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# PPF dependent fixed point theorems for $\alpha_c$-admissible rational type contractive mappings in Banach spaces

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**Abstract**

In this paper, we prove some PPF dependent fixed point theorems in the Razumikhin class for some rational type contractive mappings involving $\alpha_c$-admissible mappings where the domain and range of the mappings are not the same. As applications of these results, we derive some PPF dependent fixed point theorems for these nonself-contractions whenever the range space is endowed with a graph. Our results extend and generalize some results in the literature.

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**Keywords:** fixed point; complete metric space; PPF dependent fixed point; $\alpha_c$-admissible mapping; rational type contractive mapping; Banach space

## 1 Introduction

The fixed point theory in Banach spaces plays an important role and is useful in mathematics. In fact, fixed point theory can be applied for solving equilibrium problems, variational inequalities and optimization problems. In particular, a very powerful tool is the Banach fixed point theorem, which was generalized and extended in various directions (see [1–37]). In 1977, Bernfeld *et al.* [2] introduced the concept of PPF dependent fixed point or the fixed point with PPF dependence which is a fixed point for mappings that have different domains and ranges. They also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. Very recently, some authors established the existence and uniqueness of PPF dependent fixed point for different types of contractive mappings and generalized some results of Bernfeld *et al.* [2] (see [1, 4, 12, 15, 20], and [33]).

In order to generalize the Banach contraction principle, Geraghty [9] proved the following theorem.

**Theorem 1** (Geraghty [9]) *Let* $(X, d)$ *be a complete metric space and* $T : X \to X$ *be an operator. Suppose that there exists* $\beta : [0, +\infty) \to [0, 1)$ *satisfying the condition*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0, \quad \textit{as } n \to +\infty.$$

*If $T$ satisfies the following inequality*:

$$d(Tx, Ty) \leq \beta\big(d(x,y)\big)d(x,y) \quad \textit{for all } x, y \in X, \tag{1.1}$$

*then $T$ has a unique fixed point*.

Throughout this paper, let $(E, \|\cdot\|_E)$ be a Banach space, $I$ denotes a closed interval $[a,b]$ in $\mathbb{R}$ and $E_0 = C(I,E)$ denotes the set of all continuous $E$-valued functions on $I$ equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I}\big\|\phi(t)\big\|_E.$$

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in $E_0$ is defined by

$$\mathcal{R}_c = \big\{\phi \in E_0 : \|\phi\|_{E_0} = \big\|\phi(c)\big\|_E\big\}.$$

Clearly, every constant function from $I$ to $E$ is a member of $\mathcal{R}_c$. It is easy to see that the class $\mathcal{R}_c$ is algebraically closed with respect to difference, *i.e.*, $\phi - \xi \in \mathcal{R}_c$ when $\phi, \xi \in \mathcal{R}_c$. Also the class $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on $E_0$ generated by the norm $\|\cdot\|_{E_0}$.

**Definition 1** ([2]) A mapping $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

**Definition 2** ([2]) The mapping $T : E_0 \to E$ is called a Banach type contraction if there exists $k \in [0,1)$ such that

$$\|T\phi - T\xi\|_E \leq k\|\phi - \xi\|_{E_0}$$

for all $\phi, \xi \in E_0$.

Samet in 2012 introduced the concepts of $\alpha$-$\psi$-contractive and $\alpha$-admissible mappings. Karapınar and Samet generalized these notions to obtain other fixed point results. Many authors generalized these notions to obtain fixed point results (see [18, 19, 21–23], and [32]).

Samet *et al.* [31], defined the notion of $\alpha$-admissible mappings as follows:

**Definition 3** ([31]) Let $T$ be a self-mapping on $X$ and $\alpha : X \times X \to [0,\infty)$ be a function. We say that $T$ is an $\alpha$-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

**Definition 4** ([17]) Let $f : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that $f$ is a triangular $\alpha$-admissible mapping if

(T1) $\alpha(x,y) \geq 1$ implies $\alpha(fx, fy) \geq 1$, $x, y \in X$,

(T2) $\begin{cases}\alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1\end{cases}$ implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

The concept of $\alpha_c$-admissible mapping was introduced by Agarwal *et al.* in 2013 (see [1]).

**Definition 5** ([1]) Let $c \in I$, $T : E_0 \to E$, and $\alpha : E \times E \to [0, \infty)$. We say $T$ is an $\alpha_c$-admissible mapping if for $\phi, \xi \in E_0$

$$\alpha\bigl(\phi(c), \xi(c)\bigr) \geq 1 \quad \Longrightarrow \quad \alpha(T\phi, T\xi) \geq 1. \tag{1.2}$$

**Definition 6** ([4]) Let $c \in I$, $T : E_0 \to E$, and $\alpha : E \times E \to [0, \infty)$. We say $T$ is a triangular $\alpha_c$-admissible mapping if

(T1) $\alpha(\phi(c), \xi(c)) \geq 1$ implies $\alpha(T\phi, T\xi) \geq 1$,

(T2) $\alpha(\phi(c), \mu(c)) \geq 1$ and $\alpha(\mu(c), \xi(c)) \geq 1$ implies $\alpha(\phi(c), \xi(c)) \geq 1$

for $\phi, \xi, \mu \in E_0$.

**Lemma 1** ([4]) *Let $T : E_0 \to E$ be a triangular $\alpha_c$-admissible mapping. Define the sequence $\{\phi_n\}$ in the following way*:

$$T\phi_{n-1} = \phi_n(c)$$

*for all $n \in \mathbb{N}$, where $\phi_0 \in \mathcal{R}_c$ is such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Then*

$$\alpha\bigl(\phi_n(c), \phi_m(c)\bigr) \geq 1 \quad \textit{for all } m, n \in \mathbb{N} \textit{ with } m < n.$$

## 2 Main results

Let $\mathcal{F}$ denotes the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0, \quad \text{as } n \to +\infty. \tag{2.1}$$

**Definition 7** Let $T : E_0 \to E$ be a nonself-mapping and $\alpha : E \times E \to [0, \infty)$ be a function. We say $T$ is a rational Geraghty contraction of type $I$ if there exist $\beta \in \mathcal{F}$ and $c \in I$ such that

$$\alpha\bigl(\phi(c), T\phi\bigr)\alpha\bigl(\xi(c), T\xi\bigr)\|T\phi - T\xi\|_E \leq \beta\bigl(M\bigl(\phi(c), \xi(c)\bigr)\bigr)M\bigl(\phi(c), \xi(c)\bigr)$$

for all $\phi, \xi \in E_0$, where

$$M\bigl(\phi(c), \xi(c)\bigr) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \frac{\|\phi(c) - T\phi\|_E \|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}.$$

**Theorem 2** *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following assertions*:

(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference*,

(b) *$T$ is an $\alpha_c$-admissible*,

(c) *$T$ is a rational Geraghty contractive mapping of type $I$*,

(d) *if* $\{\phi_n\}$ *is a sequence in* $E_0$ *such that* $\phi_n \to \phi$ *as* $n \to \infty$ *and* $\alpha(\phi_n(c), T\phi_n) \geq 1$, *then* $\alpha(\phi(c), T\phi) \geq 1$ *for all* $n \in \mathbb{N}$,

(e) *there exists* $\phi_0 \in \mathcal{R}_c$ *such that* $\alpha(\phi_0(c), T\phi_0) \geq 1$.

*Then* $T$ *has a unique PPF dependent fixed point* $\phi^* \in \mathcal{R}_c$. *Moreover, for a fixed* $\phi_0 \in \mathcal{R}_c$, *if the sequence* $\{\phi_n\}$ *of iterates of* $T$ *is defined by* $T\phi_{n-1} = \phi_n(c)$ *for all* $n \in \mathbb{N}$, *then* $\{\phi_n\}$ *converges to* $\phi^* \in \mathcal{R}_c$.

*Proof* Let $\phi_0$ is a point in $\mathcal{R}_c \subset E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that $x_1 = \phi_1(c)$. Since $\phi_1 \in \mathcal{R}_c \subset E_0$ and, by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that $x_2 = \phi_2(c)$. Continuing this process, by induction, we can build the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subset E_0$ such that $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \bigl\|\phi_{n-1}(c) - \phi_n(c)\bigr\|_E \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is $\alpha_c$-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we deduce that

$$\alpha\bigl(\phi_1(c), T\phi_1\bigr) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

By continuing this process, we get $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. Since $T$ is a rational Geraghty contraction of type $I$, we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \bigl\|\phi_n(c) - \phi_{n+1}(c)\bigr\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \alpha\bigl(\phi_{n-1}(c), T\phi_{n-1}\bigr)\alpha\bigl(\phi_n(c), T\phi_n\bigr)\|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \beta\bigl(M\bigl(\phi_{n-1}(c), \phi_n(c)\bigr)\bigr)M\bigl(\phi_{n-1}(c), \phi_n(c)\bigr).\end{aligned} \tag{2.2}$$

On the other hand,

$$\begin{aligned}M\bigl(\phi_{n-1}(c), \phi_n(c)\bigr) &= \max\biggl\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\qquad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}}, \\ &\qquad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|T\phi_{n-1} - T\phi_n\|_E}\biggr\} \\ &= \max\biggl\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\qquad \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}}, \\ &\qquad \frac{\|\phi_{n-1}(c) - \phi_n\|_E\|\phi_n(c) - \phi_{n+1}\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E}\biggr\} \\ &\leq \max\bigl\{\|\phi_{n-1} - \phi_n\|_{E_0}, \bigl\|\phi_n(c) - \phi_{n+1}(c)\bigr\|_E\bigr\} \\ &= \max\bigl\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\bigr\}.\end{aligned}$$

If

$$\max\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0}\} = \|\phi_n-\phi_{n+1}\|_{E_0},$$

from (2.2) we have

$$\|\phi_n-\phi_{n+1}\|_{E_0} \le \beta(\|\phi_n-\phi_{n+1}\|_{E_0})\|\phi_n-\phi_{n+1}\|_{E_0} < \|\phi_n-\phi_{n+1}\|_{E_0}, \tag{2.3}$$

which is a contradiction. So,

$$\max\{\|\phi_{n-1}-\phi_n\|_{E_0}, \|\phi_n-\phi_{n+1}\|_{E_0}\} = \|\phi_{n-1}-\phi_n\|_{E_0}.$$

By (2.2) we conclude

$$\|\phi_n-\phi_{n+1}\|_{E_0} \le \beta(\|\phi_{n-1}-\phi_n\|_{E_0})\|\phi_{n-1}-\phi_n\|_{E_0} < \|\phi_{n-1}-\phi_n\|_{E_0} \tag{2.4}$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{\|\phi_n-\phi_{n+1}\|_{E_0}\}$ is decreasing in $\mathbb{R}_+$. So, it is convergent. Suppose that there exists $r \ge 0$ such that $\lim_{n\to+\infty}\|\phi_n-\phi_{n+1}\|_{E_0} = r$. Assume that $r > 0$. Taking the limit as $n \to +\infty$ from (2.4) we conclude

$$r \le \lim_{n\to+\infty}\beta(\|\phi_{n-1}-\phi_n\|_{E_0})r,$$

which implies $1 \le \lim_{n\to+\infty}\beta(\|\phi_{n-1}-\phi_n\|_{E_0})$. So,

$$\lim_{n\to+\infty}\beta(\|\phi_{n-1}-\phi_n\|_{E_0}) = 1,$$

and since $\beta \in \mathcal{F}$, $\lim_{n\to+\infty}\|\phi_{n-1}-\phi_n\|_{E_0} = 0$, which is a contradiction. Hence, $r = 0$. This means

$$\lim_{n\to+\infty}\|\phi_{n-1}-\phi_n\|_{E_0} = 0. \tag{2.5}$$

We prove that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. Assume that $\{\phi_n\}$ is not a Cauchy sequence, then

$$\lim_{m,n\to+\infty}\|\phi_m-\phi_n\|_{E_0} > 0. \tag{2.6}$$

Since $T$ is a rational Geraghty contraction of type $I$, we have

$$\begin{aligned}
\|\phi_n-\phi_m\|_{E_0} &\le \|\phi_n-\phi_{n+1}\|_{E_0} + \|\phi_{n+1}-\phi_{m+1}\|_{E_0} + \|\phi_{m+1}-\phi_m\|_{E_0} \\
&\le \|\phi_n-\phi_{n+1}\|_{E_0} + \alpha(\phi_n(c), T\phi_n)\alpha(\phi_m(c), T\phi_m)\|T\phi_n - T\phi_m\|_{E_0} \\
&\quad + \|\phi_{m+1}-\phi_m\|_{E_0} \\
&\le \|\phi_n-\phi_{n+1}\|_{E_0} + \beta(M(\phi_n(c), \phi_m(c)))M(\phi_n(c), \phi_m(c)) \\
&\quad + \|\phi_{m+1}-\phi_m\|_{E_0}.
\end{aligned}$$

Taking the limit when $m, n \to \infty$ in the above inequality and applying (2.5) we deduce

$$\lim_{m,n\to\infty}\|\phi_n-\phi_m\|_{E_0} \le \lim_{m,n\to\infty}\beta(M(\phi_n(c), \phi_m(c)))\lim_{m,n\to\infty}M(\phi_n(c), \phi_m(c)), \tag{2.7}$$

where

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq M\big(\phi_n(c), \phi_m(c)\big) \\
&= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\quad \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|T\phi_n - T\phi_m\|_E}\bigg\} \\
&= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\quad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_{n+1}(c) - \phi_{m+1}(c)\|_E}\bigg\} \\
&= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\quad \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_{n+1} - \phi_{m+1}\|_{E_0}}\bigg\}.
\end{aligned} \tag{2.8}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.5), we get

$$\lim_{m,n\to+\infty} M\big(\phi_n(c), \phi_m(c)\big) = \lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}. \tag{2.9}$$

So, by (2.7) and (2.9), we have

$$\limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \leq \limsup_{m,n\to+\infty} \beta\big(\|\phi_n - \phi_m\|_{E_0}\big) \limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}$$

and hence from (2.6) we get $1 \leq \limsup_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0})$. This means

$$\lim_{m,n\to+\infty} \beta\big(\|\phi_m - \phi_n\|_{E_0}\big) = 1$$

and since $\beta \in \mathcal{F}$, we conclude

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} = 0,$$

which is a contradiction. Consequently,

$$\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} = 0$$

and hence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By Completeness of $E_0$, we find that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, this means $\phi_n \to \phi^*$, as $n \to +\infty$. Since $\mathcal{R}_c$ is topologically closed, we deduce, $\phi^* \in \mathcal{R}_c$. By condition $b$, we have $\alpha(\phi^*(c), T\phi^*) \geq 1$. Now, since $T$ is a rational Geraghty contraction of type $I$, we have

$$\begin{aligned}
&\big\|T\phi^* - \phi^*(c)\big\|_E \\
&\quad \leq \big\|T\phi^* - \phi_n(c)\big\|_E + \big\|\phi_n(c) - \phi^*(c)\big\|_E \\
&\quad = \big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0}
\end{aligned}$$

$$\begin{aligned}&\leq \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\phi_{n-1}(c), T\phi_{n-1}\big)\big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0}\\ &\leq \beta\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big)M\big(\phi^*(c), \phi_{n-1}(c)\big).\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\big\|T\phi^* - \phi^*(c)\big\|_E \leq \lim_{n\to\infty} \beta\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big) \lim_{n\to\infty} M\big(\phi^*(c), \phi_{n-1}(c)\big). \tag{2.10}$$

But

$$\begin{aligned}\lim_{n\to\infty} M\big(\phi^*(c), \phi_{n-1}(c)\big) &= \lim_{n\to\infty} \max\bigg\{\big\|\phi^* - \phi_{n-1}\big\|_{E_0},\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}},\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|T\phi^* - T\phi_{n-1}\|_E}\bigg\}\\ &= \lim_{n\to\infty} \max\bigg\{\big\|\phi^* - \phi_{n-1}\big\|_{E_0},\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}},\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|T\phi^* - \phi_n(c)\|_E}\bigg\}\\ &= \lim_{n\to\infty} \max\bigg\{\big\|\phi^* - \phi_{n-1}\big\|_{E_0},\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}}\\ &\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|T\phi^* - \phi_n(c)\|_E}\bigg\} = 0.\end{aligned} \tag{2.11}$$

Therefore, from (2.10) and (2.11), we deduce

$$\big\|T\phi^* - \phi^*(c)\big\|_E = 0,$$

that is,

$$T\phi^* = \phi^*(c),$$

which implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Now, we show that $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. Suppose on the contrary that $\phi^*$ and $\varphi^*$ are two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. Then

$$\begin{aligned}\big\|\phi^* - \varphi^*\big\|_{E_0} &= \big\|\phi^*(c) - \varphi^*(c)\big\|_E = \big\|T\phi^* - T\varphi^*\big\|_E\\ &\leq \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\varphi^*(c), T\varphi^*\big)\big\|T\phi^* - T\varphi^*\big\|_E\\ &\leq \beta\big(M\big(\phi^*(c), \varphi^*(c)\big)\big)M\big(\phi^*(c), \varphi^*(c)\big),\end{aligned}$$

where

$$M\big(\phi^*(c),\varphi^*(c)\big) = \max\biggl\{\bigl\|\phi^*-\varphi^*\bigr\|_{E_0}, \frac{\|\phi^*(c)-T\phi^*\|_E\|\varphi^*(c)-T\varphi^*\|_E}{1+\|\phi^*-\varphi^*\|_{E_0}}, \frac{\|\phi^*(c)-T\phi^*\|_E\|\varphi^*(c)-T\varphi^*\|_E}{1+\|T\phi^*-T\varphi^*\|_E}\biggr\} = \bigl\|\phi^*-\varphi^*\bigr\|_{E_0}.$$

Therefore,

$$\bigl\|\phi^*-\varphi^*\bigr\|_{E_0} \le \beta\bigl(\bigl\|\phi^*-\varphi^*\bigr\|_{E_0}\bigr)\bigl\|\phi^*-\varphi^*\bigr\|_{E_0} < \bigl\|\phi^*-\varphi^*\bigr\|_{E_0},$$

which is a contradiction. Hence, $\phi^* = \varphi^*$. Then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. □

**Definition 8** Let $\alpha : E\times E \to [0,\infty)$ and $T : E_0 \to E$. We say that $T$ is a rational Geraghty contraction of type *II* if there exist $\beta \in \mathcal{F}$ and $c \in I$ such that

$$\alpha\bigl(\phi(c),T\phi\bigr)\alpha\bigl(\xi(c),T\xi\bigr)\|T\phi-T\xi\|_E \le \beta\bigl(M\bigl(\phi(c),\xi(c)\bigr)\bigr)M\bigl(\phi(c),\xi(c)\bigr)$$

for all $\phi, \xi \in E_0$, where

$$M\bigl(\phi(c),\xi(c)\bigr) = \max\biggl\{\|\phi-\xi\|_{E_0}, \frac{\|\phi(c)-T\phi\|_E\|\phi(c)-T\xi\|_E+\|\xi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi(c)-T\phi\|_E+\|\xi(c)-T\xi\|_E}, \frac{\|\phi(c)-T\phi\|_E\|\phi(c)-T\xi\|_E+\|\xi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E}\biggr\}.$$

**Theorem 3** *Let $T : E_0 \to E$ and $\alpha : E\times E \to [0,\infty)$ be two mappings satisfying the following assertions*:

(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference*,
(b) *$T$ is an $\alpha_c$-admissible*,
(c) *$T$ is a rational Geraghty contractive mapping of type II*,
(d) *if $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n\to\infty$ and $\alpha(\phi_n(c),T\phi_n) \ge 1$, then $\alpha(\phi(c),T\phi) \ge 1$ for all $n \in \mathbb{N}$*,
(e) *there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c),T\phi_0) \ge 1$.*

*Then $T$ has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of $T$ is defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to $\phi^* \in \mathcal{R}_c$.*

*Proof* Suppose that $\phi_0$ is a point in $\mathcal{R}_c \subset E_0$ such that $\alpha(\phi_0(c),T\phi_0) \ge 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that $x_1 = \phi_1(c)$. Since $\phi_1 \in \mathcal{R}_c \subset E_0$ and, by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that $x_2 = \phi_2(c)$. Continuing this process, by induction, we can build the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subset E_0$ such that $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$.

It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is $\alpha_c$-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we deduce that

$$\alpha(\phi_1(c), T\phi_1) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

Continuing this process, we get $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. Since $T$ is a rational Geraghty contraction of type *II*, we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \alpha(\phi_{n-1}(c), T\phi_{n-1})\alpha(\phi_n(c), T\phi_n)\|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \beta\big(M(\phi_{n-1}(c), \phi_n(c))\big)M(\phi_{n-1}(c), \phi_n(c)). \end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned} &M(\phi_{n-1}(c), \phi_n(c)) \\ &\quad = \max\bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\qquad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_n\|_E\|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1}(c) - T\phi_{n-1}\|_E + \|\phi_n(c) - T\phi_n\|_E}, \\ &\qquad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_n\|_E\|\phi_n(c) - T\phi_{n-1}\|_E}{1 + \|\phi_{n-1}(c) - T\phi_n\|_E + \|\phi_n(c) - T\phi_{n-1}\|_E}\bigg\} \\ &\quad = \max\bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\qquad \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E\|\phi_n(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_n(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E}, \\ &\qquad \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E\|\phi_n(c) - \phi_n(c)\|_E}{1 + \|\phi_{n-1}(c) - \phi_{n+1}(c)\|_E + \|\phi_n(c) - \phi_n(c)\|_E}\bigg\} \\ &\quad = \max\bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\qquad \frac{\|\phi_{n-1} - \phi_n\|_{E_0}\|\phi_{n-1} - \phi_{n+1}\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}\|\phi_n - \phi_n\|_{E_0}}{1 + \|\phi_{n-1} - \phi_n\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}}, \\ &\qquad \frac{\|\phi_{n-1} - \phi_n\|_{E_0}\|\phi_{n-1} - \phi_{n+1}\|_{E_0} + \|\phi_n - \phi_{n+1}\|_{E_0}\|\phi_n - \phi_n\|_{E_0}}{1 + \|\phi_{n-1} - \phi_{n+1}\|_{E_0} + \|\phi_n - \phi_n\|_{E_0}}\bigg\} \\ &\quad = \|\phi_{n-1} - \phi_n\|_{E_0}. \end{aligned}$$

From (2.12) we conclude

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq \beta(\|\phi_{n-1} - \phi_n\|_{E_0})\|\phi_{n-1} - \phi_n\|_{E_0} < \|\phi_{n-1} - \phi_n\|_{E_0} \tag{2.13}$$

for all $n \in \mathbb{N}$. So, the sequence $\{\|\phi_n - \phi_{n+1}\|_{E_0}\}$ is decreasing in $\mathbb{R}_+$ and there exists $r \geq 0$ such that $\lim_{n\to+\infty} \|\phi_n - \phi_{n+1}\|_{E_0} = r$. Reviewing the proof of Theorem 2, we can show that

$r = 0$, *i.e.*,

$$\lim_{n\to+\infty} \|\phi_{n-1} - \phi_n\|_{E_0} = 0. \tag{2.14}$$

Now, we prove that the sequence $\{\phi_n\}$ is Cauchy in $\mathcal{R}_c$. If not, then

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} > 0. \tag{2.15}$$

From the fact that $T$ is a rational Geraghty contraction of type *II*, we have

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\le \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{m+1}\|_{E_0} + \|\phi_{m+1} - \phi_m\|_{E_0} \\
&\le \|\phi_n - \phi_{n+1}\|_{E_0} + \alpha\big(\phi_n(c), T\phi_n\big)\alpha\big(\phi_m(c), T\phi_m\big)\|T\phi_n - T\phi_m\|_E \\
&\quad + \|\phi_{m+1} - \phi_m\|_{E_0} \\
&\le \|\phi_n - \phi_{n+1}\|_{E_0} + \beta\big(M\big(\phi_n(c), \phi_m(c)\big)\big)M\big(\phi_n(c), \phi_m(c)\big) \\
&\quad + \|\phi_{m+1} - \phi_m\|_{E_0}.
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.14) we deduce

$$\lim_{m,n\to\infty} \|\phi_n - \phi_m\|_{E_0} \le \lim_{m,n\to\infty} \beta\big(M\big(\phi_n(c), \phi_m(c)\big)\big) \lim_{m,n\to\infty} M\big(\phi_n(c), \phi_m(c)\big), \tag{2.16}$$

where

$$\begin{aligned}
&\|\phi_n - \phi_m\|_{E_0} \\
&\quad \le M\big(\phi_n(c), \phi_m(c)\big) \\
&\quad = \max\Bigg\{ \|\phi_n - \phi_m\|_{E_0}, \\
&\qquad \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_n(c) - T\phi_m\|_E + \|\phi_m(c) - T\phi_m\|_E \|\phi_m(c) - T\phi_n\|_E}{1 + \|\phi_n(c) - T\phi_n\|_E + \|\phi_m(c) - T\phi_m\|_E}, \\
&\qquad \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_n(c) - T\phi_m\|_E + \|\phi_m(c) - T\phi_m\|_E \|\phi_m(c) - T\phi_n\|_E}{1 + \|\phi_n(c) - T\phi_m\|_E + \|\phi_m(c) - T\phi_n\|_E} \Bigg\} \\
&\quad = \max\Bigg\{ \|\phi_n - \phi_m\|_{E_0}, \\
&\qquad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_n(c) - \phi_{m+1}(c)\|_E + \|\phi_m(c) - \phi_{m+1}\|_E \|\phi_m(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E + \|\phi_m(c) - \phi_{m+1}(c)\|_E}, \\
&\qquad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_n(c) - \phi_{m+1}(c)\|_E + \|\phi_m(c) - \phi_{m+1}(c)\|_E \|\phi_m(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_{m+1}(c)\|_E + \|\phi_m(c) - \phi_{n+1}(c)\|_E} \Bigg\} \\
&\quad = \max\Bigg\{ \|\phi_n - \phi_m\|_{E_0}, \\
&\qquad \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_n - \phi_{m+1}\|_{E_0} + \|\phi_m - \phi_{m+1}\|_{E_0} \|\phi_m - \phi_{n+1}\|_{E_0}}{1 + \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_m - \phi_{m+1}\|_{E_0}}, \\
&\qquad \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_n - \phi_{m+1}\|_{E_0} + \|\phi_m - \phi_{m+1}\|_{E_0} \|\phi_m - \phi_{n+1}\|_{E_0}}{1 + \|\phi_n - \phi_{m+1}\|_{E_0} + \|\phi_m - \phi_{n+1}\|_{E_0}} \Bigg\}.
\end{aligned} \tag{2.17}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.14), we get

$$\lim_{m,n\to+\infty} M\big(\phi_n(c), \phi_m(c)\big) = \lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}. \tag{2.18}$$

So, from (2.16) and (2.18), we obtain

$$\limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \le \limsup_{m,n\to+\infty} \beta\big(\|\phi_n - \phi_m\|_{E_0}\big) \limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}$$

and so by (2.15) we get, $1 \le \limsup_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0})$. That is,

$$\lim_{m,n\to+\infty} \beta\big(\|\phi_m - \phi_n\|_{E_0}\big) = 1$$

and since $\beta \in \mathcal{F}$, we deduce

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} = 0,$$

which is a contradiction. Consequently,

$$\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} = 0$$

and hence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By completeness of $E_0$, we find that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, this means that $\phi_n \to \phi^*$, as $n \to +\infty$. Since $\mathcal{R}_c$ is topologically closed, we deduce that $\phi^* \in \mathcal{R}_c$. Now, since $T$ is a rational Geraghty contraction of type *II*, we have

$$\begin{aligned}
&\big\|T\phi^* - \phi^*(c)\big\|_E \\
&\quad \le \big\|T\phi^* - \phi_n(c)\big\|_E + \big\|\phi_n(c) - \phi^*(c)\big\|_E \\
&\quad = \big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0} \\
&\quad \le \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\phi_{n-1}(c), T\phi_{n-1}\big)\big\|T\phi^* - T\phi_{n-1}\big\|_E + \big\|\phi_n - \phi^*\big\|_{E_0} \\
&\quad \le \beta\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big)M\big(\phi^*(c), \phi_{n-1}(c)\big).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\big\|T\phi^* - \phi^*(c)\big\|_E \le \lim_{n\to\infty} \beta\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big) \lim_{n\to\infty} M\big(\phi^*(c), \phi_{n-1}(c)\big). \tag{2.19}$$

But

$$\begin{aligned}
&M\big(\phi^*(c), \phi_{n-1}(c)\big) \\
&\quad = \max\bigg\{\big\|\phi^* - \phi_{n-1}\big\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi^*(c) - T\phi_{n-1}\|_E + \|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi^*(c) - T\phi_{n-1}\|_E + \|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\phi_{n-1}\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E}\bigg\}
\end{aligned}$$

$$\begin{aligned}
&= \max\Bigg\{ \|\phi^* - \phi_{n-1}\|_{E_0}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - \phi_n(c)\|_E + \|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - \phi_n(c)\|_E}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - \phi_n(c)\|_E + \|\phi_{n-1}(c) - \phi_n(c)\|_E \|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - \phi_n(c)\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E} \Bigg\} \\
&= \max\Bigg\{ \|\phi^* - \phi_{n-1}\|_{E_0}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n-1} - \phi_n\|_{E_0} \|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1} - \phi_n\|_{E_0}}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n-1} - \phi_n\|_{E_0} \|\phi_{n-1}(c) - T\phi^*\|_E}{1 + \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n-1} - T\phi^*\|_{E_0}} \Bigg\}. \qquad (2.20)
\end{aligned}$$

So,

$$\lim_{n\to\infty} M\big(\phi^*(c), \phi_{n-1}(c)\big) = 0,$$

and by (2.19) and (2.20), we conclude

$$\|T\phi^* - \phi^*(c)\|_E = 0,$$

that is,

$$T\phi^* = \phi^*(c),$$

which implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Finally, we prove the uniqueness of the PPF dependent fixed point of T in $\mathcal{R}_c$. Let $\phi^*$ and $\varphi^*$ be two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. So, we obtain

$$\begin{aligned}
\|\phi^* - \varphi^*\|_{E_0} &= \|\phi^*(c) - \varphi^*(c)\|_E \\
&= \|T\phi^* - T\varphi^*\|_E \\
&\le \alpha\big(\phi^*(c), T\phi^*\big)\alpha\big(\varphi^*(c), T\varphi^*\big)\|T\phi^* - T\varphi^*\|_E \\
&\le \beta\big(M\big(\phi^*(c), \varphi^*(c)\big)\big)M\big(\phi^*(c), \varphi^*(c)\big),
\end{aligned}$$

where

$$\begin{aligned}
M\big(\phi^*(c), \varphi^*(c)\big) &= \max\Bigg\{ \|\phi^* - \varphi^*\|_{E_0}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - T\varphi^*\|_E + \|\varphi^*(c) - T\varphi^*\|_E \|\varphi^*(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\varphi^*(c) - T\varphi^*\|_E}, \\
&\quad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi^*(c) - T\varphi^*\|_E + \|\varphi^*(c) - T\varphi^*\|_E \|\varphi^*(c) - T\phi^*\|_E}{1 + \|\phi^*(c) - T\varphi^*\|_E + \|\varphi^*(c) - T\phi^*\|_E} \Bigg\} \\
&= \|\phi^* - \varphi^*\|_{E_0}.
\end{aligned}$$

Therefore,

$$\|\phi^* - \varphi^*\|_{E_0} \le \beta\big(\|\phi^* - \varphi^*\|_{E_0}\big)\|\phi^* - \varphi^*\|_{E_0} < \|\phi^* - \varphi^*\|_{E_0},$$

which is a contradiction. Hence, $\phi^* = \varphi^*$. Therefore, $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

**Definition 9** Let $\alpha : E \times E \to [0,\infty)$ and $T : E_0 \to E$. We say that $T$ is a rational Geraghty contraction of type *III* if there exist $\beta \in \mathcal{F}$ and $c \in I$ such that

$$\alpha\bigl(\phi(c), T\phi\bigr)\alpha\bigl(\xi(c), T\xi\bigr)\|T\phi - T\xi\|_E \leq \beta\bigl(M\bigl(\phi(c),\xi(c)\bigr)\bigr)M\bigl(\phi(c),\xi(c)\bigr)$$

for all $\phi, \xi \in E_0$, where

$$M\bigl(\phi(c),\xi(c)\bigr) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0} + \|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}, \frac{\|\phi(c) - T\xi\|_E\|\phi(c) - \xi(c)\|_E}{1 + \|\phi(c) - T\phi\|_E + \|\xi(c) - T\phi\|_E + \|\xi(c) - T\xi\|_E}\right\}.$$

**Theorem 4** *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0,\infty)$ be two mappings satisfying the following assertions*:

(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference,*
(b) *$T$ is an $\alpha_c$-admissible,*
(c) *$T$ is a rational Geraghty contractive mapping of type III,*
(d) *if $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$, then $\alpha(\phi(c), T\phi) \geq 1$ for all $n \in \mathbb{N}$,*
(e) *there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.*

*Then $T$ has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of $T$ defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to the PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof* Suppose that $\phi_0$ be a point in $\mathcal{R}_c \subset E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that $x_1 = \phi_1(c)$. Since $\phi_1 \in \mathcal{R}_c \subset E_0$ and, by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that $x_2 = \phi_2(c)$. Repeating this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subset E_0$ such that $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$. From the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference it follows that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \bigl\|\phi_{n-1}(c) - \phi_n(c)\bigr\|_E \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is $\alpha_c$-admissible and $\alpha(\phi_0(c), \phi_1(c)) = \alpha(\phi_0(c), T\phi_0) \geq 1$, we deduce

$$\alpha\bigl(\phi_1(c), T\phi_1\bigr) = \alpha(T\phi_0, T\phi_1) \geq 1.$$

Continuing this process, we get $\alpha(\phi_{n-1}(c), T\phi_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. By the fact that $T$ is a rational Geraghty contraction of type *III*, we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \bigl\|\phi_n(c) - \phi_{n+1}(c)\bigr\|_E = \|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \alpha\bigl(\phi_{n-1}(c), T\phi_{n-1}\bigr)\alpha\bigl(\phi_n(c), T\phi_n\bigr)\|T\phi_{n-1} - T\phi_n\|_E \\ &\leq \beta\bigl(M\bigl(\phi_{n-1}(c),\phi_n(c)\bigr)\bigr)M\bigl(\phi_{n-1}(c),\phi_n(c)\bigr). \qquad (2.21)\end{aligned}$$

On the other hand,

$$\begin{aligned} M\bigl(\phi_{n-1}(c),\phi_n(c)\bigr) &= \max\biggl\{\|\phi_{n-1}-\phi_n\|_{E_0}, \\ &\quad \frac{\|\phi_{n-1}(c)-T\phi_{n-1}\|_E\|\phi_n(c)-T\phi_n\|_E}{1+\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_{n-1}(c)-T\phi_n\|_E+\|\phi_n(c)-T\phi_{n-1}\|_E}, \\ &\quad \frac{\|\phi_{n-1}(c)-T\phi_n\|_E\|\phi_{n-1}-\phi_n\|_{E_0}}{1+\|\phi_{n-1}(c)-T\phi_{n-1}\|_E+\|\phi_n(c)-T\phi_{n-1}\|_E+\|\phi_n(c)-T\phi_n\|_E}\biggr\} \\ &= \max\biggl\{\|\phi_{n-1}-\phi_n\|_{E_0}, \\ &\quad \frac{\|\phi_{n-1}(c)-\phi_n(c)\|_E\|\phi_n(c)-\phi_{n+1}(c)\|_E}{1+\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_{n-1}(c)-\phi_{n+1}(c)\|_E+\|\phi_n(c)-\phi_n(c)\|_E}, \\ &\quad \frac{\|\phi_{n-1}(c)-\phi_{n+1}(c)\|_E\|\phi_{n-1}-\phi_n\|_{E_0}}{1+\|\phi_{n-1}(c)-\phi_n(c)\|_E+\|\phi_n(c)-\phi_n(c)\|_E+\|\phi_n(c)-\phi_{n+1}(c)\|_E}\biggr\} \\ &\le \max\biggl\{\|\phi_{n-1}-\phi_n\|_{E_0}, \\ &\quad \frac{\|\phi_{n-1}-\phi_n\|_{E_0}(\|\phi_n-\phi_{n-1}\|_{E_0}+\|\phi_{n-1}-\phi_{n+1}\|_{E_0})}{1+\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_{n-1}-\phi_{n+1}\|_{E_0}+\|\phi_n-\phi_n\|_{E_0}}, \\ &\quad \frac{(\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_n-\phi_{n+1}\|_{E_0})\|\phi_{n-1}-\phi_n\|_{E_0}}{1+\|\phi_{n-1}-\phi_n\|_{E_0}+\|\phi_n-\phi_n\|_{E_0}+\|\phi_n-\phi_{n+1}\|_{E_0}}\biggr\} \\ &= \|\phi_{n-1}-\phi_n\|_{E_0}. \end{aligned}$$

From (2.21) we conclude

$$\|\phi_n-\phi_{n+1}\|_{E_0} \le \beta\bigl(\|\phi_{n-1}-\phi_n\|_{E_0}\bigr)\|\phi_{n-1}-\phi_n\|_{E_0} < \|\phi_{n-1}-\phi_n\|_{E_0} \tag{2.22}$$

for all $n \in \mathbb{N}$. This implies that the sequence $\{\|\phi_n-\phi_{n+1}\|_{E_0}\}$ is decreasing in $\mathbb{R}_+$. Then there exists $r \ge 0$ such that $\lim_{n\to+\infty}\|\phi_n-\phi_{n+1}\|_{E_0} = r$. Repeating the proof of Theorem 2, we conclude that $r = 0$. That is,

$$\lim_{n\to+\infty}\|\phi_{n-1}-\phi_n\|_{E_0} = 0. \tag{2.23}$$

Now, we prove that the sequence $\{\phi_n\}$ is Cauchy in $\mathcal{R}_c$. If not, then

$$\lim_{m,n\to+\infty}\|\phi_m-\phi_n\|_{E_0} > 0. \tag{2.24}$$

Since $T$ is a rational Geraghty contraction of type *III*, we have

$$\begin{aligned} \|\phi_n-\phi_m\|_{E_0} &\le \|\phi_n-\phi_{n+1}\|_{E_0}+\|\phi_{n+1}-\phi_{m+1}\|_{E_0}+\|\phi_{m+1}-\phi_m\|_{E_0} \\ &\le \|\phi_n-\phi_{n+1}\|_{E_0}+\alpha\bigl(\phi_n(c),T\phi_n\bigr)\alpha\bigl(\phi_m(c),T\phi_m\bigr)\|T\phi_n-T\phi_m\|_{E_0} \\ &\quad +\|\phi_{m+1}-\phi_m\|_{E_0} \\ &\le \beta\bigl(M\bigl(\phi_n(c),\phi_m(c)\bigr)\bigr)M\bigl(\phi_n(c),\phi_m(c)\bigr) \\ &\quad +\|\phi_{m+1}-\phi_m\|_{E_0}. \end{aligned}$$

Making $m, n \to \infty$ in the above inequality and applying (2.23) we have

$$\lim_{m,n\to\infty} \|\phi_n - \phi_m\|_{E_0} \leq \lim_{m,n\to\infty} \beta\bigl(M\bigl(\phi_n(c), \phi_m(c)\bigr)\bigr) \lim_{m,n\to\infty} M\bigl(\phi_n(c), \phi_m(c)\bigr). \tag{2.25}$$

Also,

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq M\bigl(\phi_n(c), \phi_m(c)\bigr) \\
&= \max\biggl\{ \|\phi_n - \phi_m\|_{E_0}, \\
&\quad \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|\phi_n(c) - \phi_m(c)\|_{E_0} + \|\phi_n(c) - T\phi_m\|_E + \|\phi_m(c) - T\phi_n\|_E}, \\
&\quad \frac{\|\phi_n(c) - T\phi_m\|_E \|\phi_n(c) - \phi_m(c)\|_E}{1 + \|\phi_n(c) - T\phi_n\|_E + \|\phi_m(c) - T\phi_n\|_E + \|\phi_m(c) - T\phi_m\|_E} \biggr\} \\
&= \max\biggl\{ \|\phi_n - \phi_m\|_{E_0}, \\
&\quad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_m(c)\|_{E_0} + \|\phi_n(c) - \phi_{m+1}(c)\|_E + \|\phi_m(c) - \phi_{n+1}(c)\|_E}, \\
&\quad \frac{\|\phi_n(c) - \phi_{m+1}(c)\|_E \|\phi_n(c) - \phi_m(c)\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E + \|\phi_m(c) - \phi_{n+1}(c)\|_E + \|\phi_m(c) - \phi_{m+1}(c)\|_E} \biggr\} \\
&\leq \max\biggl\{ \|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_n - \phi_m\|_{E_0} + \|\phi_n - \phi_{m+1}\|_{E_0} + \|\phi_m - \phi_{n+1}\|_{E_0}}, \\
&\quad \frac{(\|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_m - \phi_{n+1}\|_{E_0} + \|\phi_m - \phi_{m+1}\|_{E_0}) \|\phi_n - \phi_m\|_{E_0}}{1 + \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_m - \phi_{n+1}\|_{E_0} + \|\phi_m - \phi_{m+1}\|_{E_0}} \biggr\}.
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.23), we get

$$\lim_{m,n\to+\infty} M(\phi_n, \phi_m) = \lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}. \tag{2.26}$$

Hence, from (2.25) and (2.26), we obtain

$$\limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \leq \limsup_{m,n\to+\infty} \beta\bigl(\|\phi_n - \phi_m\|_{E_0}\bigr) \limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}$$

and so by (2.24) we get $1 \leq \limsup_{m,n\to+\infty} \beta(\|\phi_n - \phi_m\|_{E_0})$. That is,

$$\lim_{m,n\to+\infty} \beta\bigl(\|\phi_m - \phi_n\|_{E_0}\bigr) = 1$$

and since $\beta \in \mathcal{F}$, we deduce

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} = 0,$$

which is a contradiction. Consequently,

$$\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} = 0$$

and hence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. Completeness of $E_0$ shows that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, this means that $\phi_n \to \phi^*$, as $n \to +\infty$. Since $\mathcal{R}_c$ is topologically closed, we deduce that $\phi^* \in \mathcal{R}_c$. Now, since $T$ is a rational Geraghty contraction of type *III*, we have

$$\begin{aligned}
&\bigl\|T\phi^* - \phi^*(c)\bigr\|_E \\
&\quad\le \bigl\|T\phi^* - \phi_n(c)\bigr\|_E + \bigl\|\phi_n(c) - \phi^*(c)\bigr\|_E \\
&\quad= \bigl\|T\phi^* - T\phi_{n-1}\bigr\|_E + \bigl\|\phi_n - \phi^*\bigr\|_{E_0} \\
&\quad\le \alpha\bigl(\phi^*(c), T\phi^*\bigr)\alpha\bigl(\phi_{n-1}(c), T\phi_{n-1}\bigr)\bigl\|T\phi^* - T\phi_{n-1}\bigr\|_E + \bigl\|\phi_n - \phi^*\bigr\|_{E_0} \\
&\quad\le \beta\bigl(M\bigl(\phi^*(c), \phi_{n-1}(c)\bigr)\bigr)M\bigl(\phi^*(c), \phi_{n-1}(c)\bigr).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\bigl\|T\phi^* - \phi^*(c)\bigr\|_E \le \lim_{n\to\infty} \beta\bigl(M\bigl(\phi^*(c), \phi_{n-1}(c)\bigr)\bigr) \lim_{n\to\infty} M\bigl(\phi^*(c), \phi_{n-1}(c)\bigr). \tag{2.27}$$

But

$$\begin{aligned}
M\bigl(\phi^*(c), \phi_{n-1}(c)\bigr) &= \max\Biggl\{\bigl\|\phi^* - \phi_{n-1}\bigr\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0} + \|\phi^*(c) - T\phi_{n-1}\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E}, \\
&\qquad \frac{\|\phi^*(c) - T\phi_{n-1}\|_E\|\phi^* - \phi_{n-1}\|_{E_0}}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}\Biggr\} \\
&= \max\Biggl\{\bigl\|\phi^* - \phi_{n-1}\bigr\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0} + \|\phi^*(c) - \phi_n(c)\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E}, \\
&\qquad \frac{\|\phi^*(c) - \phi_n(c)\|_E\|\phi^* - \phi_{n-1}\|_{E_0}}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - \phi_n(c)\|_E}\Biggr\} \\
&= \max\Biggl\{\bigl\|\phi^* - \phi_{n-1}\bigr\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|\phi^* - \phi_{n-1}\|_{E_0} + \|\phi^* - \phi_n\|_{E_0} + \|\phi_{n-1}(c) - T\phi^*\|_E}, \\
&\qquad \frac{\|\phi^* - \phi_n\|_{E_0}\|\phi^* - \phi_{n-1}\|_{E_0}}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\phi_{n-1}(c) - T\phi^*\|_E + \|\phi_{n-1} - \phi_n\|_{E_0}}\Biggr\}.
\end{aligned} \tag{2.28}$$

Therefore, from (2.27) and (2.28), we deduce that

$$\bigl\|T\phi^* - \phi^*(c)\bigr\|_E = 0,$$

that is,

$$T\phi^* = \phi^*(c),$$

which implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Suppose that $\phi^*$ and $\varphi^*$ are two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. So,

$$\begin{aligned}\|\phi^* - \varphi^*\|_{E_0} &= \|\phi^*(c) - \varphi^*(c)\|_E = \|T\phi^* - T\varphi^*\|_E \\ &\leq \alpha(\phi^*(c), T\phi^*)\alpha(\varphi^*(c), T\varphi^*)\|T\phi^* - T\varphi^*\|_E \\ &\leq \beta(M(\phi^*(c), \varphi^*(c)))M(\phi^*(c), \varphi^*(c)),\end{aligned}$$

where

$$\begin{aligned}M(\phi^*(c), \varphi^*(c)) = \max\biggl\{ &\|\phi^* - \varphi^*\|_{E_0}, \\ &\frac{\|\phi^*(c) - T\phi^*\|_E\|\varphi^*(c) - T\varphi^*\|_E}{1 + \|\phi^* - \varphi^*\|_{E_0} + \|\phi^*(c) - T\varphi^*\|_E + \|\varphi^*(c) - T\phi^*\|_E}, \\ &\frac{\|\phi^*(c) - T\varphi^*\|_E\|\phi^* - \varphi^*\|_{E_0}}{1 + \|\phi^*(c) - T\phi^*\|_E + \|\varphi^*(c) - T\phi^*\|_E + \|\varphi^*(c) - T\varphi^*\|_E}\biggr\} \\ = \|\phi^* - \varphi^*\|_{E_0}.\end{aligned}$$

Therefore,

$$\|\phi^* - \varphi^*\|_{E_0} \leq \beta(\|\phi^* - \varphi^*\|_{E_0})\|\phi^* - \varphi^*\|_{E_0} < \|\phi^* - \varphi^*\|_{E_0},$$

which is a contradiction. Hence, $\phi^* = \varphi^*$. Then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. □

**Corollary 1** *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following assertions*:

(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference,*

(b) *$T$ is an $\alpha_c$-admissible,*

(c) *assume that*

$$\alpha(\phi(c), T\phi)\alpha(\xi(c), T\xi)\|T\phi - T\xi\|_E \leq rM(\phi(c), \xi(c))$$

*for all $\phi, \xi \in E_0$, where $0 \leq r < 1$ and*

$$\begin{aligned}M(\phi(c), \xi(c)) = \max\biggl\{ &\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \\ &\frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\biggr\}\end{aligned}$$

*or*

$$\begin{aligned}M(\phi(c), \xi(c)) = \max\biggl\{ &\|\phi - \xi\|_{E_0}, \\ &\frac{\|\phi(c) - T\phi\|_E\|\phi(c) - T\xi\|_E + \|\xi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi - T\phi\|_E + \|\xi - T\xi\|_E}, \\ &\frac{\|\phi(c) - T\phi\|_E\|\phi(c) - T\xi\|_E + \|\xi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi - T\xi\|_E + \|\xi - T\phi\|_E}\biggr\},\end{aligned}$$

*or*

$$M\big(\phi(c),\xi(c)\big)=\max\Bigg\{\|\phi-\xi\|_{E_0},\\ \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}+\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E},\\ \frac{\|\phi(c)-T\xi\|_E\|\phi-\xi\|_{E_0}}{1+\|\phi(c)-T\phi\|_E+\|\xi(c)-T\phi\|_E+\|\xi(c)-T\xi\|_E}\Bigg\},$$

(d) *if* $\{\phi_n\}$ *is a sequence in* $E_0$ *such that* $\phi_n\to\phi$ *as* $n\to\infty$ *and* $\alpha(\phi_n(c),T\phi_n)\geq 1$, *then* $\alpha(\phi(c),T\phi)\geq 1$ *for all* $n\in\mathbb{N}$,

(e) *there exists* $\phi_0\in\mathcal{R}_c$ *such that* $\alpha(\phi_0(c),T\phi_0)\geq 1$.

*Then* $T$ *has a unique PPF dependent fixed point* $\phi^*\in\mathcal{R}_c$.

**Corollary 2** *Let* $T:E_0\to E$, $\alpha:E\times E\to[0,\infty)$ *be two mappings satisfying the following assertions*:

(a) *there exists* $c\in I$ *such that* $\mathcal{R}_c$ *is topologically closed and algebraically closed with respect to difference*,

(b) $T$ *is an* $\alpha_c$*-admissible*,

(c) *assume that*

$$\alpha\big(\phi(c),T\phi\big)\alpha\big(\xi(c),T\xi\big)\|T\phi-T\xi\|_E\\ \leq a\|\phi-\xi\|_{E_0}+b\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}}+c\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}$$

*or*

$$\|T\phi-T\xi\|_E\leq a\|\phi-\xi\|_{E_0}\\ +b\frac{\|\phi(c)-T\phi\|_E\|\phi(c)-T\xi\|_E+\|\xi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi(c)-T\phi\|_E+\|\xi(c)-T\xi\|_E}\\ +c\frac{\|\phi(c)-T\phi\|_E\|\phi(c)-T\xi\|_E+\|\xi(c)-T\xi\|_E\|\xi(c)-T\phi\|_E}{1+\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E},$$

*or*

$$\|T\phi-T\xi\|_E\leq a\|\phi-\xi\|_{E_0}\\ +b\frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}+\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E}\\ +c\frac{\|\phi(c)-T\xi\|_E\|\phi-\xi\|_{E_0}}{1+\|\phi(c)-T\phi\|_E+\|\xi(c)-T\phi\|_E+\|\xi(c)-T\xi\|_E}$$

*for all* $\phi,\xi\in E_0$, *where* $a,b,c\geq 0$, $0\leq a+b+c<1$ *and* $c\in I$,

(d) *if* $\{\phi_n\}$ *is a sequence in* $E_0$ *such that* $\phi_n\to\phi$ *as* $n\to\infty$ *and* $\alpha(\phi_n(c),T\phi_n)\geq 1$, *then* $\alpha(\phi(c),T\phi)\geq 1$ *for all* $n\in\mathbb{N}$,

(e) *there exists* $\phi_0\in\mathcal{R}_c$ *such that* $\alpha(\phi_0(c),T\phi_0)\geq 1$.

*Then* $T$ *has a unique PPF dependent fixed point* $\phi^*\in\mathcal{R}_c$. *Moreover, for a fixed* $\phi_0\in\mathcal{R}_c$, *if the sequence* $\{\phi_n\}$ *of iterates of* $T$ *is defined by* $T\phi_{n-1}=\phi_n(c)$ *for all* $n\in\mathbb{N}$, *then* $\{\phi_n\}$ *converges to a PPF dependent fixed point of* $T$ *in* $\mathcal{R}_c$.

Let $\Psi$ be the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

**Lemma 2** (Berinde [3], Rus [28]) *If $\psi \in \Psi$, then the following are satisfied*:
(a) $\psi(t) < t$ *for all* $t > 0$;
(b) $\psi(0) = 0$.

As an example $\psi_1(t) = kt$ for all $t \geq 0$, where $k \in [0,1)$ and $\psi_2(t) = \ln(t+1)$ for all $t \geq 0$, are in $\Psi$.

**Theorem 5** *Let $T : E_0 \to E$ and $\alpha : E \times E \to [0,\infty)$ be two mappings satisfying the following assertions*:
(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference,*
(b) *$T$ is a triangular $\alpha_c$-admissible,*
(c) *suppose that there exists $\psi \in \Psi$ such that*

$$\alpha\big(\phi(c),\xi(c)\big)\|T\phi - T\xi\|_E \leq \psi\big(M\big(\phi(c),\xi(c)\big)\big), \tag{2.29}$$

*where*

$$M\big(\phi(c),\xi(c)\big) = \max\left\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\right\}$$

*for all $\phi, \xi \in E_0$,*
(d) *if $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$, then $\alpha(\phi(c), T\phi) \geq 1$ for all $n \in \mathbb{N}$,*
(e) *there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.*

*Then $T$ has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of $T$ is defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to the PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

*Proof* Suppose that $\phi_0$ is a point in $\mathcal{R}_c \subset E_0$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that $x_1 = \phi_1(c)$. Since $\phi_1 \in \mathcal{R}_c \subset E_0$ and, by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that $x_2 = \phi_2(c)$. Inductively, we can build the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subset E_0$ such that $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$. From Lemma 1, we have $\alpha(\phi_m(c), \phi_n(c)) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$. It follows from the fact that $\mathcal{R}_c$ is algebraically closed with respect to difference that

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \big\|\phi_{n-1}(c) - \phi_n(c)\big\|_E \quad \text{for all } n \in \mathbb{N}.$$

Now, by (2.29) we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_E &= \|T\phi_{n-1}, T\phi_n\|_E \le \alpha\big(\phi_{n-1}(c), \phi_n(c)\big)\|T\phi_{n-1}, T\phi_n\|_E \\ &\le \psi\big(M\big(\phi_{n-1}(c), \phi_n(c)\big)\big), \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} M\big(\phi_{n-1}(c), \phi_n(c)\big) &= \max\bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\quad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}}, \\ &\quad \frac{\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|T\phi_{n-1} - T\phi_n\|_E}\bigg\} \\ &= \max\bigg\{\|\phi_{n-1} - \phi_n\|_{E_0}, \\ &\quad \frac{\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_{n-1} - \phi_n\|_{E_0}}, \\ &\quad \frac{\|\phi_{n-1}(c) - \phi_n\|_E\|\phi_n(c) - \phi_{n+1}\|_E}{1 + \|\phi_n - \phi_{n+1}\|_E}\bigg\} \\ &\le \max\big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \big\|\phi_n(c) - \phi_{n+1}(c)\big\|_E\big\} \\ &= \max\big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\big\}. \end{aligned}$$

If

$$\max\big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\big\} = \|\phi_n - \phi_{n+1}\|_{E_0}$$

from (2.30) we have

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &\le \psi\big(M\big(\phi_{n-1}(c), \phi_n(c)\big)\big) = \psi\big(M\big(\|\phi_n - \phi_{n+1}\|_{E_0}\big)\big) \\ &< \|\phi_n - \phi_{n+1}\|_{E_0}, \end{aligned} \tag{2.31}$$

which is a contradiction. So,

$$\max\big\{\|\phi_{n-1} - \phi_n\|_{E_0}, \|\phi_n - \phi_{n+1}\|_{E_0}\big\} = \|\phi_{n-1} - \phi_n\|_{E_0}.$$

By (2.30), we conclude

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &\le \psi\big(M\big(\phi_{n-1}(c), \phi_n(c)\big)\big) = \psi\big(M\big(\|\phi_{n-1} - \phi_n\|_{E_0}\big)\big) \\ &< \|\phi_{n-1} - \phi_n\|_{E_0}. \end{aligned} \tag{2.32}$$

By induction, we get

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le \psi^n\big(\|\phi_0 - \phi_1\|_{E_0}\big)$$

for all $n \in \mathbb{N}$. As $\psi \in \Psi$, we conclude

$$\lim_{n\to+\infty} \|\phi_n - \phi_{n+1}\|_{E_0} = 0. \tag{2.33}$$

We prove that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. Assume that $\{\phi_n\}$ is not a Cauchy sequence, then

$$\lim_{m,n\to+\infty} \|\phi_m - \phi_n\|_{E_0} > 0. \tag{2.34}$$

By (2.29), we have

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\le \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{m+1}\|_{E_0} + \|\phi_{m+1} - \phi_m\|_{E_0} \\
&\le \|\phi_n - \phi_{n+1}\|_{E_0} + \alpha\big(\phi_n(c), \phi_m(c)\big)\|T\phi_n - T\phi_m\|_{E_0} \\
&\quad + \|\phi_{m+1} - \phi_m\|_{E_0} \\
&\le \|\phi_n - \phi_{n+1}\|_{E_0} + \psi\big(M\big(\phi_n(c), \phi_m(c)\big)\big) + \|\phi_{m+1} - \phi_m\|_{E_0}.
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.33) we have

$$\lim_{m,n\to\infty} \|\phi_n - \phi_m\|_{E_0} \le \lim_{m,n\to\infty} \psi\big(M\big(\phi_n(c), \phi_m(c)\big)\big), \tag{2.35}$$

where

$$\begin{aligned}
M\big(\phi_n(c), \phi_m(c)\big) &= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\qquad \frac{\|\phi_n(c) - T\phi_n\|_E \|\phi_m(c) - T\phi_m\|_E}{1 + \|T\phi_n - T\phi_m\|_E}\bigg\} \\
&= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \\
&\qquad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\qquad \frac{\|\phi_n(c) - \phi_{n+1}(c)\|_E \|\phi_m(c) - \phi_{m+1}(c)\|_E}{1 + \|\phi_{n+1}(c) - \phi_{m+1}(c)\|_E}\bigg\} \\
&= \max\bigg\{\|\phi_n - \phi_m\|_{E_0}, \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_n - \phi_m\|_{E_0}}, \\
&\qquad \frac{\|\phi_n - \phi_{n+1}\|_{E_0} \|\phi_m - \phi_{m+1}\|_{E_0}}{1 + \|\phi_{n+1} - \phi_{m+1}\|_{E_0}}\bigg\}.
\end{aligned} \tag{2.36}$$

Letting $m, n \to \infty$ in the above inequality and applying (2.33), we get

$$\lim_{m,n\to+\infty} M\big(\phi_n(c), \phi_m(c)\big) = \lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0}. \tag{2.37}$$

So, by (2.35) and (2.37), we have

$$\limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} \le \limsup_{m,n\to+\infty} \psi\big(\|\phi_n - \phi_m\|_{E_0}\big) < \limsup_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0},$$

which is a contradiction. Consequently,

$$\lim_{m,n\to+\infty} \|\phi_n - \phi_m\|_{E_0} = 0.$$

Hence, $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. Completeness of $E_0$ shows that $\{\phi_n\}$ converges to a point $\phi^* \in E_0$, that is, $\phi_n \to \phi^*$ as $n \to \infty$. Since $\mathcal{R}_c$ is topologically closed, we deduce, $\phi^* \in \mathcal{R}_c$. Now, by (2.29), we get

$$\begin{aligned}
&\|T\phi^* - \phi^*(c)\|_E \\
&\quad \le \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E \\
&\quad = \|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\quad \le \alpha\big(\phi^*(c), \phi_{n-1}(c)\big)\|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\quad \le \psi\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get

$$\|T\phi^* - \phi^*(c)\|_E \le \lim_{n\to\infty} \psi\big(M\big(\phi^*(c), \phi_{n-1}(c)\big)\big). \tag{2.38}$$

But

$$\begin{aligned}
\lim_{n\to\infty} M\big(\phi^*(c), \phi_{n-1}(c)\big) &= \lim_{n\to\infty} \max\bigg\{ \|\phi^* - \phi_{n-1}\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|T\phi^* - T\phi_{n-1}\|_E} \bigg\} \\
&= \lim_{n\to\infty} \max\bigg\{ \|\phi^* - \phi_{n-1}\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|T\phi^* - \phi_n(c)\|_E} \bigg\} \\
&= \lim_{n\to\infty} \max\bigg\{ \|\phi^* - \phi_{n-1}\|_{E_0}, \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|\phi^* - \phi_{n-1}\|_{E_0}} \\
&\qquad \frac{\|\phi^*(c) - T\phi^*\|_E \|\phi_{n-1} - \phi_n\|_{E_0}}{1 + \|T\phi^* - \phi_n(c)\|_E} \bigg\} = 0.
\end{aligned} \tag{2.39}$$

Therefore, from (2.38) and (2.39), we deduce

$$\|T\phi^* - \phi^*(c)\|_E = 0,$$

that is,

$$T\phi^* = \phi^*(c),$$

which implies that $\phi^*$ is a PPF dependent fixed point of $T$ in $\mathcal{R}_c$. Suppose that $\phi^*$ and $\varphi^*$ are two PPF dependent fixed points of $T$ in $\mathcal{R}_c$ such that $\phi^* \neq \varphi^*$. So,

$$\begin{aligned} \|\phi^* - \varphi^*\|_{E_0} &= \|\phi^*(c) - \varphi^*(c)\|_E = \|T\phi^* - T\varphi^*\|_E \\ &\leq \alpha\big(\phi^*(c), \varphi^*(c)\big)\|T\phi^* - T\varphi^*\|_E \\ &\leq \psi\big(M\big(\phi^*(c), \varphi^*(c)\big)\big), \end{aligned}$$

where

$$\begin{aligned} M\big(\phi^*(c), \varphi^*(c)\big) &= \max\bigg\{\|\phi^* - \varphi^*\|_{E_0}, \frac{\|\phi^*(c) - T\phi^*\|_E\|\varphi^*(c) - T\varphi^*\|_E}{1 + \|\phi^* - \varphi^*\|_{E_0}}, \\ &\quad \frac{\|\phi^*(c) - T\phi^*\|_E\|\varphi^*(c) - T\varphi^*\|_E}{1 + \|T\phi^* - T\varphi^*\|_E}\bigg\} \\ &= \|\phi^* - \varphi^*\|_{E_0}. \end{aligned}$$

Therefore,

$$\|\phi^* - \varphi^*\|_{E_0} \leq \psi\big(\|\phi^* - \varphi^*\|_{E_0}\big) < \|\phi^* - \varphi^*\|_{E_0},$$

which is a contradiction. Hence, $\phi^* = \varphi^*$. Then $T$ has a unique PPF dependent fixed point in $\mathcal{R}_c$. □

Now, in Theorem 5 we take $\psi(t) = rt$, where $0 \leq r < 1$ and we have the following corollary.

**Corollary 3** *Let $T : E_0 \to E$, $\alpha : E \times E \to [0, \infty)$ be two mappings satisfying the following assertions*:

(a) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference*,

(b) *$T$ is a triangular $\alpha_c$-admissible*,

(c)

$$\alpha\big(\phi(c), \xi(c)\big)\|T\phi - T\xi\| \leq rM\big(\phi(c), \xi(c)\big), \tag{2.40}$$

*where*

$$\begin{aligned} M\big(\phi(c), \xi(c)\big) = \max\bigg\{&\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \\ &\frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\bigg\} \end{aligned}$$

*for all $\phi, \xi \in E_0$*,

(d) *if $\phi_n$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and $\alpha(\phi_n(c), T\phi_n) \geq 1$, then $\alpha(\phi(c), T\phi) \geq 1$ for all $n \in \mathbb{N}$*,

(e) *there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$.*

*Then $T$ has a unique PPF dependent fixed point $\phi^* \in \mathcal{R}_c$. Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if the sequence $\{\phi_n\}$ of iterates of $T$ is defined by $T\phi_{n-1} = \phi_n(c)$ for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of $T$ in $\mathcal{R}_c$.*

## 3 Some results in Banach spaces endowed with a graph

Consistent with Jachymski [13], let $(X,d)$ be a metric space and $\Delta$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We suppose that $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G),E(G))$. Moreover, we may treat $G$ as a weighted graph (see [14, p.309]) by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$, then a path in $G$ from $x$ to $y$ of length $N$ ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$, and $(x_{i-1},x_i) \in E(G)$ for $i = 1,\dots,N$. Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if $(X,d)$ is endowed with a graph. The first result in this direction was given by Jachymski [13].

**Definition 10** ([13]) Let $(X,d)$ be a metric space endowed with a graph $G$. We say that a self-mapping $T : X \to X$ is a Banach $G$-contraction or simply a $G$-contraction if $T$ preserves the edges of $G$, that is,

$$(x,y) \in E(G) \quad \Longrightarrow \quad (Tx,Ty) \in E(G) \quad \text{for all } x,y \in X$$

and $T$ decreases the weights of the edges of $G$ in the following way:

$$\exists\alpha \in (0,1) \text{ such that for all } x,y \in X, \quad (x,y) \in E(G) \quad \Longrightarrow \quad d(Tx,Ty) \le \alpha d(x,y).$$

**Theorem 6** *Let $T : E_0 \to E$ and $E$ endowed with a graph $G$. Suppose that the following assertions hold*:

(i) *there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference*,

(ii) *if $(\phi(c),\xi(c)) \in E(G)$, then $(T\phi,T\xi) \in E(G)$*,

(iii) *assume that*

$$\|T\phi - T\xi\|_E \le \beta\big(M\big(\phi(c),\xi(c)\big)\big)M\big(\phi(c),\xi(c)\big),$$

*where*

$$M\big(\phi(c),\xi(c)\big) = \max\biggl\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|\phi - \xi\|_{E_0}}, \frac{\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E}\biggr\}$$

*or*

$$M\big(\phi(c),\xi(c)\big) = \max\biggl\{\|\phi - \xi\|_{E_0}, \frac{\|\phi(c) - T\phi\|_E\|\phi(c) - T\xi\|_E + \|\xi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi(c) - T\phi\|_E + \|\xi(c) - T\xi\|_E}, \frac{\|\phi(c) - T\phi\|_E\|\phi(c) - T\xi\|_E + \|\xi(c) - T\xi\|_E\|\xi(c) - T\phi\|_E}{1 + \|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E}\biggr\},$$

*or*

$$M\big(\phi(c),\xi(c)\big) = \max\left\{\|\phi-\xi\|_{E_0}, \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}+\|\phi(c)-T\xi\|_E+\|\xi(c)-T\phi\|_E}, \frac{\|\phi(c)-T\xi\|_E\|\phi-\xi\|_{E_0}}{1+\|\phi(c)-T\phi\|_E+\|\xi(c)-T\phi\|_E+\|\xi(c)-T\xi\|_E}\right\},$$

(iv) *if* $\{\phi_n\}$ *is a sequence in* $E_0$ *such that* $\phi_n \to \phi$ *as* $n \to \infty$ *and* $(\phi_n(c), T\phi_n(c)) \in E(G)$, *then* $(\phi(c), T\phi(c)) \in E(G)$ *for all* $n \in \mathbb{N}$,

(v) *there exists* $\phi_0 \in \mathcal{R}_c$ *such that* $(\phi_0(c), T\phi_0) \in E(G)$.

*Then* $T$ *has a unique PPF dependent fixed point* $\phi^*$ *in* $\mathcal{R}_c$.

*Proof* Define $\alpha : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Now, we show that $T$ is an $\alpha_c$-admissible mapping. Suppose that $\alpha(\phi(c),\psi(c)) \geq 1$. Therefore, we have $(\phi(c),\psi(c)) \in E(G)$. From (ii), we get $(T\phi, T\psi) \in E(G)$. So, $\alpha(T\phi, T\psi) \geq 1$ and $T$ is an $\alpha_c$-admissible mapping. By the definition of $\alpha$ and from (iii), we have

$$\alpha\big(\phi(c),T\phi\big)\alpha\big(\xi(c),T\xi\big)\|T\phi-T\xi\|_E \leq \beta\big(M\big(\phi(c),\xi(c)\big)\big)M\big(\phi(c),\xi(c)\big).$$

From (v), there exists $\phi_0 \in \mathcal{R}_c$ such that $\alpha(\phi_0(c), T\phi_0) \geq 1$. Let $\{\phi_n\}$ is a sequence in $E_0$ such that $\phi_n \to \phi$ as $n \to \infty$ and $(\phi_n(c), T\phi_n) \in E(G)$ for all $n \in \mathbb{N}$, then $\alpha(\phi_n(c), T\phi_n) \geq 1$. Thus, from (iv) we get, $(\phi(c), T\phi(c)) \in E(G)$. That is, $\alpha(\phi(c), T\phi(c)) \geq 1$. Therefore all conditions of Theorems 2, 3, 4 hold true and $T$ has a PPF dependent fixed point. □

**Theorem 7** *Let* $T : E_0 \to E$ *and* $E$ *be endowed with a graph* $G$ *and for all* $(\phi(c),\xi(c)) \in E(G)$ *and* $(\xi(c),\psi(c)) \in E(G)$, *we have* $(\phi(c),\psi(c)) \in E(G)$. *Suppose that the following assertions hold*:

(i) *there exists* $c \in I$ *such that* $\mathcal{R}_c$ *is topologically closed and algebraically closed with respect to difference*,

(ii) *if* $(\phi(c),\xi(c)) \in E(G)$, *then* $(T\phi, T\xi) \in E(G)$,

(iii) *assume that for* $\psi \in \Psi$, *we have*

$$\|T\phi - T\xi\| \leq \psi\big(M\big(\phi(c),\xi(c)\big)\big), \tag{3.1}$$

*where*

$$M\big(\phi(c),\xi(c)\big) = \max\left\{\|\phi-\xi\|_{E_0}, \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|\phi-\xi\|_{E_0}}, \frac{\|\phi(c)-T\phi\|_E\|\xi(c)-T\xi\|_E}{1+\|T\phi-T\xi\|_E}\right\}$$

*for all* $\phi, \xi \in E_0$,

(iv) *if* $\{\phi_n\}$ *is a sequence in* $E_0$ *such that* $\phi_n \to \phi$ *as* $n \to \infty$ *and* $(\phi_n(c), T\phi_n(c)) \in E(G)$, *then* $(\phi(c), T\phi(c)) \in E(G)$, *for all* $n \in \mathbb{N}$,

(v) *there exists* $\phi_0 \in \mathcal{R}_c$ *such that* $(\phi_0(c), T\phi_0) \in E(G)$.

*Then* $T$ *has a unique PPF dependent fixed point* $\phi^*$ *in* $\mathcal{R}_c$.

## 4 Application

In this section, we present an application of our Theorem 5 to establish PPF dependent solution of a nonlinear integral equation. Let $\Omega_0 = C(J, \mathbb{R})$ where $J := [j, 0]$ with $j \in \mathbb{R}_-$. $\Omega_0$ is a Banach space with the following norm:

$$\|\phi\|_{\Omega_0} = \sup_{t \in J} |\phi(t)|.$$

For $\zeta \in C(I, \mathbb{R})$ consider the following nonlinear integral problem:

$$\phi(t) = \zeta(0) + \int_0^T G(T, s) f(s, \phi_s)\, ds, \tag{4.1}$$

where $t \in I = [0, T]$, $\phi_t(a) = \phi(t + a)$ with $a \in J$ and $f \in C(I \times C(J, \mathbb{R}), \mathbb{R})$ and $G \in C(I \times I, \mathbb{R}_+)$.

Let

$$\hat{E} = \{\hat{\phi} = (\phi_t)_{t \in I} : \phi_t \in \Omega_0, \phi \in C(I, \mathbb{R})\}$$

and

$$\|\hat{\phi}\|_{\hat{E}} := \sup_{t \in I} \|\phi_t\|_{\Omega_0}.$$

This means that

$$\hat{\phi} \in C(J, \mathbb{R}).$$

In [20], it is shown that $\hat{E}$ is complete. Next, we define the function $S : \hat{E} \to \mathbb{R}$ by

$$S\hat{\phi} = S(\phi_t)_{t \in I} = \zeta(0) + \int_0^T G(T, s) f(s, \phi_s)\, ds.$$

We will consider (4.1) under the following assumptions:

(i) $(\sup_{t \in I} \int_0^t G(t, s)\, ds) \leq 1$,

(ii) there exist $\psi \in \Psi$ and $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for all $t \in I$, $\hat{\phi}, \hat{\varsigma} \in \hat{E}$ with $\theta(\hat{\phi}(t), \hat{\varsigma}(t)) \geq 0$ we have

$$|f(t, \phi) - f(t, \varsigma)| \leq \psi(|\phi(0) - \varsigma(0)|),$$

(iii) if $\theta(\hat{\phi}(t), \hat{\psi}(t)) \geq 0$, then $\theta(S\hat{\phi}, S\hat{\psi}) \geq 0$,

(iv) if $\theta(\hat{\phi}(t), \hat{\mu}(t)) \geq 0$ and $\theta(\hat{\mu}(t), \hat{\psi}(t)) \geq 0$, then $\theta(\hat{\phi}(t), \hat{\psi}(t)) \geq 0$,

(v) there exists $\phi_0 \in \hat{E}$ such that $\theta(\phi_0(t), S\phi_0) \geq 0$,

(vi) if $\{\hat{\phi}_n\}$ is a sequence in $\hat{E}$ such that $\hat{\phi}_n \to \hat{\phi}$ as $n \to \infty$ and $\theta(\hat{\phi}_n(t), S\hat{\phi}_n) \geq 0$ for all $n$, then $\theta(\hat{\phi}(t), S\hat{\phi}) \geq 0$.

**Theorem 8** *Under assumptions* (i)-(vi), *the integral equation* (4.1) *has a solution on* $J \cup I$.

*Proof* For $\hat{\phi}, \hat{\varsigma} \in \hat{E}$ with $\theta(\hat{\phi}(t), \hat{\varsigma}(t)) \geq 0$ from (ii), we have

$$\begin{aligned}
|S\hat{\phi} - S\hat{\varsigma}| &= \left| \int_0^T G(T,s) f(s, \phi_s)\, ds - \int_0^T G(T,s) f(s, \varsigma_s)\, ds \right| \\
&= \left| \int_0^T G(T,s)\bigl(f(s,\phi_s) - f(s,\varsigma_s)\bigr)\, ds \right| \\
&\leq \int_0^T \left| G(T,s)\bigl(f(s,\phi_s) - f(s,\varsigma_s)\bigr)\, ds \right| \\
&\leq \int_0^T G(T,s) \left| \bigl(f(s,\phi_s) - f(s,\varsigma_s)\bigr) \right| ds \\
&\leq \int_0^T G(T,s) \psi\bigl(\left|\phi_s(0) - \varsigma_s(0)\right|\bigr)\, ds \\
&= \int_0^T G(T,s) \psi\bigl(\left|\phi(s) - \varsigma(s)\right|\bigr)\, ds \\
&\leq \int_0^T G(T,s) \psi\bigl(\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}\bigr)\, ds \\
&= \psi\bigl(\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}\bigr) \left( \int_0^T G(T,s)\, ds \right) \\
&\leq \psi\bigl(\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}\bigr) \left[ \sup_{t \in I} \int_0^t G(T,s)\, ds \right] \\
&\leq \psi\bigl(\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}\bigr).
\end{aligned}$$

Now, we define $\alpha : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$\alpha\bigl(\hat{\phi}(t), \hat{\psi}(t)\bigr) = \begin{cases} 1 & \text{if } \theta(\hat{\phi}(t), \hat{\psi}(t)) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for all $\hat{\phi}, \hat{\varsigma} \in \hat{E}$, we have

$$\alpha\bigl(\hat{\phi}(t), \hat{\varsigma}(t)\bigr)|S\hat{\phi} - S\hat{\varsigma}| \leq \psi\bigl(\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}\bigr) \leq \psi\bigl(M\bigl(\hat{\phi}(t), \hat{\varsigma}(t)\bigr)\bigr),$$

where

$$\begin{aligned}
M\bigl(\hat{\phi}(t), \hat{\varsigma}(t)\bigr) = \max\biggl\{ &\|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}, \frac{\|\hat{\phi}(t) - S\hat{\phi}\|_{\mathbb{R}} \|\hat{\varsigma}(t) - S\hat{\varsigma}\|_{\mathbb{R}}}{1 + \|\hat{\phi} - \hat{\varsigma}\|_{\hat{E}}}, \\
&\frac{\|\hat{\phi}(t) - S\hat{\phi}\|_{\mathbb{R}} \|\hat{\varsigma}(t) - S\hat{\varsigma}\|_{\mathbb{R}}}{1 + \|S\hat{\phi} - S\hat{\varsigma}\|_{\mathbb{R}}} \biggr\}.
\end{aligned}$$

From the conditions (iii) and (iv), we deduce that $S$ is a triangular $\alpha_c$-admissible mapping. By the condition (vi), we conclude that if a sequence $\{\hat{\phi}_n\}$ is such that $\hat{\phi}_n \to \hat{\phi}$ as $n \to$

$\infty$ and $\alpha(\hat{\phi}_n(t), S\hat{\phi}_n) \geq 1$ for all $n$, then $\alpha(\hat{\phi}(t), S\hat{\phi}) \geq 1$ and by (v), there is $\phi_0 \in \hat{E}$ such that $\alpha(\phi_0(t), S\phi_0) \geq 1$. The Razumikhin $\mathcal{R}_0$ is $C(I, \mathbb{R})$, which is topologically closed and algebraically closed with respect to difference. Hence, the hypotheses of Theorem 5 are satisfied with $c = 0$. So, there exists a fixed point $\hat{\phi}^* \in \hat{E}$ such that $S\hat{\phi}^* = \hat{\phi}^*(0)$. This means that

$$\zeta(0) + \int_0^T G(T,s) f\big(s, \phi_s^*\big)\, ds = \big(\phi_t^*(0)\big)_{t\in I} = \big(\phi^*(t)\big)_{t\in I}. \qquad \square$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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