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New feasible iterative algorithms and strong convergence theorems for bilevel split equilibrium problems

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Abstract
In this paper, we first introduce and investigate a bilevel split equilibrium problem (BSEP) which can be regarded as a new development in the field of equilibrium problems. We provide some new feasible iterative algorithms for BSEP and establish strong convergence theorems for these iterative algorithms in different spaces.

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Keywords: metric projection; adjoint operator; equilibrium problem (EP); split equilibrium problem (SEP); bilevel split equilibrium problem (BSEP); bilevel convex optimization problem (BCOP); the common solution of equilibrium problems (CEP); feasible iterative algorithm; strong convergence theorem

1 Introduction and preliminaries

Let K be a closed convex subset of a real Hilbert space H. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product of H and the norm of H, respectively. For each point $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\|x - P_K x\| \le \|x - y\| \quad \text{for all } y \in K.$$

The mapping P_K is called the *metric projection* from H onto K. It is well known that P_K has the following properties:

(i) $\langle x - y, P_K x - P_K y\rangle \ge \|P_K x - P_K y\|^2$ for every $x, y \in H$.
(ii) For $x \in H$ and $z \in K$, $z = P_K(x) \Leftrightarrow \langle x - z, z - y\rangle \ge 0$ for all $y \in K$.
(iii) For $x \in H$ and $y \in K$,

$$\|y - P_K(x)\|^2 + \|x - P_K(x)\|^2 \le \|x - y\|^2. \tag{1.1}$$

Let H_1 and H_2 be two Hilbert spaces. Let $A : H_1 \to H_2$ and $A^* : H_2 \to H_1$ be two bounded linear operators. A^* is called the adjoint operator (or adjoint) of A if

$$\langle Az, w\rangle = \langle z, A^* w\rangle \quad \text{for all } z \in H_1 \text{ and } w \in H_2.$$

It is known that the adjoint operator of a bounded linear operator on a Hilbert space always exists and is bounded linear and unique. Moreover, it is not hard to show that if A^* is an adjoint operator of A, then $\|A\| = \|A^*\|$. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

Example 1.1 ([1]) Let $H_2 = \mathbb{R}$ with the standard norm $|\cdot|$ and $H_1 = \mathbb{R}^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for some $\alpha = (a_1, a_2) \in \mathbb{R}^2$. $\langle x, y\rangle = xy$ denotes the inner product of H_2 for some $x, y \in H_2$, and $\langle \alpha, \beta\rangle = \sum_{i=1}^{2} a_i b_i$ denotes the inner product of H_1 for some $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in H_1$. Let $A\alpha = a_2 - a_1$ for $\alpha = (a_1, a_2) \in H_1$ and $Bx = (-x, x)$ for $x \in H_2$, then B is an adjoint operator of A.

Example 1.2 ([1]) Let $H_1 = \mathbb{R}^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for some $\alpha = (a_1, a_2) \in \mathbb{R}^2$ and $H_2 = \mathbb{R}^3$ with the norm $\|\gamma\| = (c_1^2 + c_2^2 + c_3^2)^{\frac{1}{2}}$ for some $\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$. Let $\langle \alpha, \beta\rangle = \sum_{i=1}^{2} a_i b_i$ and $\langle \gamma, \eta\rangle = \sum_{i=1}^{3} c_i d_i$ denote the inner product of H_1 and H_2, respectively, where $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in H_1$, $\gamma = (c_1, c_2, c_3), \eta = (d_1, d_2, d_3) \in H_2$. Let $A\alpha = (a_2, a_1, a_1 - a_2)$ for $\alpha = (a_1, a_2) \in H_1$ and $B\gamma = (c_2 + c_3, c_1 - c_3)$ for $\gamma = (c_1, c_2, c_3) \in H_2$. Obviously, B is an adjoint operator of A.

Let f be a bi-function from $C \times C$ to $\mathbb{R}$. The classical equilibrium problem (EP, for short) is defined as follows.

(EP) Find $p \in C$ such that $f(p, y) \geq 0$, $\forall y \in C$.

The set of such solutions is denoted by EP(f), that is, $\mathrm{EP}(f) = \{u \in C : f(u, v) \geq 0, \forall v \in C\}$. In fact, equilibrium problem has an important relationship with variational inequality problem. For example, let $T : C \to H$ be a nonlinear mapping satisfying $\langle Tx, y - x\rangle \geq 0$ for all $x, y \in C$. Then $x \in \mathrm{EP}(f)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$. It is known that the EP is an important mathematical model for nonlinear analysis and applied sciences which is generalized to many new mathematical models and includes many important problems arising in physics, engineering, science optimization, economics, network, game theory, complementary problems, variational inequalities problems, saddle point problems, fixed point problems and others; for details, one can refer to [2–8] and references therein. Many authors have proposed some useful methods to solve the EP; see, for instance, [2–5, 9–17] and references therein.

Recent investigations and developments in equilibrium theory as well as optimization theory have been applied to connect fundamental sciences with the real world. According to our experience, useful methods of real world problems often need to be used to solve several problems arising in different spaces. In view of this, recent studies focus on split problems which are more closed in the real world applications; see, for instance, [1, 18–24] and the references therein. Recently, He [1] considered the following split equilibrium problem. Let H_1 and H_2 be two real Hilbert spaces. Let C be a closed convex subset of H_1 and K be a closed convex subset of H_2. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be two bi-functions, and $A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem (SEP, in short) is defined as follows:

(SEP) Find $p \in C$ such that $f(p, y) \geq 0$, $\forall y \in C$, and $u := Ap$ satisfying $g(u, v) \geq 0$, $\forall v \in K$.

In [1], the author established weak convergence algorithms and strong convergence algorithms for SEP (see [1] for more details).

Motivated and inspired by the works mentioned above, in this paper we shall introduce and investigate the following new problem. Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C be a closed convex subset of H_1, Q be a closed convex subset of H_2 and K be a closed convex subset of H_3. Let $f : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ and $h : K \times K \to \mathbb{R}$ be three bi-functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with theirs adjoint operators A^* and B^*, respectively. The mathematical model about *bilevel split equilibrium problem* (BSEP, in short) is defined as follows:

(BSEP) Find $p \in C$ and $q \in Q$ such that
(i) $f(p,x) \geq 0$ and $g(q,y) \geq 0$ for all $x \in C$ and $y \in Q$;
(ii) $Ap = Bq := u$;
(iii) $h(u,z) \geq 0$ for all $z \in K$.

In fact, BSEP can be regarded as a new development in the field of equilibrium problems and contains several important problems as special cases. It was profoundly believed that BSEP will motivate and inspire further scientific activities in the fields of equilibrium problems, optimization problems, game problems, complementary problems, variational inequalities problems, fixed point problems and their applications.

Example A Let H_1, H_2 and H_3 be three real Hilbert spaces. Let $C \subset H_1$, $Q \subset H_2$ and $K \subset H_3$ be three closed convex sets. Let $f^* : C \to \mathbb{R}$, $g^* : Q \to \mathbb{R}$ and $h^* : K \to \mathbb{R}$ be three convex functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with their adjoint operators A^* and B^*, respectively. Let

$$f(x,\alpha) = f^*(x) - f^*(\alpha) \quad \text{for } x,\alpha \in C,$$
$$g(y,\beta) = g^*(y) - g^*(\beta) \quad \text{for } y,\beta \in Q,$$

and

$$h(z,\eta) = h^*(z) - h^*(\eta) \quad \text{for } z,\eta \in K.$$

Then BSEP reduces the bilevel convex optimization problem (BCOP):

(BCOP) Find $p \in C$ and $q \in Q$ such that $u := Ap = Bq \in K, f^*(x) \geq f^*(p), g^*(y) \geq g^*(q)$ and $h^*(z) \geq h^*(u)$ for all $x \in C$, $y \in Q$ and $z \in K$.

Example B Let H_1, H_2 and H_3 be three real Hilbert spaces. Let $C \subset H_1$, $Q \subset H_2$ and $K \subset H_3$ be three closed convex sets. Let $T : C \to H_1$, $S : Q \to H_2$ and $G : K \to H_3$ be three nonlinear operators. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with their adjoint operators A^* and B^*, respectively. If $f(p,x) = \langle Tp, x-p\rangle$, $g(q,y) = \langle Sq, y-q\rangle$ and $h(u,z) = \langle Tu, z-u\rangle$, then BSEP reduces to the bilevel split variational inequality problem (BSVI):

(BSVI) Find $p \in C$ and $q \in Q$ such that $u := Ap = Bq \in K$ satisfying $\langle Tp, x-p\rangle \geq 0$, $\langle Sq, y-q\rangle \geq 0$ and $\langle Tu, z-u\rangle \geq 0$ for all $x \in C$, $y \in Q$ and $z \in K$.

Example C Let H_1 and H_2 be two real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Let $C \subset H_1$, $Q \subset H_1$ and $K \subset H_2$ be three closed convex sets. If $H_1 = H_2$ and $A = B$, then BSEP reduces to the following split equilibrium problem (1) (SEP$^{(1)}$):

(SEP$^{(1)}$) Find $p \in C$ and $q \in Q$ such that $u := Bp = Bq \in K$ satisfying $f(p,x) \geq 0$, $g(q,y) \geq 0$ and $h(u,z) \geq 0$ for all $x \in C$, $y \in Q$ and $z \in K$.

Example D Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $C \subset H_1$, $Q \subset H_2$ and $K \subset H_2$ be three closed convex sets with $Q \cap K \neq \emptyset$. If $H_2 = H_3$ and $B = I$ (identity operator), then BSEP reduces to the following split equilibrium problem (2) (SEP$^{(2)}$):

(SEP$^{(2)}$) Find $p \in C$ such that $u := Ap \in Q \cap K$ satisfying $f(p,x) \geq 0$, $g(u,y) \geq 0$ and $h(u,z) \geq 0$ for each $x \in C$, $y \in Q$ and $z \in K$.

Especially, if $g(p,y) \equiv 0$ for all $p, y \in Q$, then (SEP$^{(1)}$) reduces to finding $p \in C$ such that $u := Ap \in K$ satisfying $f(p,x) \geq 0$ and $h(u,z) \geq 0$ for all $x \in C$ and $z \in K$, which was studied in [1].

Example E Let H_1 and H_2 be two real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $C \subset H_1$, $Q \subset H_1$ and $K \subset H_2$ be three closed convex sets with $C \cap Q \neq \emptyset$. If $H_1 = H_2$ and $A = I$ (identity operator), then BSEP reduces to the following split equilibrium problems (3) (SEP$^{(3)}$):

(SEP$^{(3)}$) Find $p \in C \cap Q$ such that $u := Bp \in K$ satisfying $f(p,x) \geq 0$, $g(p,y) \geq 0$ and $h(u,z) \geq 0$ for all $x \in C$, $y \in Q$ and $z \in K$.

Example F In Example A, if $H_1 = H_2 = H_3 := H$, $C = Q = K \subset H$ and $A = B = I$ (identity operator), then BSEP reduces to the common solution of equilibrium problems (CEP):

(CEP) Find $p \in C$ such that $f(p,x) \geq 0$, $g(p,y) \geq 0$ and $h(p,z) \geq 0$ for each $x, y, z \in C$.

The paper is divided into four sections. In Sections 1 and 2, we first introduce and investigate a bilevel split equilibrium problem (BSEP) and then provide some new feasible iterative algorithms for BSEP and establish strong convergence theorems for these iterative algorithms in different spaces. In Section 3, we give the proof of the main result Theorem 2.1 in detail. Finally, an example illustrating Theorem 2.1 is given in Section 4.

2 Feasible iterative algorithms for BSEP and their strong convergence theorems

In 1994, Blum and Oettli [2] established the following important existence theorem which plays a key role in solving equilibrium problems, variational inequality problems and optimization problems.

Lemma 2.1 (Blum and Oettli [2]) *Let K be a nonempty closed convex subset of H and F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions.*

(A1) $F(x,x) = 0$ *for all* $x \in K$;

(A2) *F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in K$;*
(A3) *for each $x, y, z \in K$,*

$$\limsup_{t\to 0^+} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) *for each $x \in K$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.*
Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \text{for all } y \in K.$$

In this paper, we first introduce a new iterative algorithm for BSEP and establish a strong convergence theorem for this iterative algorithm. Here, the space $H_1 \times H_2$ denotes the product space of two real Hilbert spaces H_1 and H_2, which is endowed with the usual linear operation and norm, namely, for $(x,y), (\bar{x},\bar{y}) \in H_1 \times H_2$ and $a, b \in \mathbb{R}$,

$$a(x,y) + b(\bar{x},\bar{y}) = (ax + b\bar{x}, ay + b\bar{y})$$

and

$$\bigl\|(x,y)\bigr\| = \|x\| + \|y\|.$$

Theorem 2.1 *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C be a closed convex subset of H_1, Q be a closed convex subset of H_2 and K be a closed convex subset of H_3. Let $f: C \times C \to \mathbb{R}$, $g: Q \times Q \to \mathbb{R}$ and $h: K \times K \to \mathbb{R}$ be three bi-functions. $A: H_1 \to H_3$ and $B: H_2 \to H_3$ are two bounded linear operators with their adjoint operators A^* and B^*, respectively. Suppose that all the bi-functions f, g and h satisfy conditions* (A1)-(A4). *Let $x_1 \in C$, $y_1 \in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad v_n = T^g_{r_n} y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}Au_n + \frac{1}{2}Bv_n), \\ l_n = P_C(u_n - \xi A^*(Au_n - w_n)), \qquad k_n = P_Q(v_n - \xi B^*(Bv_n - w_n)), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \\ \qquad\qquad \leq \|u_n - x\|^2 + \|v_n - y\|^2 \leq \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\xi \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ and $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C and P_Q are two projection operators from H_1 into C and from H_2 into Q, respectively. Suppose that

$$\Omega = \bigl\{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : Ap = Bq \in \mathrm{EP}(h)\bigr\} \neq \emptyset.$$

Then there exists $(p,q) \in \Omega$ such that
(a) *$(x_n, y_n) \to (p,q)$ as $n \to \infty$;*
(b) *$(u_n, v_n) \to (p,q)$ as $n \to \infty$;*
(c) *$w_n \to w^* := Ap = Bq \in \mathrm{EP}(h)$ as $n \to \infty$.*

The following conclusion is immediate from Theorem 2.1 by putting $A = B$.

Corollary 2.1 *Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two closed convex subsets of H_1 and K be a closed convex subset of H_2. Let $f: C\times C\to\mathbb{R}$, $g: Q\times Q\to\mathbb{R}$ and $h: K\times K\to\mathbb{R}$ be three bi-functions. $B: H_1\to H_2$ is a bounded linear operator with its adjoint operator B^*. Suppose that all the bi-functions f, g and h satisfy conditions* (A1)-(A4). *Let $x_1\in C$, $y_1\in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n}x_n, \qquad v_n = T^g_{r_n}y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}Bu_n + \frac{1}{2}Bv_n), \\ l_n = P_C(u_n - \xi B^*(Bu_n - w_n)), \qquad k_n = P_Q(v_n - \xi B^*(Bv_n - w_n)), \\ C_{n+1}\times Q_{n+1} = \{(x,y)\in C_n\times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \le \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n\in\mathbb{N}, \end{cases}$$

where $\xi\in(0,\frac{1}{\|B\|^2})$ and $\{r_n\}\subset(0,+\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C and P_Q are two projection operators from H_1 into C and from H_1 into Q, respectively. Suppose that

$$\Omega = \{(p,q)\in \mathrm{EP}(f)\times \mathrm{EP}(g) : Bp = Bq\in \mathrm{EP}(h)\}\neq\emptyset.$$

Then there exists $(p,q)\in\Omega$ such that

(a) $(x_n, y_n)\to(p,q)$ *as* $n\to\infty$;

(b) $(u_n, v_n)\to(p,q)$ *as* $n\to\infty$;

(c) $w_n\to w^* := Bp = Bq\in \mathrm{EP}(h)$ *as* $n\to\infty$.

If $H_2 = H_3$ and $B = I$, then Theorem 2.1 reduces to the following corollary.

Corollary 2.2 *Let H_1 and H_2 be two real Hilbert spaces. Let $C\subset H_1$ and $Q,K\subset H_2$ be three closed convex sets. Let $f: C\times C\to\mathbb{R}$, $g: Q\times Q\to\mathbb{R}$ and $h: K\times K\to\mathbb{R}$ be three bi-functions. $A: H_1\to H_2$ is a bounded linear operator with its adjoint operator A^*. Suppose that all the bi-functions f, g and h satisfy conditions* (A1)-(A4). *Let $x_1\in C$, $y_1\in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n}x_n, \qquad v_n = T^g_{r_n}y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}Au_n + \frac{1}{2}v_n), \\ l_n = P_C(u_n - \xi A^*(Au_n - w_n)), \qquad k_n = v_n - \xi(v_n - w_n), \\ C_{n+1}\times Q_{n+1} = \{(x,y)\in C_n\times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \le \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n\in\mathbb{N}, \end{cases}$$

where $\xi\in(0,\min\{1,\frac{1}{\|A\|^2}\})$ and $\{r_n\}\subset(0,+\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C and P_Q are two projection operators from H_1 into C and from H_2 into Q, respectively. Suppose that

$$\Omega = \{p\in \mathrm{EP}(f) : Ap\in \mathrm{EP}(g)\cap \mathrm{EP}(h)\}\neq\emptyset.$$

Then there exists $p \in \Omega$ *such that*

(a) $x_n \to p$ *as* $n \to \infty$;

(b) $u_n \to p$ *as* $n \to \infty$;

(c) $v_n, y_n, w_n \to w^* := Ap$ *as* $n \to \infty$.

If $H_1 = H_2$ and $A = I$, then Theorem 2.1 reduces to the following corollary.

Corollary 2.3 *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* $C, Q \subset H_1$ *and* $K \subset H_2$ *be three closed convex sets. Let* $f : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ *and* $h : K \times K \to \mathbb{R}$ *be three bi-functions.* $B : H_1 \to H_2$ *is a bounded linear operator with its adjoint operator* B^*. *Suppose that all the bi-functions* f, g *and* h *satisfy conditions* (A1)-(A4). *Let* $x_1 \in C$, $y_1 \in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad v_n = T^g_{r_n} y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}u_n + \frac{1}{2}Bv_n), \\ l_n = u_n - \xi(u_n - w_n), \qquad k_n = P_q(v_n - \xi B^*(Bv_n - w_n)), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \le \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \min\{1, \frac{1}{\|B\|^2}\})$ *and* $\{r_n\} \subset (0, +\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *and* P_Q *are two projection operators from* H_1 *into* C *and from* H_2 *into* Q, *respectively. Suppose that*

$$\Omega = \{p \in \mathrm{EP}(f) \cap \mathrm{EP}(g) : Ap \in \mathrm{EP}(h)\} \neq \emptyset.$$

Then there exists $p \in \Omega$ *such that*

(a) $x_n, u_n \to p$ *as* $n \to \infty$;

(b) $y_n, v_n \to p$ *as* $n \to \infty$;

(c) $w_n \to w^* := Ap$ *as* $n \to \infty$.

Putting $A = B = I$ (identical operator), $H_1 = H_2 = H_3 = H$ and $C = Q = K$, then we have the following result.

Corollary 2.4 *Let* H *be a real Hilbert space. Let* C *be a closed convex subset of* H. *Let* $f, g, h : C \times C \to \mathbb{R}$ *be three bi-functions. Suppose that all the bi-functions* f, g *and* h *satisfy conditions* (A1)-(A4). *Let* $x_1, y_1 \in C$, $C_1 = C$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad v_n = T^g_{r_n} y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}u_n + \frac{1}{2}v_n), \\ l_n = u_n - \xi(u_n - w_n), \qquad k_n = v_n - \xi(v_n - w_n), \\ C_{n+1} \times C_{n+1} = \{(x,y) \in C_n \times C_n : \|l_n - x\|^2 + \|k_n - y\|^2 \le \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{C_{n+1}}(y_1), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0,1)$ *and* $\{r_n\} \subset (0,+\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *is a projection operator from* H *into* C. *Suppose that*

$$\Omega = \{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : p = q \in \mathrm{EP}(h)\} \neq \emptyset.$$

Then there exists $(p,q) \in \Omega$ *such that*

(a) $(x_n, y_n) \to (p,q)$ *as* $n \to \infty$;

(b) $(u_n, v_n) \to (p,q)$ *as* $n \to \infty$;

(c) $w_n \to w^* := p = q \in \mathrm{EP}(h)$ *as* $n \to \infty$.

Remark 2.1 In Corollary 2.2, it is obvious that

$$\Omega = \{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : p = q \in \mathrm{EP}(h)\} \neq \emptyset$$

implies

$$\Omega = \{p \in \mathrm{EP}(f) \cap \mathrm{EP}(g) \cap \mathrm{EP}(h)\} \neq \emptyset.$$

Hence, the problem studied in Corollary 2.4 is still the study of a common solution of three equilibrium problems in essence.

If C, Q, K are linear subspaces of a real Hilbert space, then we have the following corollaries from Theorem 2.1 and Corollary 2.1.

Corollary 2.5 *Let* H_1, H_2 *and* H_3 *be three real Hilbert spaces. Let* $C \subset H_1$, $Q \subset H_2$ *and* $K \subset H_3$ *be three linear subspaces. Let* $f : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ *and* $h : K \times K \to \mathbb{R}$ *be three bi-functions.* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *are two bounded linear operators with their adjoint operators* A^* *and* B^*, *respectively. Suppose that all the bi-functions* f, g *and* h *satisfy conditions* (A1)-(A4). *Let* $x_1 \in C$, $y_1 \in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ *and* $\{w_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad v_n = T^g_{r_n} y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}Au_n + \frac{1}{2}Bv_n), \\ l_n = u_n - \xi A^*(Au_n - w_n), \qquad k_n = v_n - \xi B^*(Bv_n - w_n), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \leq \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \leq \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ *and* $\{r_n\} \subset (0,+\infty)$ *with* $\liminf_{n\to+\infty} r_n > 0$, P_C *and* P_Q *are two projection operators from* H_1 *into* C *and from* H_2 *into* Q, *respectively. Suppose that*

$$\Omega = \{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : Ap = Bq \in \mathrm{EP}(h)\} \neq \emptyset.$$

Then there exists $(p,q) \in \Omega$ *such that*

(a) $(x_n, y_n) \to (p,q)$ *as* $n \to \infty$;

(b) $(u_n, v_n) \to (p, q)$ *as* $n \to \infty$;
(c) $w_n \to w^* := Ap = Bq \in \mathrm{EP}(h)$ *as* $n \to \infty$.

Corollary 2.6 *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$, $Q \subset H_1$ and $K \subset H_2$ be three linear subspaces. Let $f : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ and $h : K \times K \to \mathbb{R}$ be three bi-functions. $B : H_1 \to H_2$ is a bounded linear operator with its adjoint operator B^*. Suppose that all the bi-functions f, g and h satisfy conditions* (A1)-(A4). *Let $x_1 \in C$, $y_1 \in Q$, $C_1 = C$, $Q_1 = Q$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences generated by*

$$\begin{cases} u_n = T^f_{r_n} x_n, \qquad v_n = T^g_{r_n} y_n, \qquad w_n = T^h_{r_n}(\frac{1}{2}Bu_n + \frac{1}{2}Bv_n), \\ l_n = u_n - \xi B^*(Bu_n - w_n), \qquad k_n = v_n - \xi B^*(Bv_n - w_n), \\ C_{n+1} \times Q_{n+1} = \{(x,y) \in C_n \times Q_n : \|l_n - x\|^2 + \|k_n - y\|^2 \le \|u_n - x\|^2 + \|v_n - y\|^2 \\ \qquad\qquad \le \|x_n - x\|^2 + \|y_n - y\|^2\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \\ y_{n+1} = P_{Q_{n+1}}(y_1), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\xi \in (0, \frac{1}{\|B\|^2})$ and $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to+\infty} r_n > 0$, P_C and P_Q are two projection operators from H_1 into C and from H_1 into Q, respectively. Suppose that

$$\Omega = \{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : Bp = Bq \in \mathrm{EP}(h)\} \neq \emptyset.$$

Then there exists $(p, q) \in \Omega$ such that
(a) $(x_n, y_n) \to (p, q)$ *as* $n \to \infty$;
(b) $(u_n, v_n) \to (p, q)$ *as* $n \to \infty$;
(c) $w_n \to w^* := Bp = Bq \in \mathrm{EP}(h)$ *as* $n \to \infty$.

Remark 2.2 In fact, the problem studied by Corollaries 2.1-2.3 and Corollary 2.6 is (SEP).

In order to prove Theorem 2.1, we need the following crucial known results.

Lemma 2.2 (see [10]) *Let K be a nonempty closed convex subset of H, and let F be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying* (A1)-(A4). *For $r > 0$, define a mapping $T^F_r : H \to K$ as follows*:

$$T^F_r(x) = \left\{ z \in K : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in K \right\} \tag{2.2}$$

for all $x \in H$. Then the following hold:
(i) T^F_r *is single-valued and* $\mathcal{F}(T^F_r) = \mathrm{EP}(F)$ *for* $\forall r > 0$ *and* $\mathrm{EP}(F)$ *is closed and convex*;
(ii) T^F_r *is firmly nonexpansive, that is, for any $x, y \in H$,*
$\|T^F_r x - T^F_r y\|^2 \le \langle T^F_r x - T^F_r y, x - y\rangle$.

Lemma 2.3 ([20]) *Let F^F_r be the same as in Lemma* 2.2. *If $\mathcal{F}(T^F_r) = \mathrm{EP}(F) \neq \emptyset$, then, for any $x \in H$ and $x^* \in \mathcal{F}(T^F_r)$, $\|T^F_r x - x\|^2 \le \|x - x^*\|^2 - \|T^F_r x - x^*\|^2$.*

Lemma 2.4 ([1, 19]) *Let the mapping T_r^F be defined as in Lemma* 2.2. *Then, for $r, s > 0$ and $x, y \in H$,*

$$\left\| T_r^F(x) - T_s^F(y) \right\| \le \|x - y\| + \frac{|s-r|}{s} \left\| T_s^F(y) - y \right\|.$$

In particular, $\|T_r^F(x) - T_r^F(y)\| \le \|x - y\|$ for any $r > 0$ and $x, y \in H$, that is, T_r^F is nonexpansive for any $r > 0$.

Lemma 2.5 (see, *e.g.*, [25]) *Let H be a real Hilbert space. Then the following hold*:
(a) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$ *for all $x, y \in H$*;
(b) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ *for all $x, y \in H$ and $\alpha \in [0, 1]$*.

3 Proof of Theorem 2.1

Applying Lemmas 2.1 and 2.2, we know that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are all well defined. It is also easy to verify that C_n, Q_n, $C_n \times Q_n$ are closed convex sets for $n \in \mathbb{N}$.

Now, we claim $C_n \times Q_n \neq \emptyset$ for all $n \in \mathbb{N}$. Indeed, it suffices to prove that $\Omega \subset C_n \times Q_n$ for all $n \in \mathbb{N}$. Let $(x^*, y^*) \in \Omega$. Then $x^* \in \mathrm{EP}(f)$, $y^* \in \mathrm{EP}(g)$ and

$$w^* := Ax^* = By^* \in \mathrm{EP}(h).$$

Let $n \in \mathbb{N}$ be given. By Lemma 2.3, we have

$$\begin{aligned} &\left\| u_n - x^* \right\| \le \left\| x_n - x^* \right\|, \qquad \left\| v_n - y^* \right\| \le \left\| y_n - y^* \right\|, \\ &\left\| w_n - w^* \right\| \le \left\| \frac{Au_n + Bv_n}{2} - w^* \right\|, \\ &\left\| w_n - w^* \right\|^2 \le \frac{1}{2} \left\| Au_n - w^* \right\|^2 + \frac{1}{2} \left\| Bv_n - w^* \right\|^2 \quad \text{(by Lemma 2.5)}. \end{aligned} \tag{3.1}$$

From (2.1), (3.1) and Lemma 2.5, we obtain

$$\begin{aligned} \left\| l_n - x^* \right\|^2 &= \left\| P_C\big(u_n - \xi A^*(Au_n - w_n)\big) - x^* \right\|^2 \le \left\| u_n - x^* - \xi A^*(Au_n - w_n) \right\|^2 \\ &= \left\| u_n - x^* \right\|^2 + \left\| \xi A^*(Au_n - w_n) \right\|^2 - 2\xi \left\langle u_n - x^*, A^*(Au_n - w_n) \right\rangle \\ &= \left\| u_n - x^* \right\|^2 + \left\| \xi A^*(Au_n - w_n) \right\|^2 - 2\xi \left\langle Au_n - Ax^*, Au_n - w_n \right\rangle \\ &= \left\| u_n - x^* \right\|^2 + \left\| \xi A^*(Au_n - w_n) \right\|^2 - 2\xi \left\langle Au_n - w^*, Au_n - w_n \right\rangle \\ &= \left\| u_n - x^* \right\|^2 + \left\| \xi A^*(Au_n - w_n) \right\|^2 - \xi \left\| Au_n - w^* \right\|^2 \\ &\quad - \xi \|Au_n - w_n\|^2 + \xi \left\| w_n - w^* \right\|^2 \\ &\le \left\| u_n - x^* \right\|^2 - \xi \big(1 - \xi \left\| A^* \right\|^2\big) \|Au_n - w_n\|^2 \\ &\quad - \xi \left\| Au_n - w^* \right\|^2 + \xi \left\| w_n - w^* \right\|^2 \\ &\le \left\| u_n - x^* \right\|^2 - \xi \big(1 - \xi \left\| A^* \right\|^2\big) \|Au_n - w_n\|^2 - \xi \left\| Au_n - w^* \right\|^2 \\ &\quad + \frac{\xi}{2} \left\| Au_n - w^* \right\|^2 + \frac{\xi}{2} \left\| Bv_n - w^* \right\|^2 \end{aligned}$$

$$\begin{aligned}
&= \|u_n - x^*\|^2 - \xi\left(1 - \xi\|A^*\|^2\right)\|Au_n - w_n\|^2 \\
&\quad - \frac{\xi}{2}\|Au_n - w^*\|^2 + \frac{\xi}{2}\|Bv_n - w^*\|^2 \qquad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
\|k_n - y^*\|^2 &= \left\|P_Q\left(v_n - \xi B^*(Bv_n - w_n)\right) - y^*\right\|^2 \leq \left\|v_n - y^* - \xi B^*(Bv_n - w_n)\right\|^2 \\
&= \|v_n - y^*\|^2 + \left\|\xi B^*(Bv_n - w_n)\right\|^2 - 2\xi\left\langle v_n - y^*, B^*(Bv_n - w_n)\right\rangle \\
&= \|v_n - y^*\|^2 + \left\|\xi B^*(Bv_n - w_n)\right\|^2 - 2\xi\left\langle Bv_n - By^*, Bv_n - w_n\right\rangle \\
&= \|v_n - y^*\|^2 + \left\|\xi B^*(Bv_n - w_n)\right\|^2 - 2\xi\left\langle Bv_n - w^*, Bv_n - w_n\right\rangle \\
&= \|v_n - y^*\|^2 + \left\|\xi B^*(Bv_n - w_n)\right\|^2 - \xi\|Bv_n - w^*\|^2 \\
&\quad - \xi\|Bv_n - w_n\|^2 + \xi\|w_n - w^*\|^2 \\
&\leq \|v_n - y^*\|^2 - \xi\left(1 - \xi\|B^*\|^2\right)\|Bv_n - w_n\|^2 - \xi\|Bv_n - w^*\|^2 + \xi\|w_n - w^*\|^2 \\
&\leq \|v_n - y^*\|^2 - \xi\left(1 - \xi\|B^*\|^2\right)\|Bv_n - w_n\|^2 - \xi\|Bv_n - w^*\|^2 \\
&\quad + \frac{\xi}{2}\|Au_n - w^*\|^2 + \frac{\xi}{2}\|Bv_n - w^*\|^2 \\
&= \|v_n - y^*\|^2 - \xi\left(1 - \xi\|B^*\|^2\right)\|Bv_n - w_n\|^2 \\
&\quad - \frac{\xi}{2}\|Bv_n - w^*\|^2 + \frac{\xi}{2}\|Au_n - w^*\|^2. \qquad (3.3)
\end{aligned}$$

By taking into account inequalities (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned}
\|l_n - x^*\|^2 + \|k_n - y^*\|^2 &\leq \|u_n - x^*\|^2 + \|v_n - y^*\|^2 - \xi\left(1 - \xi\|A^*\|^2\right)\|Au_n - w_n\|^2 \\
&\quad - \xi\left(1 - \xi\|B^*\|^2\right)\|Bv_n - w_n\|^2 \\
&\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \xi\left(1 - \xi\|A^*\|^2\right)\|Au_n - w_n\|^2 \\
&\quad - \xi\left(1 - \xi\|B^*\|^2\right)\|Bv_n - w_n\|^2, \qquad (3.4)
\end{aligned}$$

which implies

$$\|l_n - x^*\|^2 + \|k_n - y^*\|^2 \leq \|u_n - x^*\|^2 + \|v_n - y^*\|^2 \leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2. \qquad (3.5)$$

Inequality (3.5) shows that $(x^*, y^*) \in C_n \times Q_n$. Hence $\Omega \subset C_n \times Q_n$ and $C_n \times Q_n \neq \emptyset$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, since $\Omega \subset C_n \times Q_n$, $C_{n+1} \subset C_n$, we have

$$x_{n+1} = P_{C_{n+1}}(x_1) \subset C_n.$$

Similarly, since $Q_{n+1} \subset Q_n$, we have

$$y_{n+1} = P_{Q_{n+1}}(y_1) \subset Q_n.$$

So, for any $(x^*, y^*) \in \Omega$, we get

$$\|x_{n+1} - x_1\| \leq \|x^* - x_1\|$$

and

$$\|y_{n+1} - y_1\| \le \|y^* - y_1\|.$$

The last inequalities deduce that $\{x_n\}$ and $\{y_n\}$ are bounded and hence show that $\{k_n\}$, $\{l_n\}$, $\{u_n\}$ and $\{v_n\}$ are all bounded. For some $n \in \mathbb{N}$ with $n > 1$, from $x_n = P_{C_n}(x_1) \subset C_n$, $y_n = P_{Q_n}(y_1) \subset Q_n$ and (1.1), we have

$$\|x_{n+1} - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_{n+1} - P_{C_n}(x_1)\|^2 + \|x_1 - P_{C_n}(x_1)\|^2 \le \|x_{n+1} - x_1\|^2,$$
$$\|y_{n+1} - y_n\|^2 + \|y_1 - y_n\|^2 = \|y_{n+1} - P_{C_n}(y_1)\|^2 + \|y_1 - P_{C_n}(y_1)\|^2 \le \|y_{n+1} - y_1\|^2,$$

which yields that

$$\|x_1 - x_n\| \le \|x_{n+1} - x_1\|, \qquad \|y_1 - y_n\| \le \|y_{n+1} - y_1\|.$$

Together with the boundedness of $\{x_n\}$ and $\{y_n\}$, we know $\lim_{n\to\infty} \|x_n - x_1\|$ and $\lim_{n\to\infty} \|y_n - y_1\|$ exist. For some $k, n \in \mathbb{N}$ with $k > n > 1$, due to $x_k = P_{C_k}(x_1) \subset C_n$, $y_k = P_{Q_k}(y_1) \subset Q_n$ and (1.1), we have

$$\begin{aligned} &\|x_k - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_k - P_{C_n}(x_1)\|^2 + \|x_1 - P_{C_n}(x_1)\|^2 \le \|x_k - x_1\|^2, \\ &\|y_k - y_n\|^2 + \|y_1 - y_n\|^2 = \|y_k - P_{Q_n}(y_1)\|^2 + \|y_1 - P_{Q_n}(y_1)\|^2 \le \|y_k - y_1\|^2. \end{aligned} \tag{3.6}$$

By (3.6), we have $\lim_{n\to\infty} \|x_n - x_k\| = 0$ and $\lim_{n\to\infty} \|y_n - y_k\| = 0$. Hence $\{x_n\}$ and $\{y_n\}$ are all Cauchy sequences. Let $x_n \to p$ and $y_n \to q$ for some $(p, q) \in C \times Q$. We want to prove that $(p, q) \in \Omega$. For any $n \in \mathbb{N}$, since

$$(x_{n+1}, y_{n+1}) \in C_{n+1} \times Q_{n+1} \subset C_n \times Q_n,$$

from (2.1), we have

$$\begin{aligned} \|l_n - x_{n+1}\|^2 + \|k_n - y_{n+1}\|^2 &\le \|u_n - x_{n+1}\|^2 + \|v_n - y_{n+1}\|^2 \\ &\le \|x_n - x_{n+1}\|^2 + \|y_n - y_{n+1}\|^2. \end{aligned} \tag{3.7}$$

By taking the limit from both sides of (3.7), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \|l_n - x_{n+1}\| = \lim_{n\to\infty} \|k_n - y_{n+1}\| = 0, \\ &\lim_{n\to\infty} \|u_n - x_{n+1}\| = \lim_{n\to\infty} \|v_n - y_{n+1}\| = 0. \end{aligned} \tag{3.8}$$

Moreover, by (3.8), we get

$$\begin{aligned} &\lim_{n\to\infty} \|l_n - u_n\| = \lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|l_n - x_n\| = 0, \\ &\lim_{n\to\infty} \|k_n - v_n\| = \lim_{n\to\infty} \|v_n - y_n\| = \lim_{n\to\infty} \|k_n - y_n\| = 0. \end{aligned} \tag{3.9}$$

Since $\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|v_n - y_n\| = 0$, we have $u_n \to p$ and $v_n \to q$ as $n \to \infty$. Moreover, we obtain $Au_n \to Ap$ and $Bv_n \to Bq$ as $n \to \infty$.

Now, we claim $p \in \mathrm{EP}(f)$ and $q \in \mathrm{EP}(g)$. In fact, for $r > 0$, by Lemma 2.4, we have

$$\begin{aligned}
\|T_r^f p - p\| &= \|T_r^f p - T_{r_n}^f x_n + T_{r_n}^f x_n - x_n + x_n - p\| \\
&\le \|x_n - p\| + \frac{|r_n - r|}{r_n}\|T_{r_n}^f x_n - x_n\| + \|T_{r_n}^f x_n - x_n\| + \|x_n - p\| \\
&= \|x_n - p\| + \frac{|r_n - r|}{r_n}\|u_n - x_n\| + \|u_n - x_n\| + \|x_n - p\| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\|T_r^g q - q\| &\le \|T_r^g q - T_{r_n}^g y_n + T_{r_n}^g y_n - y_n + y_n - q\| \\
&\le \|y_n - q\| + \frac{|r_n - r|}{r_n}\|T_{r_n}^g y_n - y_n\| + \|T_{r_n}^g y_n - y_n\| + \|y_n - q\| \\
&= \|y_n - q\| + \frac{|r_n - r|}{r_n}\|v_n - y_n\| + \|v_n - y_n\| + \|y_n - q\| \to 0.
\end{aligned}$$

So, $p \in \mathrm{EP}(f)$ and $q \in \mathrm{EP}(g)$.

Finally, we prove $Ap = Bq \in \mathrm{EP}(h)$. Setting

$$\theta = \min\{\xi(1 - \xi\|A^*\|^2), \xi(1 - \xi\|B^*\|^2)\}.$$

Then, for any $n \in \mathbb{N}$, by (3.4) and (3.9), we have

$$\begin{aligned}
&\theta\|Au_n - w_n\|^2 + \theta\|Bv_n - w_n\|^2 \\
&\quad\le \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - \|l_n - x^*\|^2 - \|k_n - y^*\|^2 \\
&\quad= \{\|x_n - x^*\| - \|l_n - x^*\|\}\{\|x_n - x^*\| + \|l_n - x^*\|\} \\
&\qquad+ \{\|y_n - y^*\| - \|k_n - y^*\|\}\{\|y_n - y^*\| + \|k_n - y^*\|\} \\
&\quad\le \|l_n - x_n\|\{\|x_n - x^*\| + \|l_n - x^*\|\} + \|k_n - y_n\|\{\|y_n - y^*\| + \|k_n - y^*\|\} \\
&\quad\to 0.
\end{aligned} \tag{3.10}$$

Hence (3.10) implies

$$\lim_{n\to\infty}\|Au_n - w_n\| = \lim_{n\to\infty}\|Bv_n - w_n\| = 0, \quad \text{and} \quad \lim_{n\to\infty}\|Au_n - Bv_n\| = 0. \tag{3.11}$$

Since $Au_n \to Ap$, $Bv_n \to Bq$ and (3.11), we obtain $Ap = Bq$ and $w_n \to w^*$, where $w^* := Ap = Bq$. On the other hand, for $r > 0$, by Lemma 2.4 again, we have

$$\begin{aligned}
&\|T_r^h w^* - w^*\| \\
&\quad= \left\|T_r^h w^* - T_{r_n}^h \frac{Au_n + Bv_n}{2} + T_{r_n}^h \frac{Au_n + Bv_n}{2} - \frac{Au_n + Bv_n}{2} + \frac{Au_n + Bv_n}{2} - w^*\right\| \\
&\quad\le \left\|\frac{Au_n + Bv_n}{2} - w^*\right\| + \frac{|r_n - r|}{r_n}\left\|T_{r_n}^h \frac{Au_n + Bv_n}{2} - \frac{Au_n + Bv_n}{2}\right\| \\
&\qquad+ \left\|T_{r_n}^h \frac{Au_n + Bv_n}{2} - \frac{Au_n + Bv_n}{2}\right\| + \left\|\frac{Au_n + Bv_n}{2} - w^*\right\|
\end{aligned}$$

$$= 2\left\| \frac{Au_n + Bv_n}{2} - w^* \right\| + \frac{|r_n - r|}{r_n} \left\| w_n - \frac{Au_n + Bv_n}{2} \right\| + \left\| w_n - \frac{Au_n + Bv_n}{2} \right\| \to 0.$$

Hence $w^* \in \mathrm{EP}(h)$, namely $Ap = Bq \in \mathrm{EP}(h)$. Therefore, conclusions (a), (b) and (c) are all proved. The proof is completed.

4 An example of Theorem 2.1

In this section, we give an example illustrating Theorem 2.1.

Example 4.1 Let $H_1 = R^2$, $H_2 = R^3$ and $H_3 = R^4$ be three real Hilbert spaces with the standard norm and inner product. For each $\alpha = (\alpha_1, \alpha_2) \in R^2$ and $\nu = (z_1, z_2, z_3, z_4) \in R^4$, define

$$A\alpha = (\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 - \alpha_2)$$

and

$$A^*\nu = (z_1 + z_3 + z_4, z_2 + z_3 - z_4).$$

Then A is a bounded linear operator from R^2 into R^4 with $\|A\| = \sqrt{3}$, and A^* is an adjoint operator of A with $\|A^*\| = \sqrt{3}$. For each $\beta = (\beta_1, \beta_2, \beta_3) \in R^3$ and $\nu = (z_1, z_2, z_3, z_4) \in R^4$, let

$$B\beta = (\beta_1, \beta_2, \beta_3, \beta_1 - \beta_2)$$

and

$$B^*\nu = (z_1 + z_4, z_2 - z_4, z_3).$$

Then B is a bounded linear operator from R^3 into R^4 with $\|B\| = \sqrt{3}$, and B^* is an adjoint operator of B with $\|B^*\| = \sqrt{3}$. Put

$$C := \{\alpha = (\alpha_1, \alpha_2) \in R^2 : -1 \le \alpha_1 \le 2, 3 \le \alpha_2 \le 4\},$$
$$Q := \{\beta = (\beta_1, \beta_2, \beta_3) \in R^3 : -1 \le \beta_1 \le 1, 3 \le \beta_2 \le 4, 3 \le \beta_3 \le 5\}$$

and

$$K := \{z = (z_1, z_2, z_3, z_4) \in R^4 : 0 \le z_1 \le 1, 3 \le z_2 \le 6, 3 \le z_3 \le 5, -5 \le z_4 \le -3\}.$$

For each $\alpha = (\alpha_1, \alpha_2) \in C$, $\beta = (\beta_1, \beta_2, \beta_3) \in Q$ and $z = (z_1, z_2, z_3, z_4) \in K$, define

$$f^*(\alpha) = \alpha_1^2 + \alpha_2^2,$$
$$g^*(\beta) = \beta_1^2 + \beta_2^2 + \beta_3^2$$

and

$$h^*(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

For each $\alpha, x \in C$, let

$$f(\alpha, x) = f^*(x) - f^*(\alpha).$$

For each $\beta, y \in Q$, let

$$g(\beta, y) = g^*(y) - g^*(\beta).$$

For each $\eta, z \in K$, let

$$h(\eta, z) = h^*(z) - h^*(\eta).$$

It is not hard to verify that f, g and h satisfy conditions (A1)-(A4) with $\mathrm{EP}(f) = \{p = (0,3)\}$, $\mathrm{EP}(g) = \{q = (0,3,3)\}$, $\mathrm{EP}(h) = \{(0,3,3,-3)\}$ and

$$\Omega = \bigl\{(p,q) \in \mathrm{EP}(f) \times \mathrm{EP}(g) : Ap = Bq \in \mathrm{EP}(h)\bigr\} \neq \emptyset.$$

Let $C_1 = C$, $Q_1 = Q$, $\xi = \frac{1}{6}$ and $r_n \equiv 1$ for $n \in \mathbb{N}$. Thus, for each $\bar{x} = (a,b) \in C$ and $\bar{y} = (c,d,e) \in Q$ with $c > 0$, we have the following:

- $u = (\frac{a}{3}, 3) = T^f_{r_n}\bar{x}$,
- $v = (\frac{c}{3}, 3, 3) = T^g_{r_n}\bar{y}$,
- $w = (\frac{a+c}{6}, 3, 3, -3) = T^h_{r_n}(\frac{1}{2}A\bar{x} + \frac{1}{2}B\bar{y})$,
- $l = P_C(u - \frac{1}{6}A^*(Au - w)) = (\frac{7a+c}{36}, 3)$,
- $k = P_Q(v - \frac{1}{6}B^*(Bv - w)) = (\frac{9c+a}{36}, 3 + \frac{c}{18}, 3)$.

For $x_1 = (a_1, b_1) \in C$ and $y_1 = (c_1, d_1, e_1) \in Q$ with $15c_1 \geq a_1 > 0$, $17a_1 \geq c_1$ and $d_1 > 3 + \frac{c_1}{18}$, we obtain the following:

- $u_1 = (\frac{a_1}{3}, 3)$,
- $v_1 = (\frac{c_1}{3}, 3, 3)$,
- $w_1 = (\frac{a_1+c_1}{6}, 3, 3, -3)$,
- $l_1 = (\frac{7a_1+c_1}{36}, 3)$,
- $k_1 = (\frac{9c_1+a_1}{36}, 3 + \frac{c_1}{18}, 3)$,
- $C_2 = \{\alpha = (\alpha_1, \alpha_2) \in C_1 : -1 \leq \alpha_1 \leq \frac{19a_1+c_1}{72}, 3 \leq \alpha_2 \leq \frac{b_1+3}{2}\}$,
- $x_2 = P_{C_2}(x_1) = (\frac{19a_1+c_1}{72}, \frac{b_1+3}{2}) := (a_2, b_2)$,
- $Q_2 = \{\beta = (\beta_1, \beta_2, \beta_3) \in Q_1 : -1 \leq \beta_1 \leq \frac{21c_1+a_1}{72}, 3 + \frac{c_1}{36} \leq \beta_2 \leq \frac{d_1+3}{2}, 3 \leq \beta_3 \leq \frac{e_1+3}{2}\}$,
- $y_2 = P_{Q_2}(y_1) = (\frac{21c_1+a_1}{72}, \frac{d_1+3}{2}, \frac{e_1+3}{2}) := (c_2, d_2, e_2)$.

From x_2, y_2, we have $u_2 = (\frac{1}{3}a_2, 3)$, $v_2 = (\frac{1}{3}c_2, 3, 3)$ and $w_2 = (\frac{1}{6}(a_2 + b_2), 3, 3, -3)$. Since

$$d_2 > 3 + \frac{c_2}{18},$$

$$15c_2 \geq a_2 > 0,$$

and

$$17a_2 \geq c_2,$$

we get the following:

- $C_3 = \{\alpha = (\alpha_1, \alpha_2) \in C_2 : -1 \le \alpha_1 \le \frac{19a_2+c_2}{72}, 3 \le \alpha_2 \le \frac{1}{2}(b_2 + 3)\}$,
- $x_3 = (\frac{19a_2+c_2}{72}, \frac{1}{2}(b_2 + 3)) := (a_3, b_3)$,
- $Q_3 = \{\beta = (\beta_1, \beta_2, \beta_3) \in Q_1 : -1 \le \beta_1 \le \frac{21c_2+a_2}{72}, 3 + \frac{c_2}{36} \le \beta_2 \le \frac{d_2+3}{2}, 3 \le \beta_3 \le \frac{e_2+3}{2}\}$,
- $y_3 = (\frac{21c_2+a_2}{72}, \frac{d_2+3}{2}, \frac{e_2+3}{2}) := (c_3, d_3, e_3)$.

Similarly, for $n \in \mathbb{N}$ with $n > 1$, we obtain

- $C_{n+1} = \{\alpha = (\alpha_1, \alpha_2) \in C_n : -1 \le \alpha_1 \le \frac{19a_{n-1}+c_{n-1}}{72}, 3 \le \alpha_2 \le \frac{1}{2}(b_{n-1} + 3)\}$,
- $x_{n+1} = (\frac{19a_{n-1}+c_{n-1}}{72}, \frac{1}{2}(b_{n-1} + 3))$,
- $Q_{n+1} = \{\beta = (\beta_1, \beta_2, \beta_3) \in Q_n : -1 \le \beta_1 \le \frac{21c_{n-1}+a_{n-1}}{72}, 3 + \frac{c_{n-1}}{36} \le \beta_2 \le \frac{d_{n-1}+3}{2}, 3 \le \beta_3 \le \frac{e_{n-1}+3}{2}\}$,
- $y_{n+1} = (\frac{21c_{n-1}+a_{n-1}}{72}, \frac{d_{n-1}+3}{2}, \frac{e_{n-1}+3}{2})$,
- $u_{n+1} = (\frac{1}{3}a_{n-1}, 3)$,
- $v_{n+1} = (\frac{1}{3}c_{n-1}, 3, 3)$,
- $w_{n+1} = (\frac{c_{n-1}+a_{n-1}}{6}, 3, 3, -3)$.

By mathematical induction, we know that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ and $\{e_n\}$ all are decreasing sequences. Moreover, $a_n \to 0$, $b_n \to 3$, $c_n \to 0$, $d_n \to 3$ and $e_n \to 3$ as $n \to \infty$. So, we have $u_n \to (0, 3)$, $v_n \to (0, 3, 3)$, $w_n \to (0, 3, 3, -3)$, $x_n \to (0, 3)$ and $y_n \to (0, 3, 3)$ as $n \to \infty$.

5 Conclusion

In this paper, we first introduce and investigate BSEP which can be regarded as a new development in the field of equilibrium problems. We provide some new iterative algorithms for BSEP and establish strong convergence theorems for these iterative algorithms in different spaces. An example illustrating Theorem 2.1 is also given.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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