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Convergence theorems for equilibrium problem and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense

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Abstract

In this paper, we introduce an iterative process which converges strongly to a common element of the set of fixed points of an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and the solution set of generalized equilibrium problem in Banach spaces. Our theorems improve, generalize, and extend several results recently announced.

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1 Introduction

Let E be a real Banach space with the dual space E^*. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonlinear mapping. We denote by $F(T)$ the set of fixed points of T.

A mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. In uniformly convex Banach spaces, they proved that if C is nonempty, bounded, closed, and convex, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

A mapping T is said to be asymptotically nonexpansive in the intermediate sense (see [2]) if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.1}$$

If $F(T) \ne \phi$ and (1.1) holds for all $x \in K$, $y \in F(T)$, then T is called asymptotically quasi-nonexpansive in the intermediate sense. It is well known that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self-mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point (see [3]). It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Iterative approximation of a fixed point for asymptotically nonexpansive mappings in Hilbert or Banach spaces has been studied extensively by many authors (see [4–6] and the references therein).

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E,$$

is studied by Alber [7]. It follows from the definition of ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|x\| + \|y\|\bigr)^2, \quad \forall x,y \in E. \tag{1.2}$$

Remark 1.1

(i) If E is a reflexive, strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.
(ii) If E is a real Hilbert space, then $\phi(x,y) = \|x - y\|^2$.

Let E be reflexive, strictly convex and smooth Banach space. The generalized projection mapping, introduced by Alber [7], is a mapping $\Pi_C : E \to C$ that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$, that is, $\Pi_C x = \bar{x}$, where is $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y,x).$$

A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\tilde{F}(T)$. A mapping T is called relatively nonexpansive (see [8]) if $\tilde{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

Recently, Matsushita and Takahashi [9] proved strong convergence theorems for approximation of fixed points of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach space. More precisely, they proved the following theorem.

Theorem 1.1 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z,y_n) \le \phi(z,x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0,1,2,\dots, \end{cases} \tag{1.3}$$

where J is the normalized duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

In [10], Hao introduced the following iterative scheme for approximating a fixed point of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense in a reflexive, strictly convex and smooth Banach space E: $x_0 \in E$, $C_1 = C$, $x_1 = \Pi_{C_1}x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \quad n = 1, 2, \ldots, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$.

Motivated and inspired by the works mentioned above, in this paper, we introduce a new iterative scheme of the generalized f-projection operator for finding a common element of the set of fixed points of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and the solution set of generalized equilibrium problem in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property.

2 Preliminaries

Let E be a real Banach space with the norm $\|\cdot\|$ and let E^* be the dual space of E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

By the Hahn-Banach theorem, $J(x)$ is nonempty.

A Banach space E is called strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ with $x \neq y$, where $U = \{x \in E : \|x\| = 1\}$ is the unit sphere of E. A Banach space E is called smooth if the limit

$$\lim_{t\to\infty} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also called uniformly smooth if the limit exists uniformly for all $x, y \in U$. In this paper, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Remark 2.1 The basic properties of a Banach space E related to the normalized duality mapping J are as follows (see [11]):

(1) If E is a smooth Banach space, then J is single-valued and semicontinuous;
(2) If E is a uniformly smooth Banach space, then J is uniformly norm-to-norm continuous on each bounded subset of E;
(3) If E is a uniformly smooth Banach space, then E is smooth and reflexive;
(4) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous;
(5) E is a uniformly smooth Banach space if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

Definition 2.1 A mapping $T : C \to C$ is said to be

(1) quasi-ϕ-nonexpansive if $F(T) \neq \phi$ and

$$\phi(p, Tx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(T)$;

(2) asymptotically quasi-ϕ-nonexpansive in the intermediate sense if $F(T) \neq \phi$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p, x)\bigr) \leq 0$$

put

$$\xi_n = \max\Bigl\{0, \sup_{p\in F(T), x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p, x)\bigr)\Bigr\}.$$

Remark 2.2 From the definition of asymptotically quasi-ϕ-nonexpansiveness in the intermediate sense, it is obvious that $\xi_n \to 0$ as $n \to \infty$ and

$$\phi\bigl(p, T^n x\bigr) \leq \phi(p, x) + \xi_n, \quad \forall p \in F(T), x \in C.$$

Recall that T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty} \bigl\{ \bigl\| T^{n+1} x - T^n x \bigr\| : x \in K \bigr\} = 0.$$

Definition 2.2 A mapping $T : C \to C$ is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $Tx = y$.

Following Alber [7], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \Bigl\{ u \in C : \phi(u, x) = \min_{y\in C} \phi(y, x) \Bigr\}, \quad \forall x \in E.$$

In 2006, Wu and Huang [12] introduced a generalized f-projection operator in a Banach space, which extends the definition of the generalized projection Π_C. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(y, \overline{w}) = \|y\|^2 - 2\langle y, \overline{w}\rangle + \|\overline{w}\|^2 + 2\rho f(y)$$

for all $(y, \overline{w}) \in C \times E^*$, where ρ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From the definition of G, it is easy to see the following properties:

(i) $G(y, \overline{w})$ is convex and continuous with respect to $\overline{w}$ when y is fixed;

(ii) $G(y, \overline{w})$ is convex and lower semicontinuous with respect to y when $\overline{w}$ is fixed.

Definition 2.3 ([13]) Let E be a real smooth Banach space and let C be a nonempty closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \Bigl\{ u \in C : G(u, Jx) = \inf_{y\in C} G(y, Jx), \forall x \in E \Bigr\}.$$

Lemma 2.1 ([14]) *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha$$

for all $x \in E$.

Lemma 2.2 ([13]) *Let E be a reflexive smooth Banach space and let C be a nonempty closed convex subset of E. The following statements hold*:

(1) *$\Pi_C^f x$ is a nonempty closed convex subset of C for all $x \in E$;*

(2) *For all $x \in F$, $\overline{x} \in \Pi_C^f x$ if and only if*

$$\langle \overline{x} - y, Jx - J\overline{x} \rangle + \rho f(y) - \rho f(\overline{x}) \geq 0$$

for all $y \in C$;

(3) *If E is strictly convex, then Π_C^f is a single-valued mapping.*

Let θ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem is to find $\overline{x} \in C$ such that

$$\theta(\overline{x}, y) \geq 0 \tag{2.1}$$

for all $y \in C$. The set of solutions of (2.1) is denoted by $EP(\theta)$.

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone; *i.e.*, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta\bigl(tz + (1 - t)x, y\bigr) \leq \theta(x, y);$$

(A4) for all $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.3 ([15]) *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E and let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *For all $r > 0$ and $x \in E$, define a mapping $T_r^\theta : E \to C$ as follows*:

$$T_r^\theta x = \left\{ z \in C : \theta(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following conclusions hold:

(1) *T_r^θ is single-valued;*

(2) *T_r^θ is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\bigl\langle T_r^\theta x - T_r^\theta y, JT_r^\theta x - JT_r^\theta y \bigr\rangle \leq \bigl\langle T_r^\theta x - T_r^\theta y, Jx - Jy \bigr\rangle;$$

(3) *$F(T_r^\theta) = EP(\theta)$ is closed and convex;*

(4) T_r^θ *is quasi-ϕ-nonexpansive*;
(5) $\phi(q, T_r^\theta x) + \phi(T_r^\theta x, x) \le \phi(q, x)$, $\forall q \in F(T_r^\theta)$.

Lemma 2.4 ([10]) *Let E be a reflexive, strictly convex and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.5 ([13]) *Let E be a real reflexive smooth Banach space and let C be a nonempty closed and convex subset of E. Then, for any $x \in E$ and $\overline{x} \in \Pi_C^f x$,*

$$\phi(y, \overline{x}) + G(\overline{x}, Jx) \le G(y, Jx)$$

for all $y \in C$.

3 Main results

Theorem 3.1 *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is asymptotically regular on C, $\mathcal{F} = F(T) \cap EP(\theta)$ is nonempty, and $F(T)$ is bounded. Let $f : E \to \mathbb{R}^+$ be a convex and lower semicontinuous function with $C \subset \operatorname{int}(D(f))$ and $f(0) = 0$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ and $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0, 1)$ satisfying the following conditions*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n), \\ u_n \in C \text{ such that } \theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le \alpha_n G(z, Jx_1) + (1 - \alpha_n) G(z, Jx_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$ and $\Pi_{C_{n+1}}^f$ is the generalized f-projection operator. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}^f x_1$.

Proof It follows from Lemma 2.3 and Lemma 2.4 that $\mathcal{F}$ is a closed convex subset of C, so that $\Pi_{\mathcal{F}}^f x$ is well defined for any $x \in C$.

We split the proof into six steps.

Step 1. We first show that C_n is nonempty, closed, and convex for all $n \ge 1$.

In fact, it is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 2$. For $z_1, z_2 \in C_{n+1}$, we see that $z_1, z_2 \in C_n$. It follows that $z = tz_1 + (1 -$

$t)z_2 \in C_n$, where $t \in (0,1)$. Notice that

$$G(z_1, Ju_n) \le \alpha_n G(z_1, Jx_1) + (1-\alpha_n)G(z_1, Jx_n) + \xi_n,$$

and

$$G(z_2, Ju_n) \le \alpha_n G(z_2, Jx_1) + (1-\alpha_n)G(z_2, Jx_n) + \xi_n.$$

The above inequalities are equivalent to

$$\begin{aligned} &2\alpha_n\langle z_1, Jx_1\rangle + 2(1-\alpha_n)\langle z_1, Jx_n\rangle - 2\langle z_1, Ju_n\rangle \\ &\quad \le \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|u_n\|^2 + \xi_n \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &2\alpha_n\langle z_2, Jx_1\rangle + 2(1-\alpha_n)\langle z_2, Jx_n\rangle - 2\langle z_2, Ju_n\rangle \\ &\quad \le \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|u_n\|^2 + \xi_n. \end{aligned} \tag{3.3}$$

Multiplying t and $1-t$ on both sides of (3.2) and (3.3), respectively, we obtain

$$\begin{aligned} &2\alpha_n\langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Ju_n\rangle \\ &\quad \le \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|u_n\|^2 + \xi_n. \end{aligned}$$

Hence we have

$$G(z, Ju_n) \le \alpha_n G(z, Jx_1) + (1-\alpha_n)G(z, Jx_n) + \xi_n.$$

This implies that C_{n+1} is closed and convex for all $n \ge 1$. This shows that $\Pi^f_{C_{n+1}}x_1$ is well defined.

Step 2. We show that $\mathcal{F} \subset C_n$ for all $n \ge 1$.

For $n = 1$, we have $\mathcal{F} \subset C_1 = C$. Now, assume that $\mathcal{F} \subset C_n$ for some $n \ge 2$. Let $q \in \mathcal{F}$. Since T is asymptotically quasi-ϕ-nonexpansive with intermediate sense, we have from Remark 2.2 and Lemma 2.3 that

$$\begin{aligned} G(q, Ju_n) &= G\big(q, JT^{\theta}_{r_n}y_n\big) \\ &= \phi\big(q, T^{\theta}_{r_n}y_n\big) + 2\rho f(q) \\ &\le \phi(q, y_n) + 2\rho f(q) \\ &= G(q, Jy_n) \\ &= G\big(q, \alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n\big) \\ &= \|q\|^2 - 2\alpha_n\langle q, Jx_1\rangle - 2\beta_n\big\langle q, JT^n x_n\big\rangle - 2\gamma_n\langle q, Jx_n\rangle \\ &\quad + \big\|\alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n\big\|^2 + 2\rho f(q) \\ &\le \|q\|^2 - 2\alpha_n\langle q, Jx_1\rangle - 2\beta_n\big\langle q, JT^n x_n\big\rangle - 2\gamma_n\langle q, Jx_n\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n \|Jx_1\|^2 + \beta_n \|JT^n x_n\|^2 + \gamma_n \|Jx_n\|^2 + 2\rho f(q) \\
&= \alpha_n G(q, Jx_1) + \beta_n G(q, JT^n x_n) + \gamma_n G(q, Jx_n) \\
&= \alpha_n G(q, Jx_1) + \beta_n \{\phi(q, T^n x_n) + 2\rho f(q)\} + \gamma_n G(q, Jx_n) \\
&\le \alpha_n G(q, Jx_1) + \beta_n \{\phi(q, x_n) + \xi_n + 2\rho f(q)\} + \gamma_n G(q, Jx_n) \\
&\le \alpha_n G(q, Jx_1) + \beta_n G(q, Jx_n) + \gamma_n G(q, Jx_n) + \xi_n \\
&= \alpha_n G(q, Jx_1) + (1 - \alpha_n) G(q, Jx_n) + \xi_n,
\end{aligned}
$$

which shows that $q \in C_{n+1}$. This implies that $\mathcal{F} \subset C_{n+1}$ and so $\mathcal{F} \subset C_n$ for all $n \ge 1$.

Step 3. We prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} G(x_n, Jx_1)$ exists.

By Lemma 2.1, we have the result that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$
f(x) \ge \langle x, x^* \rangle + \alpha.
$$

Since $x_n \in C_n \subset E$, it follows that

$$
\begin{aligned}
G(x_n, Jx_1) &= \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho f(x_n) \\
&\ge \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho \langle x_n, x^*\rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^*\rangle + \|x_1\|^2 + 2\rho\alpha \\
&\ge \|x_n\|^2 - 2\|x_n\| \|Jx_1 - \rho x^*\| + \|x_1\|^2 + 2\rho\alpha \\
&= (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha.
\end{aligned}
$$

For all $q \in \mathcal{F}$ and $x_n = \Pi_{C_n}^{f_1} x_1$, we have

$$
\begin{aligned}
G(q, Jx_1) &\ge G(x_n, Jx_1) \\
&\ge (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\| - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha.
\end{aligned}
$$

This implies that the sequence $\{x_n\}$ is bounded and so is $\{G(x_n, Jx_1)\}$. From (1.2) and Lemma 2.5, we obtain

$$
0 \le (\|x_{n+1}\| - \|x_n\|)^2 \le \phi(x_{n+1}, x_n) \le G(x_{n+1}, Jx_1) - G(x_n, Jx_1). \tag{3.4}
$$

This shows that $\{G(x_n, Jx_1)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} G(x_n, Jx_1)$ exists.

Step 4. Next, we prove that $x_n \to \bar{x}$, $y_n \to \bar{x}$, and $u_n \to \bar{x}$ as $n \to \infty$, where $\bar{x}$ is some point in C.

By (3.4), we obtain

$$
\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}
$$

Since $\{x_n\}$ is bounded and E is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$ as $n \to \infty$. Since C_n is closed and convex, we find that $\bar{x} \in C_n$. From the weak lower semicontinuity of the norm and $x_n = \Pi^f_{C_n} x_1$, we obtain

$$\begin{aligned} G(\overline{x}, Jx_1) &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx_1\rangle + \|x_1\|^2 + 2\rho f(\overline{x}) \\ &\le \liminf_{n\to\infty} \bigl\{ \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho f(x_n) \bigr\} \\ &= \liminf_{n\to\infty} G(x_n, Jx_1) \\ &\le \limsup_{n\to\infty} G(x_n, Jx_1) \\ &\le G(\overline{x}, Jx_1), \end{aligned}$$

which implies that $\lim_{n\to\infty} G(x_n, Jx_1) = G(\overline{x}, Jx_1)$. From Lemma 2.5, we obtain

$$\begin{aligned} 0 &\le \bigl(\|\overline{x}\| - \|x_n\|\bigr)^2 \\ &\le \phi(\overline{x}, x_n) \\ &\le G(\overline{x}, Jx_1) - G(x_n, Jx_1). \end{aligned}$$

Hence we have $\lim_{n\to\infty} \|x_n\| = \|\overline{x}\|$. In view of the Kadec-Klee property of E, we find that

$$\lim_{n\to\infty} x_n = \overline{x}. \tag{3.6}$$

And we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Since J is uniformly norm-to-norm continuous, it follows that

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0.$$

From $x_{n+1} = \Pi^f_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ and (3.1), we have

$$G(x_{n+1}, Ju_n) \le \alpha_n G(x_{n+1}, Jx_1) + (1-\alpha_n) G(x_{n+1}, Jx_n) + \xi_n.$$

This is equivalent to the following:

$$\phi(x_{n+1}, u_n) \le \alpha_n \phi(x_{n+1}, x_1) + (1-\alpha_n)\phi(x_{n+1}, x_n) + \xi_n. \tag{3.7}$$

Due to (3.5), (3.7), the assumption (ii), and Remark 2.2, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

By (1.2), it follows that

$$\|u_n\| \to \|\overline{x}\| \tag{3.8}$$

as $n \to \infty$. Since J is uniformly norm-to-norm continuous, we obtain

$$\|Ju_n\| \to \|J\overline{x}\| \tag{3.9}$$

as $n \to \infty$. This implies that $\{\|Ju_n\|\}$ is bounded in E^*. Since E^* is reflexive, we assume that $Ju_n \rightharpoonup \overline{u} \in E^*$ as $n \to \infty$. In view of $J(E) = E^*$, there exists $u \in E$ such that $Ju = \overline{u}$. This implies that $Ju_n \rightharpoonup Ju$. We have

$$\begin{aligned} \phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2. \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, this yields

$$\begin{aligned} 0 &\geq \|\overline{x}\|^2 - 2\langle \overline{x}, \overline{u}\rangle + \|\overline{u}\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Ju\rangle + \|Ju\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Ju\rangle + \|u\|^2 \\ &= \phi(\overline{x}, u), \end{aligned}$$

which shows that $\overline{x} = u$ and so $Ju_n \rightharpoonup J\overline{x}$. It follows from (3.9) and the Kadec-Klee property of E^* that $Ju_n \to J\overline{x}$ as $n \to \infty$. Since J^{-1} is norm-weak-continuous, we have

$$u_n \rightharpoonup \overline{x}. \tag{3.10}$$

From (3.8), (3.10), and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} u_n = \overline{x}. \tag{3.11}$$

On the other hand, we see from the weak lower semicontinuity of the norm that

$$\begin{aligned} \phi(q, \overline{x}) &= \|q\|^2 - 2\langle q, J\overline{x}\rangle + \|\overline{x}\|^2 \\ &\leq \liminf_{n\to\infty}\bigl(\|q\|^2 - 2\langle q, Ju_n\rangle + \|u_n\|^2\bigr) \\ &= \liminf_{n\to\infty} \phi(q, u_n) \\ &\leq \limsup_{n\to\infty} \phi(q, u_n) \\ &= \limsup_{n\to\infty}\bigl(\|q\| - 2\langle q, Ju_n\rangle + \|u_n\|^2\bigr) \\ &\leq \phi(q, \overline{x}), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \phi(q, u_n) = \phi(q, \overline{x}). \tag{3.12}$$

By (3.6) and (3.11), we obtain $\lim_{n\to\infty}\|x_n - u_n\| = 0$. The uniform continuity of J on bounded sets gives

$$\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0. \tag{3.13}$$

Now, using the definition of ϕ, we have, for all $q \in \mathcal{F}$,

$$\begin{aligned}\phi(q,x_n) - \phi(q,u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|\big(\|x_n\| + \|u_n\|\big) + 2\|q\|\|Jx_n - Ju_n\|.\end{aligned}$$

From (3.13), we obtain

$$\phi(q,x_n) - \phi(q,u_n) \to 0$$

as $n \to \infty$. By (3.12), it follows that

$$\lim_{n\to\infty}\phi(q,x_n) = \phi(q,\overline{x}). \tag{3.14}$$

Hence, for any $q \in \mathcal{F} \subset C_n$, it follows from the convexity of $\|\cdot\|^2$ and Lemma 2.3 that

$$\begin{aligned}\phi(q,u_n) &= \phi\big(q, T_{r_n}^{\theta}y_n\big) \\ &\le \phi(q,y_n) \\ &= \phi\big(q, J^{-1}\big(\alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n\big)\big) \\ &= \|q\|^2 - 2\big\langle q, \alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n\big\rangle \\ &\quad + \big\|\alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n\big\|^2 \\ &\le \|q\| - 2\alpha_n\langle q, Jx_1\rangle - 2\beta_n\big\langle q, JT^n x_n\big\rangle - 2\gamma_n\langle q, Jx_n\rangle \\ &\quad + \alpha_n\|Jx_1\|^2 + \beta_n\big\|JT^n x_n\big\|^2 + \gamma_n\|Jx_n\|^2 \\ &= \alpha_n\phi(q,x_1) + \beta_n\phi\big(q, T^n x_n\big) + \gamma_n\phi(q,x_n) \\ &\le \alpha_n\phi(q,x_1) + \beta_n\big(\phi(q,x_n) + \xi_n\big) + \gamma_n\phi(q,x_n) \\ &\le \alpha_n\phi(q,x_1) + (1-\alpha_n)\phi(q,x_n) + \xi_n.\end{aligned} \tag{3.15}$$

From (3.12), (3.14), (3.15), Remark 2.2, and the assumption (ii), we obtain

$$\lim_{n\to\infty}\phi(q,y_n) = \phi(q,\overline{x}).$$

From Lemma 2.3, we see that for any $q \in \mathcal{F}$ and $u_n = T_{r_n}^{\theta}y_n$,

$$\begin{aligned}\phi(u_n,y_n) &= \phi\big(T_{r_n}^{\theta}y_n, y_n\big) \\ &\le \phi(q,y_n) - \phi\big(q, T_{r_n}^{\theta}y_n\big) \\ &= \phi(q,y_n) - \phi(q,u_n).\end{aligned}$$

Taking $n \to \infty$ on both sides of the inequality above, we have

$$\lim_{n\to\infty} \phi(u_n, y_n) = 0.$$

From (1.2), we have $(\|u_n\| - \|y_n\|)^2 \to 0$ as $n \to \infty$. By (3.8), we have

$$\|y_n\| \to \|\overline{x}\| \tag{3.16}$$

as $n \to \infty$, and so

$$\|Jy_n\| \to \|J\overline{x}\| \tag{3.17}$$

as $n \to \infty$. That is, $\{\|Jy_n\|\}$ is bounded in E^*. Since E^* is reflexive, we can assume that $Jy_n \rightharpoonup y^* \in E^*$ as $n \to \infty$. In view of $J(E) = E^*$, there exists $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned}\phi(u_n, y_n) &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, it follows that

$$\begin{aligned}0 &\geq \|\overline{x}\|^2 - 2\langle \overline{x}, y^*\rangle + \|y^*\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jy\rangle + \|Jy\|^2 \\ &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jy\rangle + \|y\|^2 \\ &= \phi(\overline{x}, y).\end{aligned}$$

From Remark 1.1, $\overline{x} = y$, *i.e.*, $y^* = J\overline{x}$. It follows that $Jy_n \rightharpoonup J\overline{x} \in E^*$ as $n \to \infty$. From (3.17) and the Kadec-Klee property of E^*, we have

$$Jy_n \to J\overline{x}$$

as $n \to \infty$. Since J^{-1} is norm-weak*-continuous, $y_n \rightharpoonup \overline{x}$ as $n \to \infty$. From (3.16) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} y_n = \overline{x}.$$

Step 5. We show that $\overline{x} \in \mathcal{F}$.
By Step 4, we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0.$$

The uniform continuity of J on bounded sets gives

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.18}$$

From the assumption $r_n \geq a$ and (3.18), we see that $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$ as $n \to \infty$. But from (A2) and (3.1), we note that

$$\frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -\theta(u_n, y) \geq \theta(y, u_n), \quad \forall y \in C$$

and hence

$$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \geq \theta(y, u_n), \quad \forall y \in C,$$

which implied that $\theta(y, \overline{x}) \leq 0$ for all $y \in C$. Put $y_t = ty + (1-t)\overline{x}$ for all $t \in (0, 1]$ and $y \in C$. Then we get $y_t \in C$ and $\theta(y_t, \overline{x}) \leq 0$. Therefore, from (A1) and (A4), we obtain

$$\begin{aligned} 0 = \theta(y_t, y_t) &\leq t\theta(y_t, y) + (1-t)\theta(y_t, \overline{x}) \\ &\leq t\theta(y_t, y). \end{aligned}$$

Thus, $\theta(y_t, y) \geq 0$ for all $y \in C$. Furthermore, as $t \to \infty$, we have from (A3) that $\theta(\overline{x}, y) \geq 0$ for all $y \in C$. This implies that $\overline{x} \in EP(\theta)$.

Finally, we show that $\overline{x} \in F(T)$. In view of $y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JT^n x_n + \gamma_n Jx_n)$, we find that

$$Ju_n - Jy_n = \alpha_n(Ju_n - Jx_1) + \beta_n\big(Ju_n - JT^n x_n\big) + \gamma_n(Ju_n - Jx_n).$$

Hence we have

$$\begin{aligned} \beta_n\big\|Ju_n - JT^n x_n\big\| \leq{}& \|Ju_n - J\overline{x}\| + \|J\overline{x} - Jy_n\| + \alpha_n\|Ju_n - Jx_1\| \\ &+ \gamma_n\|Ju_n - Jx_n\|. \end{aligned}$$

From the assumptions (ii), (iii), and (3.13), we have

$$\lim_{n\to\infty}\big\|Ju_n - JT^n x_n\big\| = 0. \tag{3.19}$$

Notice that

$$\big\|JT^n x_n - J\overline{x}\big\| \leq \big\|JT^n x_n - Ju_n\big\| + \|Ju_n - J\overline{x}\|.$$

This implies from (3.19) that

$$\lim_{n\to\infty}\big\|JT^n x_n - J\overline{x}\big\| = 0. \tag{3.20}$$

The demicontinuity of $J^{-1} : E^* \to E$ implies that $T^n x_n \rightharpoonup \overline{x}$ as $n \to \infty$. We have

$$\big|\big\|T^n x_n\big\| - \|\overline{x}\|\big| = \big|\big\|JT^n x_n\big\| - \|J\overline{x}\|\big| \leq \big\|JT^n x_n - J\overline{x}\big\|.$$

With the aid of (3.20), we see that $\lim_{n\to\infty}\|T^n x_n\| = \|\overline{x}\|$. Since E has the Kadec-Klee property, we find that

$$\lim_{n\to\infty}\big\|T^n x_n - \overline{x}\big\| = 0. \tag{3.21}$$

Since

$$\|T^{n+1}x_n - \overline{x}\| \leq \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - \overline{x}\|,$$

we find from (3.21) and the asymptotic regularity of T that

$$\lim_{n\to\infty} \|T^{n+1}x_n - \overline{x}\| = 0,$$

i.e., $TT^n x_n - \overline{x} \to 0$ as $n \to \infty$. It follows from the closedness of T that $T\overline{x} = \overline{x}$. So, $\overline{x} \in F(T)$ and hence $\overline{x} \in \mathcal{F} = F(T) \cap EP(\theta)$.

Step 6. We show that $\overline{x} = \Pi^f_{\mathcal{F}} x_1$ and so $x_n \to \Pi^f_{\mathcal{F}} x_1$ as $n \to \infty$.

Since $\mathcal{F}$ is a closed convex set, it follows from Lemma 2.2 that $\Pi^f_{\mathcal{F}} x_1$ is single-valued, which is denoted by $\tilde{x}$. By the definition of $x_n = \Pi^f_{C_n} x_1$ and $\tilde{x} \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_1) \leq G(\tilde{x}, Jx_1)$$

for all $n \geq 1$. By the definition of G, we know that for any $x \in E$, $G(u, Jx)$ is convex and lower semicontinuous with respect to u and so

$$\begin{aligned} G(\overline{x}, Jx_1) &\leq \liminf_{n\to\infty} G(x_n, Jx_1) \\ &\leq \limsup_{n\to\infty} G(x_n, Jx_1) \\ &\leq G(\tilde{x}, Jx_1). \end{aligned}$$

From the definition of $\Pi^f_{\mathcal{F}} x_1$ and $\overline{x} \in \mathcal{F}$, we conclude that

$$\overline{x} = \tilde{x} = \Pi^f_{\mathcal{F}} x_1$$

and $x_n \to \overline{x} = \Pi^f_{\mathcal{F}} x_1$ as $n \to \infty$. This completes the proof. □

Remark 3.1

(i) If $f = 0$, then $G(x, Jy) = \phi(x, y)$ and $\Pi^f_{C_n} = \Pi_{C_n}$.

(ii) If we take $f = 0$, $\theta = 0$, $u_n = y_n$, and $\alpha_n = 0$ for all $n \in \mathbb{N}$, then the iterative scheme (3.1) reduces to the following scheme:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}(\beta_n JT^n x_n + (1 - \beta_n) Jx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$, which is the algorithm introduced by Hao [10] and an improvement to (1.3).

If T is quasi-ϕ-nonexpansive, then Theorem 3.1 is reduced to following without the boundedness of $F(T)$ and the asymptotically regularity of T.

Corollary 3.1 *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Let C be a nonempty closed convex subset of E. Let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4). *Let $T : C \to C$ be a closed and quasi-ϕ-nonexpansive mapping. Assume that $\mathcal{F} = F(T) \cap EP(\theta)$ is nonempty. Let $f : E \to \mathbb{R}^+$ be a convex and lower semicontinuous function with $C \subset \operatorname{int}(D(f))$ and $f(0) = 0$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ satisfying the following conditions*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_1 + \beta_n JTx_n + \gamma_n Jx_n) \\ u_n \in C \text{ such that } \theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le \alpha_n G(z, Jx_1) + (1 - \alpha_n)G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where $\{r_n\}$ is a real sequence in $[a, \infty)$ for some $a > 0$ and $\Pi^f_{C_{n+1}}$ is the generalized f-projection operator. Then $\{x_n\}$ converges strongly to $\Pi^f_{\mathcal{F}} x_1$.

Remark 3.2

(i) By Remark 3.1, Theorem 3.1 extends Theorem 2.1 of Hao [10].

(ii) Theorem 3.1 generalizes Theorem 3.1 of Matsushita and Takahashi [9] in the following respects:

- from the relatively nonexpansive mapping to the asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense;
- from a uniformly convex and uniformly smooth Banach space to a uniformly smooth and strictly convex Banach space with the Kadec-Klee property;

(iii) in view of the mappings and the frame work of the spaces, Theorem 3.1 generalizes and improves Theorem 3.1 of Ma *et al.* [16], Theorem 3.1 of Qin *et al.* [17], Theorem 3.1 of Qing and Lv [18] and Theorem 3.1 of Saewan [19].

We now provide a nontrivial family of mappings satisfying the conditions of Theorem 3.1.

Example 3.1 Let $E = \mathbb{R}$ with the standard norm $\|\cdot\| = |\cdot|$ and $C = [0,1]$. Let $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2}x, & x \in [0, \frac{1}{2}], \\ 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

We first show that T is an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense with $F(T) = \{0\} \neq \phi$. In fact, for $p = 0 \in F(T)$, we have

$$\begin{aligned}\phi(p, T^n x) &= \left|0 - T^n x\right|^2 \\ &= \frac{1}{2^{2n}}|x|^2 \\ &\leq |0 - x|^2 = \phi(p, x), \quad \forall x \in \left[0, \frac{1}{2}\right]\end{aligned}$$

and

$$\begin{aligned}\phi(p, T^n x) &= \left|0 - T^n x\right|^2 \\ &= 0 \\ &\leq |0 - x|^2 = \phi(p, x), \quad \forall x \in \left(\frac{1}{2}, 1\right].\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} \left(\phi(p, T^n x) - \phi(p, x)\right) \leq 0.$$

Next, we define a bifunction $\theta : C \times C \to \mathbb{R}$ satisfying the conditions (A1)-(A4) by

$$\theta(x, y) = y^2 - x^2.$$

Then the set of solutions $EP(\theta)$ to the equilibrium problem for θ is obviously $\{0\}$. Since $\mathcal{F} = F(T) \cap EP(\theta) \neq \phi$ and $F(T)$ is bounded, it follows from Theorem 3.1 that the sequence defined by (3.1) converges strongly to $\Pi^f_{\mathcal{F}} x_1$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
JUJ conceived of the study, its design, and its coordination. The author read and approved the final manuscript.

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References

1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
2. Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**(2), 169-179 (1993)
3. Kirk, WA: Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type. Isr. J. Math. **17**, 339-346 (1974)
4. Chidume, CE, Ofoedu, EU, Zegeye, H: Strong and weak convergence theorem for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **280**, 364-374 (2003)
5. Górnicki, J: Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Comment. Math. Univ. Carol. **30**, 249-252 (1989)
6. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)

7. Alber, YI: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Appl. Math., vol. 178, pp. 15-50. Dekker, New York (1996)
8. Su, Y, Wang, D, Shang, M: Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings. Fixed Point Theory Appl. **2008**, 284613 (2008)
9. Matsushita, S, Takahashi, W: A strong convergence theorem for relatively nonexpansive mappings in Banach spaces. J. Approx. Theory **134**, 257-266 (2005)
10. Hao, Y: Some results on a modified Mann iterative scheme in a reflexive Banach space. Fixed Point Theory Appl. **2013**, 227 (2013)
11. Cioranescu, I: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
12. Wu, KQ, Huang, NJ: The generalized f-projection operator with an application. Bull. Aust. Math. Soc. **73**, 307-317 (2006)
13. Li, X, Huang, N, O'Regan, D: Strong convergence theorems for relative nonexpansive mappings in Banach spaces with applications. Comput. Math. Appl. **60**, 1322-1331 (2010)
14. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)
15. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45-57 (2009)
16. Ma, Z, Wang, L, Chang, SS: Strong convergence theorem for quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. J. Inequal. Appl. **2013**, 306 (2013)
17. Qin, X, Cho, SY, Wang, L: Algorithms for treating equilibrium and fixed point problems. Fixed Point Theory Appl. **2013**, 308 (2013)
18. Qing, Y, Lv, S: A strong convergence theorem for solutions of equilibrium problems and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense. Fixed Point Theory Appl. **2013**, 305 (2013)
19. Saewan, S: Strong convergence theorem for total quasi-ϕ-asymptotically nonexpansive mappings in a Banach space. Fixed Point Theory Appl. **2013**, 297 (2013)

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