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An iterative method for common solutions of equilibrium problems and hierarchical fixed point problems

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Abstract
The purpose of this paper is to investigate the problem of finding an approximate point of the common set of solutions of an equilibrium problem and a hierarchical fixed point problem in the setting of real Hilbert spaces. We establish the strong convergence of the proposed method under some mild conditions. Several special cases are also discussed. Numerical examples are presented to illustrate the proposed method and convergence result. The results presented in this paper extend and improve some well-known results in the literature.
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1 Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $F_1 : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (in short, EP) is to find $x \in C$ such that

$$F_1(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of EP (1.1) is denoted by $EP(F_1)$.

The equilibrium problem provides a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. The theory of equilibrium problems has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences; see [1–10] and the references therein.

If $F_1(x,y) = \langle Ax, y-x\rangle$, where $A : C \to H$ is a nonlinear operator, then EP (1.1) is equivalent to find a vector $x \in C$ such that

$$\langle y-x, Ax\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

It is a well-known classical variational inequality problem. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems; see [1–27] and the references therein.

The fixed point problem for the mapping $T : C \to H$ is to find $x \in C$ such that

$$Tx = x. \tag{1.3}$$

We denote by $F(T)$ the set of solutions of (1.3). It is well known that $F(T)$ is closed and convex, and $P_F(T)$ is well defined.

Let $S : C \to H$ be a nonexpansive mapping, that is, $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. The hierarchical fixed point problem (in short, HFPP) is to find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.4}$$

It is linked with some monotone variational inequalities and convex programming problems; see [12]. Various methods have been proposed to solve HFPP (1.4); see, for example, [13–17] and the references therein. In 2010, Yao *et al.* [12] studied the following iterative algorithm to solve HFPP (1.4):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C\big[\alpha_n f(x_n) + (1 - \alpha_n) T y_n\big], \quad \forall n \geq 0, \end{aligned} \tag{1.5}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Under certain restrictions on the parameters, They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $z \in F(T)$, which is also a unique solution of the following variational inequality:

$$\big\langle (I - f)z, y - z \big\rangle \geq 0, \quad \forall y \in F(T). \tag{1.6}$$

In 2011, Ceng *et al.* [18] investigated the following iterative method:

$$x_{n+1} = P_C\big[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\big(T(y_n)\big)\big], \quad \forall n \geq 0, \tag{1.7}$$

where U is a Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (1.7) converges strongly to a unique solution of the variational inequality:

$$\big\langle \rho U(z) - \mu F(z), x - z \big\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In this paper, motivated by the work of Ceng *et al.* [18, 20], Yao *et al.* [12], Bnouhachem [19] and others, we propose an iterative method for finding an approximate element of the common set of solutions of EP (1.1) and HFPP (1.4) in the setting of real Hilbert spaces. We establish a strong convergence theorem for the sequence generated by the proposed method. The proposed method is quite general and flexible and includes several known methods for solving of variational inequality problems, equilibrium problems, and hierarchical fixed point problems; see, for example, [12, 13, 15, 18, 19, 21] and the references therein.

2 Preliminaries

We present some definitions which will be used in the sequel.

Definition 2.1 A mapping $T : C \to H$ is said to be k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

- If $k = 1$, then T is called nonexpansive.
- If $k \in (0,1)$, then T is called contraction.

Definition 2.2 A mapping $T : C \to H$ is said to be

(a) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone mapping is monotone and Lipschitz continuous. Also, for every nonexpansive mapping $T : H \to H$, we have

$$\bigl\langle \bigl(x - T(x)\bigr) - \bigl(y - T(y)\bigr), T(y) - T(x)\bigr\rangle \leq \frac{1}{2}\bigl\|\bigl(T(x) - x\bigr) - \bigl(T(y) - y\bigr)\bigr\|^2, \tag{2.1}$$

for all $(x, y) \in H \times H$. Therefore, for all $(x, y) \in H \times \operatorname{Fix}(T)$, we have

$$\bigl\langle x - T(x), y - T(x)\bigr\rangle \leq \frac{1}{2}\bigl\|T(x) - x\bigr\|^2. \tag{2.2}$$

The following lemma provides some basic properties of the projection onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities*:

(a) $\langle z - P_C[z], P_C[z] - v\rangle \geq 0$, $\forall z \in H$, $v \in C$;
(b) $\langle u - v, P_C[u] - P_C[v]\rangle \geq \|P_C[u] - P_C[v]\|^2$, $\forall u, v \in H$;
(c) $\|P_C[u] - P_C[v]\| \leq \|u - v\|$, $\forall u, v \in H$;
(d) $\|u - P_C[z]\|^2 \leq \|z - u\|^2 - \|z - P_C[z]\|^2$, $\forall z \in H$, $u \in C$.

Assumption 2.1 [1] Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(A_1) $F_1(x, x) = 0$, $\forall x \in C$;
(A_2) F_1 is monotone, that is, $F_1(x, y) + F_1(y, x) \leq 0$, $\forall x, y \in C$;

(A_3) For each $x, y, z \in C$, $\lim_{t\to 0} F_1(tz + (1-t)x, y) \leq F_1(x, y)$;
(A_4) For each $x \in C$, $y \to F_1(x, y)$ is convex and lower semicontinuous.

Lemma 2.2 [2] *Let C be a nonempty closed convex subset of a real Hilbert space H and $F_1 : C \times C \to \mathbb{R}$ satisfy conditions* (A_1)-(A_4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as*

$$T_r(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following statements hold:

(i) *T_r is nonempty single-valued*;
(ii) *T_r is firmly nonexpansive, that is,*

$$\| T_r(x) - T_r(y) \|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle, \quad \forall x, y \in H;$$

(iii) $F(T_r) = EP(F_1)$;
(iv) *$EP(F_1)$ is closed and convex.*

Lemma 2.3 [22] *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to C$ is a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *then the mapping $I - T$ is demiclosed at* 0, *that is, if $\{x_n\}$ is a sequence in C converges weakly to x and $\{(I - T)x_n\}$ converges strongly to* 0, *then $(I - T)x = 0$.*

Lemma 2.4 [18] *Let $U : C \to H$ be a τ-Lipschitzian mapping and $F : C \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Then, for $0 \leq \rho\tau < \mu\eta$, $\mu F - \rho U$ is $(\mu\eta - \rho\tau)$-strongly monotone, that is,*

$$\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y \rangle \geq (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x, y \in C.$$

Lemma 2.5 [23] *Let $\lambda \in (0, 1)$, $\mu > 0$, and $F : C \to H$ be an k-Lipschitzian and η-strongly monotone operator. In association with a nonexpansive mapping $T : C \to C$, define a mapping $T^\lambda : C \to H$ by*

$$T^\lambda x = Tx - \lambda\mu FT(x), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is,

$$\| T^\lambda x - T^\lambda y \| \leq (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.6 [25] *Let C be a closed convex subset of a real Hilbert space H and $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) *the weak w-limit set $w_w(x_n) \subset C$, where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$*
(ii) *for each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*

Then the sequence $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.7 [24] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\upsilon_n)a_n + \delta_n,$$

where $\{\upsilon_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence such that

(i) $\sum_{n=1}^{\infty} \upsilon_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\upsilon_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 An iterative method and strong convergence results

In this section, we propose and analyze an iterative method for finding the common solutions of EP (1.1) and HFPP (1.4).

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction that satisfy conditions (A_1)-(A_4), and let $S, T : C \to C$ be nonexpansive mappings such that $F(T) \cap EP(F_1) \neq \emptyset$. Let $F : C \to C$ be a k-Lipschitzian mapping and η-strongly monotone, and let $U : C \to C$ be a τ-Lipschitzian mapping.

Algorithm 3.1 For any given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ be generated by

$$\begin{cases} F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ y_n = \beta_n S x_n + (1-\beta_n)u_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)(T(y_n))], \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

Suppose that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$ and $0 \leq \rho\tau < \nu$, where $\nu = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. Also, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \gamma_n = 0$, $\gamma_n + \alpha_n < 1$,

(b) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(c) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$,

(d) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n-1} - \gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(e) $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Remark 3.1 Algorithm 3.1 can be viewed as an extension and improvement for some well-known methods.

- If $\gamma_n = 0$, then the proposed method is an extension and improvement of a method studied in [19, 26].
- If $U = f$, $F = I$, $\rho = \mu = 1$, and $\gamma_n = 0$, then we obtain an extension and improvement of a method considered in [12].
- The contractive mapping f with a coefficient $\alpha \in [0,1)$ in other papers [12, 21, 23] is extended to the cases of the Lipschitzian mapping U with a coefficient constant $\gamma \in [0,\infty)$.

Lemma 3.1 *Let $x^* \in F(T) \cap EP(F_1)$. Then $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ are bounded.*

Proof It follows from Lemma 2.2 that $u_n = T_{r_n}(x_n)$. Let $x^* \in F(T) \cap EP(F_1)$, then $x^* = T_{r_n}(x^*)$. Define $V_n = \alpha_n \rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)(T(y_n))$.

We prove that the sequence $\{x_n\}$ is bounded. Without loss of generality, we can assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. From (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|P_C[V_n] - P_C[x^*]\| \\
&\quad \leq \|\alpha_n \rho U(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \mu F)(T(y_n)) - x^*\| \\
&\quad = \|\alpha_n(\rho U(x_n) - \mu F(x^*)) + \gamma_n(x_n - x^*) + ((1-\gamma_n)I - \alpha_n \mu F)(T(y_n)) \\
&\qquad - ((1-\gamma_n)I - \alpha_n \mu F)(T(x^*))\| \\
&\quad \leq \alpha_n \|\rho U(x_n) - \mu F(x^*)\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1-\gamma_n) \times \left\| \left(I - \frac{\alpha_n \mu}{1-\gamma_n} F\right)(T(y_n)) - \left(I - \frac{\alpha_n \mu}{1-\gamma_n} F\right) T(x^*) \right\| \\
&\quad = \alpha_n \|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1-\gamma_n) \times \left\| \left(I - \frac{\alpha_n \mu}{1-\gamma_n} F\right)(T(y_n)) - \left(I - \frac{\alpha_n \mu}{1-\gamma_n} F\right) T(x^*) \right\| \\
&\quad \leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1-\gamma_n)\left(1 - \frac{\alpha_n \nu}{1-\gamma_n}\right) \|y_n - x^*\| \\
&\quad = \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1 - \gamma_n - \alpha_n \nu) \|\beta_n S x_n + (1-\beta_n) u_n - x^*\| \\
&\quad \leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1 - \gamma_n - \alpha_n \nu)(\beta_n \|S x_n - S x^*\| + \beta_n \|S x^* - x^*\| \\
&\qquad + (1-\beta_n)\|T_{r_n}(x_n) - x^*\|) \\
&\quad \leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1 - \gamma_n - \alpha_n \nu)(\beta_n \|S x_n - S x^*\| + \beta_n \|S x^* - x^*\| + (1-\beta_n)\|x_n - x^*\|) \\
&\quad \leq \alpha_n \rho \tau \|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \gamma_n \|x_n - x^*\| \\
&\qquad + (1 - \gamma_n - \alpha_n \nu)(\beta_n \|x_n - x^*\| + \beta_n \|S x^* - x^*\| + (1-\beta_n)\|x_n - x^*\|) \\
&\quad = (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| \\
&\qquad + (1 - \gamma_n - \alpha_n \nu)\beta_n \|S x^* - x^*\| \\
&\quad \leq (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n \|(\rho U - \mu F)x^*\| + \beta_n \|S x^* - x^*\| \\
&\quad \leq (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n (\|(\rho U - \mu F)x^*\| + \|S x^* - x^*\|) \\
&\quad = (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \frac{\alpha_n(\nu - \rho\tau)}{\nu - \rho\tau} (\|(\rho U - \mu F)x^*\| + \|S x^* - x^*\|) \\
&\quad \leq \max\left\{ \|x_n - x^*\|, \frac{1}{\nu - \rho\tau} (\|(\rho U - \mu F)x^*\| + \|S x^* - x^*\|) \right\},
\end{aligned}$$

where the third inequality follows from Lemma 2.5. By induction on n, we obtain

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{1}{\nu - \rho\tau}\left(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|\right)\right\},$$

for $n \geq 0$ and $x_0 \in C$. Hence, $\{x_n\}$ is bounded, and consequently, we deduce that $\{u_n\}$, $\{y_n\}$, $\{S(x_n)\}$, $\{T(y_n)\}$, $\{F(T(y_n))\}$, and $\{U(x_n)\}$ are bounded. □

Lemma 3.2 *Let $x^* \in F(T) \cap EP(F_1)$ and $\{x_n\}$ be a sequence generated by Algorithm* 3.1. *Then the following statements hold.*

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
(b) *The weak w-limit set $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\} \subset F(T)$.*

Proof From the definition of the sequence $\{y_n\}$ in Algorithm 3.1, we have

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \leq \left\|\beta_n Sx_n + (1-\beta_n)u_n - \left(\beta_{n-1}Sx_{n-1} + (1-\beta_{n-1})u_{n-1}\right)\right\| \\
&\quad = \left\|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1}\right. \\
&\qquad \left. + (1-\beta_n)(u_n - u_{n-1}) + (\beta_{n-1} - \beta_n)u_{n-1}\right\| \\
&\quad \leq \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|u_n - u_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}|\left(\|Sx_{n-1}\| + \|u_{n-1}\|\right).
\end{aligned} \tag{3.2}$$

Since $u_n = T_{r_n}(x_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1})$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.3}$$

and

$$F_1(u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C. \tag{3.4}$$

Take $y = u_{n-1}$ in (3.3) and $y = u_n$ in (3.4), we get

$$F_1(u_n, u_{n-1}) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0, \tag{3.5}$$

and

$$F_1(u_{n-1}, u_n) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0. \tag{3.6}$$

Adding (3.5) and (3.6), and using the monotonicity of F_1, we obtain

$$\left\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0,$$

which implies that

$$\begin{aligned}
0 &\le \left\langle u_n - u_{n-1}, \frac{r_n}{r_{n-1}}(u_{n-1} - x_{n-1}) - (u_n - x_n)\right\rangle \\
&= \left\langle u_{n-1} - u_n, u_n - u_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} - x_n + \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)u_{n-1} - x_n + \left(\frac{r_n}{r_{n-1}}\right)x_{n-1}\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&= \left\langle u_{n-1} - u_n, \left(1 - \frac{r_n}{r_{n-1}}\right)(u_{n-1} - x_{n-1}) + (x_{n-1} - x_n)\right\rangle \\
&\quad - \|u_n - u_{n-1}\|^2 \\
&\le \|u_{n-1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\right\} \\
&\quad - \|u_n - u_{n-1}\|^2,
\end{aligned}$$

and then

$$\|u_{n-1} - u_n\| \le \left|1 - \frac{r_n}{r_{n-1}}\right|\|u_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|.$$

Without loss of generality, assume that there exists a real number χ such that $r_n > \chi > 0$ for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \le \|x_{n-1} - x_n\| + \frac{1}{\chi}|r_{n-1} - r_n|\|u_{n-1} - x_{n-1}\|. \tag{3.7}$$

It follows from (3.2) and (3.7) that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\|x_n - x_{n-1}\| + \frac{1}{\chi}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\|\right\} \\
&\qquad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|u_{n-1}\|\big) \\
&\quad = \|x_n - x_{n-1}\| + (1 - \beta_n)\left\{\frac{1}{\chi}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\|\right\} \\
&\qquad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|u_{n-1}\|\big).
\end{aligned} \tag{3.8}$$

Next, we estimate that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|P_C[V_n] - P_C[V_{n-1}]\big\| \\
&\le \Big\|\alpha_n\rho\big(U(x_n) - U(x_{n-1})\big) + (\alpha_n - \alpha_{n-1})\rho U(x_{n-1}) + \gamma_n(x_n - x_{n-1}) \\
&\quad + (\gamma_n - \gamma_{n-1})x_{n-1} \\
&\quad + (1 - \gamma_n) \times \left[\left(I - \frac{\alpha_n\mu}{1 - \gamma_n}F\right)\big(T(y_n)\big) - \left(I - \frac{\alpha_n\mu}{1 - \gamma_n}F\right)T(y_{n-1})\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \big((1-\gamma_n)I - \alpha_n\mu F\big)\big(T(y_{n-1})\big) - \big((1-\gamma_{n-1})I - \alpha_{n-1}\mu F\big)\big(T(y_{n-1})\big)\bigg\| \\
&\le \alpha_n\rho\tau\|x_n - x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| + (1-\gamma_n)\left(1 - \frac{\alpha_n\nu}{1-\gamma_n}\right)\|y_n - y_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\big(\|x_{n-1}\| + \big\|T(y_{n-1})\big\|\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big),
\end{aligned}
\tag{3.9}
$$

where the second inequality follows from Lemma 2.5. From (3.8) and (3.9), we have

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \alpha_n\rho\tau\|x_n - x_{n-1}\| + \gamma_n\|x_n - x_{n-1}\| \\
&\qquad + (1-\gamma_n-\alpha_n\nu)\left\{\|x_n - x_{n-1}\| + \frac{1}{\chi}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| \right. \\
&\qquad \left. + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|u_{n-1}\|\big)\right\} + |\gamma_n - \gamma_{n-1}|\big(\|x_{n-1}\| + \big\|T(y_{n-1})\big\|\big) \\
&\qquad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big) \\
&\quad \le \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| + \frac{1}{\chi}|r_n - r_{n-1}|\|u_{n-1} - x_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|u_{n-1}\|\big) + |\gamma_n - \gamma_{n-1}|\big(\|x_{n-1}\| + \big\|T(y_{n-1})\big\|\big) \\
&\qquad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big) \\
&\quad \le \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| \\
&\qquad + M\left(\frac{1}{\chi}|r_{n-1} - r_n| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\alpha_n - \alpha_{n-1}|\right),
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
M = \max\Big\{&\sup_{n\ge1}\|u_{n-1} - x_{n-1}\|, \sup_{n\ge1}\big(\|Sx_{n-1}\| + \|u_{n-1}\|\big), \sup_{n\ge1}\big(\|x_{n-1}\| + \big\|T(y_{n-1})\big\|\big), \\
&\sup_{n\ge1}\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big)\Big\}.
\end{aligned}
$$

It follows from conditions (b), (d), (e) of Algorithm 3.1, and Lemma 2.7 that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.
$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since T_{r_n} is firmly nonexpansive, we have

$$
\begin{aligned}
\big\|u_n - x^*\big\|^2 &= \big\|T_{r_n}(x_n) - T_{r_n}\big(x^*\big)\big\|^2 \\
&\le \big\langle u_n - x^*, x_n - x^*\big\rangle \\
&= \frac{1}{2}\big\{\big\|u_n - x^*\big\|^2 + \big\|x_n - x^*\big\|^2 - \big\|u_n - x^* - \big(x_n - x^*\big)\big\|^2\big\}.
\end{aligned}
$$

Hence, we get

$$
\big\|u_n - x^*\big\|^2 \le \big\|x_n - x^*\big\|^2 - \big\|u_n - x_n\big\|^2.
$$

From above inequality, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \langle P_C[V_n] - x^*, x_{n+1} - x^* \rangle \\
&\quad = \langle P_C[V_n] - V_n, P_C[V_n] - x^* \rangle + \langle V_n - x^*, x_{n+1} - x^* \rangle \\
&\quad \le \langle \alpha_n(\rho U(x_n) - \mu F(x^*)) + \gamma_n(x_n - x^*) + ((1-\gamma_n)I - \alpha_n\mu F)(T(y_n)) \\
&\qquad - ((1-\gamma_n)I - \alpha_n\mu F)(T(x^*)), x_{n+1} - x^* \rangle \\
&\quad = \langle \alpha_n\rho(U(x_n) - U(x^*)), x_{n+1} - x^* \rangle + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + \langle \gamma_n(x_n - x^*), x_{n+1} - x^* \rangle \\
&\qquad + (1-\gamma_n)\left\langle \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)(T(y_n)) - \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)(T(x^*)), x_{n+1} - x^* \right\rangle \\
&\quad \le (\alpha_n\rho\tau + \gamma_n)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + (1 - \gamma_n - \alpha_n\nu)\|y_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \le \frac{\gamma_n + \alpha_n\rho\tau}{2}\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n\nu)}{2}\left(\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad \le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\qquad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n\nu)}{2}\left(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2\right) \\
&\quad \le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\gamma_n + \alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\qquad + \alpha_n\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n\nu)}{2}\left\{\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\left(\|x_n - x^*\|^2 - \|u_n - x_n\|^2\right)\right\},
\end{aligned}
\tag{3.11}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{\gamma_n + \alpha_n\rho\tau}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^* \rangle \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n\nu)\beta_n}{1 + \alpha_n(\nu - \rho\tau)}\|Sx_n - x^*\|^2 \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n\nu)(1-\beta_n)}{1 + \alpha_n(\nu - \rho\tau)}\left\{\|x_n - x^*\|^2 - \|u_n - x_n\|^2\right\} \\
&\quad \le \frac{\gamma_n + \alpha_n\rho\tau}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&+\frac{(1-\gamma_n-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2\\
&+\|x_n-x^*\|^2-\frac{(1-\gamma_n-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{(1-\gamma_n-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|^2\\
&\quad\le\frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2\\
&\qquad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\qquad+\frac{(1-\gamma_n-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad\le\frac{\gamma_n+\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2\\
&\qquad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle\\
&\qquad+\frac{(1-\gamma_n-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2\\
&\qquad+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, we have

$$
\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.12}
$$

Since $T(x_n)\in C$, we have

$$
\begin{aligned}
&\|x_n-T(x_n)\|\\
&\quad\le\|x_n-x_{n+1}\|+\|x_{n+1}-T(x_n)\|\\
&\quad=\|x_n-x_{n+1}\|+\|P_C[V_n]-P_C[T(x_n)]\|\\
&\quad\le\|x_n-x_{n+1}\|+\|\alpha_n(\rho U(x_n)-\mu F(T(y_n)))+\gamma_n(x_n-T(y_n))\\
&\qquad+T(y_n)-T(x_n)\|\\
&\quad\le\|x_n-x_{n+1}\|+\alpha_n\|\rho U(x_n)-\mu F(T(y_n))\|+\gamma_n\|x_n-T(y_n)\|+\|y_n-x_n\|\\
&\quad\le\|x_n-x_{n+1}\|+\alpha_n\|\rho U(x_n)-\mu F(T(y_n))\|+\gamma_n\|x_n-T(y_n)\|\\
&\qquad+\|\beta_nSx_n+(1-\beta_n)u_n-x_n\|\\
&\quad\le\|x_n-x_{n+1}\|+\alpha_n\|\rho U(x_n)-\mu F(T(y_n))\|+\gamma_n\|x_n-T(y_n)\|\\
&\qquad+\beta_n\|Sx_n-x_n\|+(1-\beta_n)\|u_n-x_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\gamma_n\to 0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\|\rho U(x_n)-\mu F(T(y_n))\|$ and $\|Sx_n-x_n\|$ are bounded, and $\lim_{n\to\infty}\|u_n-x_n\|=0$, we obtain

$$\lim_{n\to\infty}\bigl\|x_n-T(x_n)\bigr\|=0.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $x_n\rightharpoonup x^*\in C$. It follows from Lemma 2.3 that $x^*\in F(T)$. Therefore, $w_w(x_n)\subset F(T)$. □

Theorem 3.1 *The sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to* $z\in F(T)\cap EP(F_1)$, *which is also a unique solution of the variational inequality*:

$$\bigl\langle\rho U(z)-\mu F(z),x-z\bigr\rangle\le 0,\quad \forall x\in F(T)\cap EP(F_1). \tag{3.13}$$

Proof From Lemma 3.2, we have $w\in F(T)$ since $\{x_n\}$ is bounded and $x_n\rightharpoonup w$. We show that $w\in EP(F_1)$. Since $u_n=T_{r_n}(x_n)$, we have

$$F_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C.$$

It follows from the monotonicity of F_1 that

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge F_1(y,u_n),\quad \forall y\in C,$$

and

$$\biggl\langle y-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\biggr\rangle\ge F_1(y,u_{n_k}),\quad \forall y\in C. \tag{3.14}$$

Since $\lim_{n\to\infty}\|u_n-x_n\|=0$ and $x_n\rightharpoonup w$, it is easy to observe that $u_{n_k}\to w$. For any $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)w$. Then we have $y_t\in C$, and from (3.14) we obtain

$$0\ge -\biggl\langle y_t-u_{n_k},\frac{u_{n_k}-x_{n_k}}{r_{n_k}}\biggr\rangle+F_1(y_t,u_{n_k}). \tag{3.15}$$

Since $u_{n_k}\to w$, it follows from (3.15) that

$$0\ge F_1(y_t,w). \tag{3.16}$$

Since F_1 satisfies (A_1)-(A_4), it follows from (3.16) that

$$0=F_1(y_t,y_t)\le tF_1(y_t,y)+(1-t)F_1(y_t,w)\le tF_1(y_t,y), \tag{3.17}$$

which implies that $F_1(y_t,y)\ge 0$. Letting $t\to 0_+$, we have

$$F_1(w,y)\ge 0,\quad \forall y\in C,$$

which implies that $w\in EP(F_1)$. Thus, we have

$$w\in F(T)\cap EP(F_1).$$

Observe that the constants satisfy $0 \leq \rho\tau < \nu$ and

$$\begin{aligned} k \geq \eta \quad &\Longleftrightarrow \quad k^2 \geq \eta^2 \\ &\Longleftrightarrow \quad 1 - 2\mu\eta + \mu^2 k^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Longleftrightarrow \quad \sqrt{1 - \mu\left(2\eta - \mu k^2\right)} \geq 1 - \mu\eta \\ &\Longleftrightarrow \quad \mu\eta \geq 1 - \sqrt{1 - \mu\left(2\eta - \mu k^2\right)} \\ &\Longleftrightarrow \quad \mu\eta \geq \nu, \end{aligned}$$

therefore, from Lemma 2.4, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$ strongly monotone, and we get the uniqueness of the solution of variational inequality (3.13), and denote it by $z \in F(T) \cap EP(F_1)$.

Next, we claim that $\limsup_{n\to\infty} \langle \rho U(z) - \mu F(z), x_n - z\rangle \leq 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\begin{aligned} \limsup_{n\to\infty} \langle \rho U(z) - \mu F(z), x_n - z\rangle &= \limsup_{k\to\infty} \langle \rho U(z) - \mu F(z), x_{n_k} - z\rangle \\ &= \langle \rho U(z) - \mu F(z), w - z\rangle \leq 0. \end{aligned}$$

Next, we show that $x_n \to z$. We have

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &\quad = \langle P_C[V_n] - z, x_{n+1} - z\rangle \\ &\quad = \langle P_C[V_n] - V_n, P_C[V_n] - z\rangle + \langle V_n - z, x_{n+1} - z\rangle \\ &\quad \leq \Big\langle \alpha_n\big(\rho U(x_n) - \mu F(z)\big) + \gamma_n(x_n - z) \\ &\qquad + (1-\gamma_n)\left[\left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)\big(T(y_n)\big) - \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)\big(T(z)\big)\right], x_{n+1} - z\Big\rangle \\ &\quad = \big\langle \alpha_n\rho\big(U(x_n) - U(z)\big), x_{n+1} - z\big\rangle \\ &\qquad + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle + \gamma_n\langle x_n - z, x_{n+1} - z\rangle \\ &\qquad + (1-\gamma_n)\left\langle\left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)\big(T(y_n)\big) - \left(I - \frac{\alpha_n\mu}{1-\gamma_n}F\right)\big(T(z)\big), x_{n+1} - z\right\rangle \\ &\quad \leq (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\ &\qquad + (1 - \gamma_n - \alpha_n\nu)\|y_n - z\|\|x_{n+1} - z\| \\ &\quad \leq (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\ &\qquad + (1 - \gamma_n - \alpha_n\nu)\big\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| \\ &\qquad + (1-\beta_n)\|u_n - z\|\big\}\|x_{n+1} - z\| \\ &\quad = (\gamma_n + \alpha_n\rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\ &\qquad + (1 - \gamma_n - \alpha_n\nu)\big\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| \\ &\qquad + (1-\beta_n)\big\|T_{r_n}(x_n) - z\big\|\big\}\|x_{n+1} - z\| \end{aligned}$$

$$
\begin{aligned}
&\leq (\gamma_n + \alpha_n \rho\tau)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \gamma_n - \alpha_n \nu)\bigl\{\beta_n\|x_n - z\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|x_n - z\|\bigr\}\|x_{n+1} - z\| \\
&= \bigl(1 - \alpha_n(\nu - \rho\tau)\bigr)\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \gamma_n - \alpha_n \nu)\beta_n\|Sz - z\|\|x_{n+1} - z\| \\
&\leq \frac{1 - \alpha_n(\nu - \rho\tau)}{2}\bigl(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\bigr) + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \gamma_n - \alpha_n \nu)\beta_n\|Sz - z\|\|x_{n+1} - z\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad\leq \frac{1 - \alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - z\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\qquad + \frac{2(1 - \gamma_n - \alpha_n \nu)\beta_n}{1 + \alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\| \\
&\quad\leq \bigl(1 - \alpha_n(\nu - \rho\tau)\bigr)\|x_n - z\|^2 \\
&\qquad + \frac{2\alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\biggl\{\frac{1}{\nu - \rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\qquad + \frac{(1 - \gamma_n - \alpha_n \nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\biggr\}.
\end{aligned}
$$

Let $\upsilon_n = \alpha_n(\nu - \rho\tau)$ and

$$
\begin{aligned}
\delta_n = {}& \frac{2\alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\biggl\{\frac{1}{\nu - \rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
& + \frac{(1 - \gamma_n - \alpha_n \nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\biggr\}.
\end{aligned}
$$

We have $\sum_{n=1}^{\infty}\alpha_n = \infty$ and

$$
\limsup_{n\to\infty}\biggl\{\frac{1}{\nu - \rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle + \frac{(1 - \gamma_n - \alpha_n \nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\biggr\} \leq 0.
$$

It follows that

$$
\sum_{n=1}^{\infty}\upsilon_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\delta_n}{\upsilon_n} \leq 0.
$$

Thus, all the conditions of Lemma 2.7 are satisfied. Hence, we deduce that $x_n \to z$. This completes the proof. □

Putting $\gamma_n = 0$ in Algorithm 3.1, we obtain the following result, which can be viewed as an extension and improvement of the method studied in [26].

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction that satisfies* (A_1)-(A_4) *and $S, T : C \to C$ be nonexpansive* *mappings such that* $F(T) \cap EP(F_1) \neq \emptyset$. *Let* $F : C \to C$ *be a* k*-Lipschitzian mapping and* η*-strongly monotone, and let* $U : C \to C$ *be a* τ*-Lipschitzian mapping. For a given* $x_0 \in C$, *let the iterative sequences* $\{u_n\}$, $\{x_n\}$ *and* $\{y_n\}$ *be generated by*

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;$$

$$y_n = \beta_n S x_n + (1 - \beta_n) u_n;$$

$$x_{n+1} = P_C\bigl[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\bigl(T(y_n)\bigr)\bigr], \quad \forall n \geq 0,$$

where $\{r_n\}, \{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$. *Suppose that the parameters satisfy* $0 < \mu < \frac{2\eta}{k^2}$, $0 \leq \rho\tau < \nu$, *where* $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. *Also,* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{r_n\}$ *are sequences satisfying conditions* (b)-(e) *of Algorithm* 3.1. *Then the sequence* $\{x_n\}$ *converges strongly to some element* $z \in F(T) \cap EP(F_1)$, *which is also a unique solution of the variational inequality*:

$$\bigl\langle \rho U(z) - \mu F(z), x - z\bigr\rangle \leq 0, \quad \forall x \in F(T) \cap EP(F_1).$$

Putting $U = f$, $F = I$, $\rho = \mu = 1$, and $\gamma_n = 0$, we obtain an extension and improvement of the method considered in [12].

Corollary 3.2 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F_1 : C \times C \to \mathbb{R}$ *be a bifunction that satisfies* (A_1)-(A_4) *and* $S, T : C \to C$ *be nonexpansive mappings such that* $F(T) \cap EP(F_1) \neq \emptyset$. *Let* $f : C \to C$ *be a* τ*-Lipschitzian mapping. For a given* $x_0 \in C$, *let the iterative sequences* $\{u_n\}$, $\{x_n\}$, *and* $\{y_n\}$ *be generated by*

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C;$$

$$y_n = \beta_n S x_n + (1 - \beta_n) u_n;$$

$$x_{n+1} = P_C\bigl[\alpha_n f(x_n) + (1 - \alpha_n) T(y_n)\bigr], \quad \forall n \geq 0,$$

where $\{r_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $(0,1)$ *which satisfy conditions* (b)-(e) *of Algorithm* 3.1. *Then the sequence* $\{x_n\}$ *converges strongly to some element* $z \in F(T) \cap EP(F_1)$ *which is also a unique solution of the variational inequality*:

$$\bigl\langle f(z) - z, x - z\bigr\rangle \leq 0, \quad \forall x \in F(T) \cap EP(F_1).$$

4 Examples

To illustrate Algorithm 3.1 and the convergence result, we consider the following examples.

Example 4.1 Let $\alpha_n = \frac{1}{2n}$, $\gamma_n = \frac{1}{2n}$, $\beta_n = \frac{1}{n^2}$ and $r_n = \frac{n}{n+1}$. Then we have $\alpha_n + \gamma_n = \frac{1}{n} < 1$,

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} = 0,$$

and

$$\sum_{n=1}^{\infty} \alpha_n = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

The sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy conditions (a) and (b). Since

$$\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \lim_{n\to\infty} \frac{2}{n} = 0,$$

condition (c) is satisfied. We compute

$$\alpha_{n-1} - \alpha_n = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{2n(n-1)}.$$

It is easy to show $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$. Similarly, we can show $\sum_{n=1}^{\infty} |\gamma_{n-1} - \gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$. The sequences $\{\alpha_n\}$, $\{\gamma_n\}$, and $\{\beta_n\}$ satisfy condition (d). We have

$$\liminf_{n\to\infty} r_n = \liminf_{n\to\infty} \frac{n}{n+1} = 1,$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} |r_{n-1} - r_n| &= \sum_{n=1}^{\infty} \left| \frac{n-1}{n} - \frac{n}{n+1} \right| \\ &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \end{aligned}$$

Then the sequence $\{r_n\}$ satisfies condition (e).

Let the mappings $T, F, S, U : \mathbb{R} \to \mathbb{R}$ be defined as

$$\begin{aligned} &T(x) = \frac{x}{2}, \quad \forall x \in \mathbb{R}, \\ &F(x) = \frac{2x+3}{7}, \quad \forall x \in \mathbb{R}, \\ &S(x) = \frac{x}{3}, \quad \forall x \in \mathbb{R}, \\ &U(x) = \frac{x}{14}, \quad \forall x \in \mathbb{R}, \end{aligned}$$

and let the mapping $F_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_1(x, y) = -3x^2 + xy + 2y^2, \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}.$$

It is easy to show that T and S are nonexpansive mappings, F is 1-Lipschitzian and $\frac{1}{7}$-strongly monotone and U is $\frac{1}{7}$-Lipschitzian. It is clear that

$$EP(F_1) \cap F(T) = \{0\}.$$

From the definition of F_1, we have

$$\begin{aligned} 0 &\le F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\ &= -3u_n^2 + u_n y + 2y^2 + \frac{1}{r_n}(y - u_n)(u_n - x_n). \end{aligned}$$

Then

$$\begin{aligned} 0 &\le r_n\bigl(-3u_n^2 + u_n y + 2y^2\bigr) + \bigl(yu_n - yx_n - u_n^2 + u_n x_n\bigr) \\ &= 2r_n y^2 + (r_n u_n + u_n - x_n)y - 3r_n u_n^2 - u_n^2 + u_n x_n. \end{aligned}$$

Let $B(y) = 2r_n y^2 + (r_n u_n + u_n - x_n)y - 3r_n u_n^2 - u_n^2 + u_n x_n$. Then $B(y)$ is a quadratic function of y with coefficient $a = 2r_n$, $b = r_n u_n + u_n - x_n$, $c = -3r_n u_n^2 - u_n^2 + u_n x_n$. We determine the discriminant Δ of B as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (r_n u_n + u_n - x_n)^2 - 8r_n\bigl(-3r_n u_n^2 - u_n^2 + u_n x_n\bigr) \\ &= u_n^2 + 10r_n u_n^2 + 25u_n^2 r_n^2 - 2x_n u_n - 10x_n u_n r_n + x_n^2 \\ &= (u_n + 5u_n r_n)^2 - 2x_n(u_n + 5u_n r_n) + x_n^2 \\ &= (u_n + 5u_n r_n - x_n)^2. \end{aligned}$$

We have $B(y) \ge 0$, $\forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta = 0$, and we obtain

$$u_n = \frac{x_n}{1 + 5r_n}. \tag{4.1}$$

For every $n \ge 1$, from (4.1), we rewrite (3.1) as follows:

$$\begin{cases} y_n = \frac{x_n}{3n^2} + (1 - \frac{1}{n^2})\frac{x_n}{(1+5r_n)}; \\ x_{n+1} = \rho\frac{x_n}{28n} + \frac{x_n}{2n} + (1 - \frac{1}{2n})\frac{y_n}{2} - \mu\frac{y_n+3}{14n}. \end{cases}$$

In all the tests we take $\rho = \frac{1}{15}$, $\mu = \frac{1}{7}$, and $N = 10$ for Algorithm 3.1. In this example $\eta = \frac{1}{7}$, $k = 1$, $\tau = \frac{1}{7}$. It is easy to show that the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$, $0 \le \rho\tau < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

The values of $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ with different n are reported in Tables 1 and 2. All codes were written in Matlab.

Remark 4.1 Table 1 and Figure 1 show that the sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ converge to 0, where $\{0\} = F(T) \cap EP(F_1)$.

Example 4.2 In this example we take the same mappings and parameters as in Example 4.1 except T and F_i.

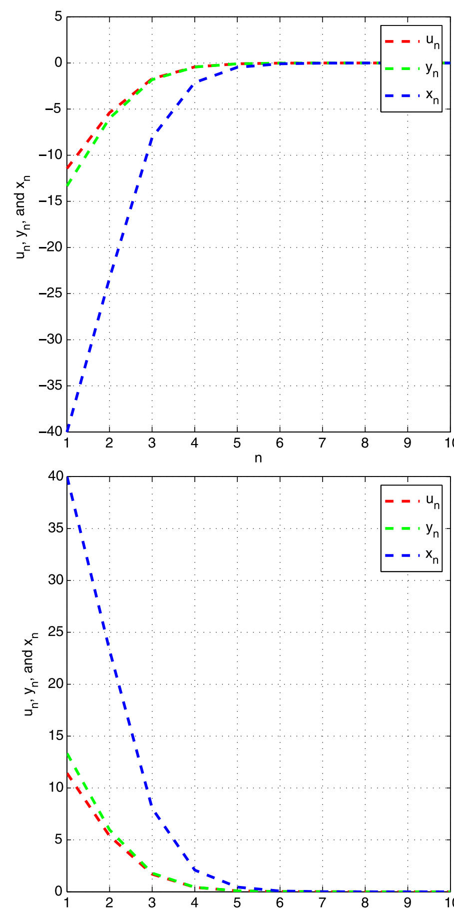
Let $T : [1, 70] \to [1, 70]$ be defined by

$$T(x) = \frac{3x + 1}{4}, \quad \forall x \in [1, 70],$$

Table 1 The values of $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial values $x_1 = -40$ and $x_1 = 40$

	$x_1 = -40$			$x_1 = 40$		
	u_n	y_n	x_n	u_n	y_n	x_n
$n = 1$	−11.428571	−13.333333	−40.000000	11.428571	13.333333	40.000000
$n = 2$	−5.382261	−5.980290	−23.323129	5.368132	5.964591	23.261905
$n = 3$	−1.702305	−1.812640	−8.085951	1.691401	1.801028	8.034153
$n = 4$	−0.422676	−0.440288	−2.113381	0.415900	0.433229	2.079500
$n = 5$	−0.089920	−0.092518	−0.464586	0.085540	0.088011	0.441955
$n = 6$	−0.017830	−0.018208	−0.094246	0.014702	0.015013	0.077709
$n = 7$	−0.003964	−0.004028	−0.021308	0.001537	0.001562	0.008260
$n = 8$	−0.001427	−0.001445	−0.007767	−0.000562	−0.000569	−0.003058
$n = 9$	−0.000907	−0.000917	−0.004990	−0.000779	−0.000787	−0.004284
$n = 10$	−0.000741	−0.000748	−0.004112	−0.000723	−0.000729	−0.004011

Figure 1 The convergence of $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial values $x_1 = -40$ and $x_1 = 40$.



and $F_1 : [1, 70] \times [1, 70] \to \mathbb{R}$ be defined by

$$F_1(x, y) = (y - x)(y + 2x - 3), \quad \forall (x, y) \in [1, 70] \times [1, 70].$$

It is clear to see that

$$EP(F_1) \cap F(T) = \{1\}.$$

By the definition of F_1, we have

$$0 \leq F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle = (y - u_n)(y + 2u_n - 3) + \frac{1}{r_n}(y - u_n)(u_n - x_n).$$

Then

$$\begin{aligned} 0 &\leq r_n(y - u_n)(y + 2u_n - 3) + \left(yu_n - yx_n - u_n^2 + u_n x_n\right) \\ &= r_n y^2 + (r_n u_n + u_n - x_n - 3r_n)y + 3r_n u_n - u_n^2 - 2r_n u_n^2 + u_n x_n. \end{aligned}$$

Let $A(y) = r_n y^2 + (r_n u_n + u_n - x_n - 3r_n)y + 3r_n u_n - u_n^2 - 2r_n u_n^2 + u_n x_n$. Then $A(y)$ is a quadratic function of y with coefficient $a = r_n$, $b = r_n u_n + u_n - x_n - 3r_n$, $c = 3r_n u_n - u_n^2 - 2r_n u_n^2 + u_n x_n$. We determine the discriminant Δ of A as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (r_n u_n + u_n - x_n - 3r_n)^2 - 4r_n\left(3r_n u_n - u_n^2 - 2r_n u_n^2 + u_n x_n\right) \\ &= 9r_n^2 - 6r_n u_n - 18r_n^2 u_n + u_n^2 + 6r_n u_n^2 + 9r_n^2 u_n^2 + 6r_n x_n - 2u_n x_n - 6r_n u_n x_n + x_n^2 \\ &= (u_n - 3r_n + 3u_n r_n - x_n)^2. \end{aligned}$$

We have $A(y) \geq 0$, $\forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta = 0$, we obtain

$$u_n = \frac{x_n + 3r_n}{1 + 3r_n}. \tag{4.2}$$

For every $n \geq 1$, from (4.2), we rewrite (3.1) as follows:

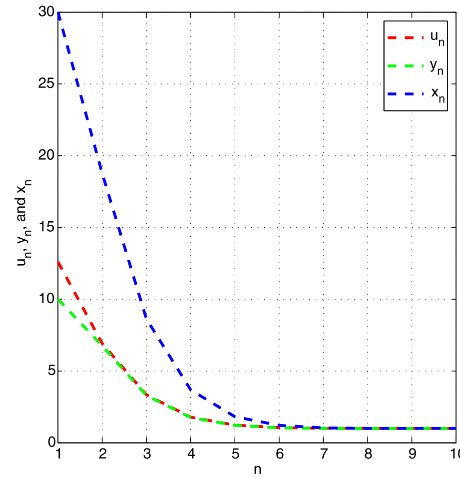
$$\begin{cases} y_n = \frac{x_n}{3n^2} + (1 - \frac{1}{n^2})(\frac{x_n + 3r_n}{1 + 3r_n}); \\ x_{n+1} = P_{[1,70]}[\rho \frac{x_n}{28n} + \frac{1}{2n}x_n + (1 - \frac{1}{2n})\frac{(3y_n + 1)}{4} - \mu \frac{3y_n + 7}{28n}]. \end{cases} \tag{4.3}$$

Remark 4.2 Table 2 and Figure 2 show that the sequences $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ converge to 1, where $\{1\} = F(T) \cap EP(F_1)$.

Table 2 The values of $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial value $x_1 = 30$

	u_n	y_n	x_n
$n = 1$	12.600000	10.000000	30.000000
$n = 2$	6.919218	6.752551	18.757653
$n = 3$	3.347083	3.294741	8.628019
$n = 4$	1.789318	1.754229	3.683683
$n = 5$	1.233421	1.208311	1.816975
$n = 6$	1.059453	1.041249	1.212331
$n = 7$	1.010462	0.996901	1.037925
$n = 8$	1.000000	0.989583	1.000000
$n = 9$	1.000000	0.991770	1.000000
$n = 10$	1.000000	0.993333	1.000000

Figure 2 The convergence of $\{u_n\}$, $\{y_n\}$, and $\{x_n\}$ with initial value $x_1 = 30$.



5 Conclusions

In this paper, we suggested and analyzed an iterative method for finding an element of the common set of solutions of (1.1) and (1.4) in real Hilbert spaces. This method can be viewed as a refinement and improvement of some existing methods for solving variational inequality problem, equilibrium problem and a hierarchical fixed point problem. Some existing methods, for example, [12, 13, 15, 18, 19, 21], can be viewed as special cases of Algorithm 3.1. Therefore, Algorithm 3.1 is expected to be widely applicable. In the hierarchical fixed point problem (1.4), if $S = I - (\rho U - \mu F)$, then we can get the variational inequality (3.13). In (3.13), if $U = 0$ then we get the variational inequality $\langle F(z), x - z\rangle \geq 0$, $\forall x \in F(T) \cap EP(F_1)$, which just is a variational inequality studied by Suzuki [23].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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