D_n : \sup_{i\geq 1} D_f(z, y_{n,i}) \leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\Big\}\\ &= \bigcap_{i\geq 1}\Big\{z \in D : \sup_{i\geq 1} D_f(z, y_{n,i}) \leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\Big\} \cap D_n\\ &= \bigcap_{i\geq 1}\big\{z \in D : \alpha_n\langle \nabla f(x_1), z - x_1\rangle + (1-\alpha_n)\langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(y_{n,i}), z - y_{n,i}\rangle\\ &\quad \leq -\alpha_n f(x_1) - (1-\alpha_n) f(x_n) + f(y_{n,i})\big\} \cap D_n.\end{aligned}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) Next, we prove that $\mathcal{F} \subset D_n$, for all $n \geq 1$.

In fact, it is obvious that $\mathcal{F} \subset D_1$. Suppose that $\mathcal{F} \subset D_n$ for some $n \geq 1$.

Letting $\omega_{n,i} = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(w_{n,i}))$. Hence for any $u \in \mathcal{F} \subset D_n$, by (3.1), we have

$$\begin{aligned}D_f(u, y_{n,i}) &= D_f\big(u, \nabla f^*\big[\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(\omega_{n,i})\big]\big)\\ &\leq \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, \omega_{n,i})\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}D_f(u, \omega_{n,i}) &= D_f\big(u, \nabla f^*\big(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(w_{n,i})\big)\big)\\ &\leq \beta_n D_f(u, x_n) + (1-\beta_n) D_f(u, w_{n,i})\\ &\leq \beta_n D_f(u, x_n) + (1-\beta_n)\big\{D_f(u, x_n) + \nu_n \zeta\big[D_f(u, x_n)\big] + \mu_n\big\}\\ &= D_f(u, x_n) + (1-\beta_n)\nu_n \zeta\big[D_f(u, x_n)\big] + (1-\beta_n)\mu_n.\end{aligned} \tag{3.3}$$

Therefore, we have

$$\begin{aligned}\sup_{i\geq 1} D_f(u, y_{n,i}) &\leq \alpha_n D_f(u, x_1) + (1-\alpha_n)\big[D_f(u, x_n) + (1-\beta_n)\nu_n \zeta\big[D_f(u, x_n)\big] + (1-\beta_n)\mu_n\big]\\ &\leq \alpha_n D_f(u, x_1) + (1-\alpha_n) D_f(u, x_n) + \nu_n \sup_{p\in\mathcal{F}} \zeta\big[D_f(p, x_n)\big]\\ &= \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n,\end{aligned} \tag{3.4}$$

where $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(D_f(p, x_n)) + \mu_n$. This shows that $u \in \mathcal{F} \subset D_{n+1}$ and so $\mathcal{F} \subset D_n$. The conclusion is proved.

(III) Now we prove that $\{x_n\}$ converges strongly to some point p^*.

Since $x_n = P^f_{D_n} x_1$, from (2.3), we have

$$\langle x_n - y, \nabla f(x_1) - \nabla f(x_n)\rangle \geq 0, \quad \forall y \in D_n.$$

Again since $\mathcal{F} \subset D_n$, we have

$$\langle x_n - u, \nabla f(x_1) - \nabla f(x_n)\rangle \geq 0, \quad \forall u \in \mathcal{F}.$$

It follows from (2.4) that, for each $u \in \mathcal{F}$ and for each $n \geq 1$,

$$D_f(x_n, x_1) = D_f\left(P^f_{D_n} x_1, x_1\right) \leq D_f(u, x_1) - D_f(u, x_n) \leq D_f(u, x_1). \tag{3.5}$$

Therefore $\{D_f(x_n, x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = P^f_{D_n} x_1$ and $x_{n+1} = P^f_{D_{n+1}} x_1 \in D_{n+1} \subset D_n$, we have $D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1)$. This implies that $\{D_f(x_n, x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty} D_f(x_n, x_1)$ exists.

By the construction of $\{D_n\}$, for any $m \geq n$, we have $D_m \subset D_n$ and $x_m = P^f_{D_m} x_1 \in D_n$. This shows that

$$D_f(x_m, x_n) = D_f\left(x_m, P^f_{D_n} x_1\right) \leq D_f(x_m, x_1) - D_f(x_n, x_1) \to 0 \quad (\text{as } n \to \infty).$$

It follows from Lemma 2.2 that $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$is a Cauchy sequence in D. Since D is complete, without loss of generality, we can assume that $\lim_{n\to\infty} x_n = p^*$ (some point in D).

By the assumption, it is easy to see that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \left[\nu_n \sup_{p\in\mathcal{F}} \zeta\left(D_f(p, x_n)\right) + \mu_n\right] = 0. \tag{3.6}$$

(IV) Now we prove that $p^* \in \mathcal{F}$.

Since $x_{n+1} \in D_{n+1}$, from (3.1), (3.5), and (3.6), we have

$$\sup_{i\geq 1} D_f(x_{n+1}, y_{n,i}) \leq \alpha_n D_f(x_{n+1}, x_1) + (1 - \alpha_n) D_f(x_{n+1}, x_n) + \xi_n \to 0. \tag{3.7}$$

Since $x_n \to p^*$, it follows from (2.6) and Lemma 2.2 that for all $i \geq 1$

$$y_{n,i} \to p^* \quad (\text{as } n \to \infty). \tag{3.8}$$

Since $\{x_n\}$ is bounded and $\{T_i\}$ is a family of uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings, we have

$$D_f(p, w_{n,i}) \leq D_f(p, x_n) + \nu_n \zeta\left[D_f(p, x_n)\right] + \mu_n, \quad \forall n, i \geq 1, p \in F(T_i), w_{n,i} \in T_i^n x_n.$$

This implies that $\{w_{n,i}\}$ is uniformly bounded.

We have

$$\begin{aligned} \|\omega_{n,i}\| &= \left\|\nabla f^*\left(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(w_{n,i})\right)\right\| \\ &\leq \beta_n \|x_n\| + (1 - \beta_n)\|w_{n,i}\| \\ &\leq \|x_n\| + \|w_{n,i}\|, \end{aligned}$$

and this implies that $\{\omega_{n,i}\}$ is also uniformly bounded.

In view of $\alpha_n \to 0$, from (3.1), we have

$$\lim_{n\to\infty} \left\|\nabla f(y_{n,i}) - \nabla f(\omega_{n,i})\right\| = \lim_{n\to\infty} \alpha_n \left\|\nabla f(x_1) - \nabla f(\omega_{n,i})\right\| = 0 \tag{3.9}$$

for each $i \geq 1$.

Since ∇f^* is uniformly continuous on each bounded subset of X^*, it follows from (3.8) and (3.9) that

$$\omega_{n,i} \to p^* \tag{3.10}$$

for each $i \geq 1$. Since ∇f is uniformly continuous on each bounded subset of X, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \| \nabla f(\omega_{n,i}) - \nabla f(p^*) \| \\ &= \lim_{n\to\infty} \| \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(w_{n,i}) - \nabla f(p^*) \| \\ &= \lim_{n\to\infty} \| \beta_n(\nabla f(x_n) - \nabla f(p^*)) + (1-\beta_n)(\nabla f(w_{n,i}) - \nabla f(p^*)) \| \\ &= \lim_{n\to\infty} (1-\beta_n) \| \nabla f(w_{n,i}) - \nabla f(p^*) \|. \end{aligned} \tag{3.11}$$

By condition (ii), we have

$$\lim_{n\to\infty} \| \nabla f(w_{n,i}) - \nabla f(p^*) \| = 0.$$

Since J is uniformly continuous, this shows that

$$\lim_{n\to\infty} w_{n,i} = p^* \tag{3.12}$$

for each $i \geq 1$. Again by the assumptions that $\{T_i\} : D \to D$ be uniformly L_i-Lipschitz continuous for each $i \geq 1$, thus we have

$$\begin{aligned} &H(T_i^{n+1}x_n, T_i^n x_n) \\ &\quad \leq H(T_i^{n+1}x_n, T_i^{n+1}x_{n+1}) + d(T_i^{n+1}x_{n+1}, x_{n+1}) \\ &\qquad + d(x_{n+1}, x_n) + d(x_n, T_i^n x_n) \\ &\quad \leq (L_i + 1) d(x_{n+1}, x_n) + d(T_i^{n+1}x_{n+1}, x_{n+1}) + d(x_n, T_i^n x_n), \end{aligned} \tag{3.13}$$

for each $i \geq 1$.

We get $\lim_{n\to\infty} \|H(T_i^{n+1}x_n) - H(T_i^n x_n)\| = 0$. Since $\lim_{n\to\infty} w_{n,i} = p^*$ and $\lim_{n\to\infty} x_n = p^*$, we have $\lim_{n\to\infty} d(H(T_i T_i^n x_n), p^*) = 0$.

In view of the closedness of T_i, it yields $d(T_i p^*, p^*) = 0$. Since $p^* \in C$, $p^* \in T_i p^*$, *i.e.*, for each $i \geq 1$, $p^* \in F(T_i)$. By the arbitrariness of $i \geq 1$, we have $p^* \in \mathcal{F}$.

(V) Finally we prove that $p^* = P^f_{\mathcal{F}} x_1$ and so $x_n \to P^f_{\mathcal{F}} x_1 = p^*$.

Let $u = P^f_{\mathcal{F}} x_1$. Since $u \in \mathcal{F} \subset D_n$ and $x_n = P^f_{D_n} x_1$, we have $D_f(x_n, x_1) \leq D_f(w, x_1)$. This implies that

$$D_f(p^*, x_1) = \lim_{n\to\infty} D_f(x_n, x_1) \leq D_f(u, x_1), \tag{3.14}$$

which yields $p^* = u = P^f_{\mathcal{F}} x_1$. Therefore, $x_n \to P^f_{\mathcal{F}} x_1$. The proof of Theorem 3.1 is completed. □

By Remark 2.2, the following corollary is obtained.

Theorem 3.2 *Let D, X, $\{\alpha_n\}$, $\{\beta_n\}$, and f be the same as in Theorem 3.1, Let $\{T_i\} : D \to N(D)$ be a family of closed and uniformly Bregman quasi-asymptotically nonexpansive multi-valued mappings with sequence $\{k_n\} \subset [1, +\infty)$, $k_n \to 1$ (as $n \to \infty$) such that, for each $i \geq 1$, $\{T_i\}$ be uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \leq \lim_{n\to\infty} \sup \beta_n < 1$.

Let x_n be a sequence generated by

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n)(\beta_n \nabla f(x_n) \\ \qquad + (1-\beta_n)\nabla f(w_{n,i})], \quad w_{n,i} \in T_i^n x_n, i \geq 1, \\ D_{n+1} = \{z \in D_n : \sup_{i\geq 1} D_f(z, y_{n,i}) \leq \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = P^f_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{3.15}$$

where $\xi_n = (k_n - 1)\sup_{p\in\mathcal{F}} \zeta(D_f(p, x_n))$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $P^f_{D_{n+1}}$ is the Bregman projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty, then $\{x_n\}$ converges strongly to $P^f_{\mathcal{F}} x_1$.

As a direct consequence of Theorem 3.1 and Remark 2.3, we obtain the convergence result concerning Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in a uniformly smooth and uniformly convex Banach space.

Theorem 3.3 *Let X be a uniformly smooth and uniformly convex Banach space and $J : X \to 2^{X^*}$ is the normalized duality mapping. Let D be a nonempty, closed, and convex subset on X and let $T : D \to N(D)$ be a family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings with sequence $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that, for each $i \geq 1$, $\{T_i\}$ be uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let x_n be a sequence generated by

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_{n,i} = J^{-1}(\alpha_n J x_1 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J w_{n,i}), \quad w_{n,i} \in T_i^n x_n, i \geq 1, \\ D_{n+1} = \{z \in D_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{3.16}$$

where $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(\phi(p, x_n)) + \mu_n$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$, $\Pi_{D_{n+1}}$ is a projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Now, we provide examples of multi-valued mappings to which the results of the paper can be applied.

Example 3.1 Let D be a unit ball in a real Hilbert space l^2, $f(x) = \|x\|^2$. Since $\nabla f(y) = 2y$, the Bregman distance with respect to f

$$D_f(x,y) = \|x\|^2 - \|y\|^2 - 2\langle y, x-y\rangle = \|x-y\|^2, \quad \forall x,y \in D. \tag{3.17}$$

Let $\{T_i\}_{i=1}^{\infty} : D \to N(D)$ be a family multi-valued mapping defined by

$$\{T_i\}_{i=1}^{\infty} : \left(x_1^{(i)}, x_2^{(i)}, \ldots\right) \to \left(0, \left(x_1^{(i)}\right)^2, a_2 x_2^{(i)}, a_3 x_3^{(i)}, \ldots\right) \in l^2, \quad \forall \left(x_1^{(i)}, x_2^{(i)}, \ldots\right) \in D,$$

where any $\{a_j\}_{j=1}^{\infty}$ is a sequence in $(0,1)$ such that $\prod_{j=2}^{\infty} a_j = \frac{1}{2}$.

From Example 2.1, we know that $\{T_i\}_{i=1}^{\infty}$ is a family of closed and uniformly Bregman totally quasi-asymptotically nonexpansive multi-valued mappings with sequence $\{\nu_n\}$, $\{\mu_n\}$, $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that, for each $i \geq 1$, $\{T_i\}_{i=1}^{\infty}$ is uniformly L_i-Lipschitz continuous. $\{\alpha_n\}$, $\{\beta_n\}$ and f are the same as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by (3.1), then $\{x_n\}$ converges strongly to $P^f_{\mathcal{F}} x_1$, where $\mathcal{F} = \bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, $P^f_{\mathcal{F}}$ is the Bregman projection of X onto $\mathcal{F}$.

4 Application

In order to emphasize the importance of Theorem 3.1, we illustrate an application with the following important example, which entails equilibrium problems in the framework of reflexive Banach spaces.

Let X be a smooth, strictly convex, and reflexive Banach space, let D be a nonempty, closed, and convex subset of X and let $G : D \times D \to R$ be a bifunction satisfying the conditions: (A1) $G(x,x) = 0$, for all $x \in D$; (A2) $G(x,y) + G(y,x) \leq 0$, for any $x, y \in D$; (A3) for each $x, y, z \in D$, $\lim_{t\to 0} G(tz + (1-t)x, y) \leq G(x,y)$; (A4) for each given $x \in D$, the function $y \longmapsto f(x,y)$ is convex and lower semicontinuous. The 'so-called' *equilibrium problem* for G is to find a $x^* \in D$ such that $G(x^*, y) \geq 0$, for each $y \in D$. The set of its solutions is denoted by $EP(G)$.

The resolvent of a bifunction G [5] is the operator $Res^f_G : X \to 2^D$ defined by

$$Res^f_G(x) = \left\{z \in D, G(z,y) + \left\langle \nabla f(z) - \nabla f(x), y - z\right\rangle \geq 0, \forall y \in D\right\}, \quad \forall x \in X. \tag{4.1}$$

If $f : X \to (-\infty, +\infty]$ is a strongly coercive and Gâteaux differentiable function, and G satisfies conditions (A1)-(A4), then $\mathrm{dom}(Res^f_G) = X$ (see [5]). We also know:

(1) Res^f_G is single-valued;
(2) Res^f_G is a Bregman firmly nonexpansive mapping, so it is a closed Bregman total quasi-asymptotically nonexpansive mapping;
(3) $F(Res^f_G) = EP(G)$;
(4) $EP(G)$ is a closed and convex subset of D;
(5) for all $x \in X$ and for all $p \in F(Res^f_G)$, we have

$$D_f\left(p, Res^f_G(x)\right) + D_f\left(Res^f_G(x), x\right) \leq D_f(p,x). \tag{4.2}$$

In addition, by Reich and Sabach [24], if f is uniformly Fréchet differentiable and bounded on bounded subsets of X, then we see that $F(Res^f_G) = \hat{F}(Res^f_G) = EP(G)$ is closed and convex. Hence, by replacing $T = Res^f_G$ in Theorem 3.1, we obtain the following result.

Theorem 4.1 *Let D, X, $\{\alpha_n\}$, $\{\beta_n\}$, and f be the same as in Theorem 3.1. Let $\{G_i : D \times D \to R\}$ be a countable families of bifunction which satisfies the conditions (A1)-(A4) such that $EP(G_i) \neq \emptyset$. Let $Res^f_{G_i}(x) : D \to 2^D$, $i = 1, 2, \dots$, be the family of mappings defined by*

$$Res^f_{G_i}(x) = \{z \in D : G_i(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in D\}, \quad x \in X. \tag{4.3}$$

Let the sequence $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ G_i(w_{n,i}, y) + \langle \nabla f(w_{n,i}) - \nabla f(x_n), y - w_{n,i} \rangle \geq 0, \quad \forall y \in D, i \geq 1, \\ y_{n,i} = \nabla f^*[\alpha_n \nabla f(x_1) + (1 - \alpha_n)(\beta_n \nabla f(x_n) \\ \qquad + (1 - \beta_n) \nabla f(w_{n,i})], \quad w_{n,i} \in Res^f_{G_i}(x_n), i \geq 1, \\ D_{n+1} = \{z \in D_n : \sup_{i \geq 1} D_f(z, y_{n,i}) \leq \alpha_n D_f(z, x_1) + (1 - \alpha_n) D_f(z, x_n) + \xi_n\}, \\ x_{n+1} = P^f_{D_{n+1}} x_1 \quad (n = 1, 2, \dots), \end{cases} \tag{4.4}$$

where $\xi_n = (k_n - 1) \sup_{p \in \mathcal{F}} \zeta(D_f(p, x_n))$, $\mathcal{F} = \bigcap_{i=1}^{\infty} F(Res^f_{G_i})$, $P^f_{D_{n+1}}$ is the Bregman projection of X onto D_{n+1}. If $\mathcal{F}$ is nonempty, then $\{x_n\}$ converges strongly to $P^f_{\mathcal{F}} x_1$, which is a common solution of the system of equilibrium problems for G_m, $m = 1, 2, \dots$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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