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General alternative regularization methods for nonexpansive mappings in Hilbert spaces

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Abstract

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $T : C \to C$ be a nonexpansive mapping with a nonempty set of fixed points $\mathrm{Fix}(T)$ and let $h : C \to C$ be a Lipschitzian strong pseudo-contraction. We first point out that the sequence generated by the usual viscosity approximation method $x_{n+1} = \lambda_n h(x_n) + (1 - \lambda_n)Tx_n$ may not converge to a fixed point of T, even not bounded. Secondly, we prove that if the sequence $(\lambda_n) \subset (0, 1)$ satisfies the conditions: (i) $\lambda_n \to 0$, (ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$ and (iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$, then the sequence (x_n) generated by a general alternative regularization method: $x_{n+1} = T(\lambda_n h(x_n) + (1 - \lambda_n)x_n)$ converges strongly to a fixed point of T, which also solves the variational inequality problem: finding an element x^* such that $\langle h(x^*) - x^*, x - x^*\rangle \le 0$ for all $x \in \mathrm{Fix}(T)$. Furthermore, we prove that if T is replaced with the sequence of average mappings $(1 - \beta_n)I + \beta_n T$ $(n \ge 0)$ such that $0 < \beta_* \le \beta_n \le \beta^* < 1$, where β_* and β^* are two positive constants, then the same convergence result holds provided conditions (i) and (ii) are satisfied. Finally, an algorithm for finding a common fixed point of a family of finite nonexpansive mappings is also proposed and its strong convergence is proved. Our results in this paper extend and improve the alternative regularization methods proposed by HK Xu.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ and let $f : C \to C$ be a α-contractive mapping, *i.e.*, there exists a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ holds for all $x, y \in C$. Let $T : C \to C$ be a nonexpansive mapping, *i.e.*, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Throughout this article, the set of fixed points of T, indicated by $\mathrm{Fix}(T) \triangleq \{x \in C \mid Tx = x\}$, is always assumed to be nonempty.

For every nonempty closed convex subset K of H, the metric (or nearest point) projection indicated by P_K from H onto K can be defined, that is, for each $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| \mid z \in K\}$. It is well known (*e.g.*, see [1]) that P_K is nonexpansive and a characteristic inequality holds: given $x \in H$ and $z \in K$, then $z = P_K x$

if and only if

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in K.$$

Since $\mathrm{Fix}(T)$ is a closed convex subset of H, so the metric projection $P_{\mathrm{Fix}(T)}$ is valid.

Recall that a mapping $h : C \to C$ is said to be L-Lipschitzian, if there exists a positive constant L such that

$$\|h(x) - h(y)\| \le L\|x - y\|, \quad \forall x, y \in C,$$

and a mapping $h : C \to C$ is said to be α-strongly pseudo-contractive, if there exists a constant $\alpha \in [0,1)$ such that

$$\langle h(x) - h(y), x - y\rangle \le \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

In this case, we also call h a α-strong pseudo-contraction.

It is very easy to see that a α-contractive mapping is a α-strongly pseudo-contractive and α-Lipschitzian mapping, *i.e.*, the class of contractive mappings is a proper subset of the class of Lipschitzian strong pseudo-contractions. The class of Lipschitzian strong pseudo-contractions will be used repeatedly in the sequel.

Recall that a mapping $F : C \to H$ is said to be η-strongly monotone, if there exists a positive constant η such that

$$\langle F(x) - F(y), x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C.$$

The variational inequality problem [2] can mathematically be formulated as the problem of finding a point $x^* \in K$ with the property

$$\langle Fx^*, x - x^*\rangle \ge 0, \quad \forall x \in K,$$

where K is a nonempty closed convex subset of H and $F : K \to H$ is a nonlinear operator. It is well known that [3] if $F : K \to H$ is a Lipschitzian and strongly monotone operator, then the variational inequality problem has a unique solution.

Many iteration processes are often used to approximate a fixed point of a nonexpansive mapping in a Hilbert space or a Banach space (for example, see [1] and [4–20]). One of them is now known as Halpern's iteration process [4] and is defined as follows: take an initial guess $x_0 \in C$ arbitrarily and define (x_n) recursively by

$$x_{n+1} = \lambda_n u + (1 - \lambda_n)Tx_n, \quad n \ge 0, \tag{1.1}$$

where (λ_n) is a sequence in the interval $[0,1]$ and u is some given element in C. For Halpern's iteration process, a classical result in the setting of Hilbert spaces is as follows.

Theorem 1.1 ([4]) *If* (λ_n) *satisfies the conditions*:

(i) $\lambda_n \to 0$ $(n \to \infty)$;
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ *or* $\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}=1$;

then the sequence (x_n) *generated by* (1.1) *converges strongly to a fixed point* x^* *of* T *such that* $x^*=P_{\mathrm{Fix}(T)}u$, *that is,*

$$\langle u-x^*,x-x^*\rangle\leq 0,\quad x\in\mathrm{Fix}(T).$$

Xu [5] proposed an alternative regularization method:

$$x_{n+1}=T\bigl(\lambda_n u+(1-\lambda_n)x_n\bigr),\quad n\geq 0 \tag{1.2}$$

and studied its strong convergence in the setting of Hilbert spaces and Banach spaces, respectively. Indeed, in the setting of Hilbert spaces, one can proved that for algorithm (1.2) the same convergence result as that of Theorem 1.1 holds under conditions (i)-(iii) above.

As an extension to Halpern's iteration process, Moudafi proposed [15] the viscosity approximation method: take an initial guess $x_0\in C$ arbitrarily and define (x_n) recursively by

$$x_{n+1}=\lambda_n f(x_n)+(1-\lambda_n)Tx_n,\quad n\geq 0, \tag{1.3}$$

where (λ_n) is a sequence in the interval $[0,1]$. Moudafi proved the following result in Hilbert spaces.

Theorem 1.2 ([15]) *If* (λ_n) *satisfies the same conditions* (i)-(iii) *as above, then the sequence* (x_n) *generated by* (1.3) *converges strongly to a fixed point* x^* *of* T, *which also solves the variational inequality problem*: *finding an element* $x^*\in\mathrm{Fix}(T)$ *such that*

$$\langle f(x^*)-x^*,x-x^*\rangle\leq 0,\quad x\in\mathrm{Fix}(T).$$

Xu studied the viscosity approximation method in the setting of Banach spaces and obtained the strong convergence theorems [16].

Similar to algorithm (1.2), we can naturally consider a general alternative regularization method: take an initial guess $x_0\in C$ arbitrarily and define (x_n) recursively by

$$x_{n+1}=T\bigl(\lambda_n f(x_n)+(1-\lambda_n)x_n\bigr),\quad n\geq 0, \tag{1.4}$$

where (λ_n) is a sequence in the interval $[0,1]$. In fact, in the setting of Hilbert spaces, it is not difficult to prove by an argument very similar to the proof of Theorem 1.2 that for algorithm (1.4) the same result as that of Theorem 1.2 holds under conditions (i)-(iii) above.

The main purpose of this paper is to consider a very natural question: if algorithms (1.3) and (1.4) can be extended to more general cases, more precisely, can we replace a contractive mapping f with a Lipschitzian strong pseudo-contraction h so that the same convergence result as that of Theorem 1.2 is still guaranteed under conditions (i)-(iii) as above? The answer to this question is negative for algorithm (1.3) unfortunately but is sure for algorithm (1.4) fortunately. In this sense, it seems reasonable to deem that algorithm (1.4) is better that algorithm (1.3).

Now we illustrate a fact via a very simple example that if f in algorithm (1.3) is replaced with a Lipschitzian and strongly pseudo-contractive mapping h, then strong convergence (even boundedness) of the iteration sequence (x_n) may not be guaranteed, in general. Indeed, take $H = \mathbb{R}^2$, $C = \mathbb{R}^2$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ u \end{pmatrix}, \quad \forall x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2.$$

Noting that T is just a rotation operator over $\mathbb{R}^2$, we see that T is a nonexpansive mapping. Moreover, noting the fact that

$$\langle Tx, x \rangle = 0$$

holds for all $x \in \mathbb{R}^2$, it is easy to see that for any positive constant κ and any $\alpha \in [0,1)$, $h \triangleq \kappa T$ is a κ-Lipschitzian and α-strongly pseudo-contractive mapping.

If f in algorithm (1.3) is replaced with $h = T$ (*i.e.*, $\kappa = 1$), then (1.3) is of the form:

$$x_{n+1} = Tx_n, \quad n \geq 0. \tag{1.5}$$

Since T is a rotation operator over $\mathbb{R}^2$, the sequence (x_n) generated by (1.5) does not converge to $\binom{0}{0}$, the unique fixed point of T, unless we take the initial guess $x_0 = \binom{0}{0}$. On the other hand, if f in algorithm (1.3) is replaced with $h = 2T$ (*i.e.*, $\kappa = 2$), thus (1.3) is rendered in the form

$$x_{n+1} = (1 + \lambda_n)Tx_n, \quad n \geq 0. \tag{1.6}$$

Consequently, noting that $\|Tx\| = \|x\|$, $\forall x \in \mathbb{R}^2$, we have

$$\begin{aligned} \|x_{n+1}\| &= (1 + \lambda_n)\|Tx_n\| \\ &= (1 + \lambda_n)\|x_n\| \\ &= (1 + \lambda_n)(1 + \lambda_{n-1}) \cdots (1 + \lambda_0)\|x_0\|. \end{aligned}$$

Since $\sum_{n=0}^{+\infty} \lambda_n = \infty$, $\lim_{n\to\infty}(1 + \lambda_n)(1 + \lambda_{n-1}) \cdots (1 + \lambda_0) = \infty$ holds and thus this implies that the sequence (x_n) generated by (1.6) is not bounded provided $x_0 \neq \binom{0}{0}$.

The rest of this paper is organized as follows. In order to prove our main results, some useful facts and tools are listed as lemmas in the next section. In Section 3, we prove that if a contractive mapping f in algorithm (1.4) is replaced with a Lipschitzian strong pseudo-contraction h, then the same convergence result as that of Theorem 1.2 is still guaranteed under conditions (i)-(iii) as above. Furthermore, we prove that if T is replaced with the sequence of average mappings $(1 - \beta_n)I + \beta_n T$ $(n \geq 0)$ such that $0 < \beta_* \leq \beta_n \leq \beta^* < 1$, where β_* and β^* are two positive constants, then the same result still holds provided conditions (i) and (ii) are satisfied. In the last section, an algorithm for finding a common fixed point of a family of finite nonexpansive mappings is also proposed and its strong convergence is proved.

We will use the notations:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x \mid \exists (x_{n_k}) \subset (x_n) \text{ such that } x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of (x_n).
3. $A \triangleq B$ means that B is the definition of A.

2 Preliminaries

We need some facts and tools in a real Hilbert space H, which are listed as lemmas below.

Lemma 2.1 *The following relation holds in a real Hilbert space H:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad x, y \in H.$$

Lemma 2.2 ([14, 21]) *Assume (a_n) is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \sigma_n, \quad n = 0, 1, 2 \ldots.$$

If $(\gamma_n)_{n=1}^\infty \subset (0,1)$, $(\delta_n)_{n=1}^\infty$ and $(\sigma_n)_{n=1}^\infty$ satisfy the conditions:

(i) $\sum_{n=1}^\infty \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$,
(iii) $\sum_{n=1}^\infty |\sigma_n| < \infty$,

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([22]) *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *If a sequence (x_n) in C is such that $x_n \rightharpoonup z$ and $\|x_n - Tx_n\| \to 0$, then $z = Tz$.*

Lemma 2.4 *For each $\lambda \in [0,1]$, the following identity holds in a real Hilbert space H:*

$$\|\lambda u + (1-\lambda)v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)\|u-v\|^2, \quad u, v \in H.$$

Lemma 2.5 ([23]) *Assume (s_n) is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\delta_n, \quad n \ge 0, \tag{2.1}$$

$$s_{n+1} \le s_n - \eta_n + \alpha_n, \quad n \ge 0, \tag{2.2}$$

where (γ_n) is a sequence in $(0,1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n) and (α_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^\infty \gamma_n = \infty$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence* $(n_k) \subset (n)$.

Then $\lim_{n\to\infty} s_n = 0$.

3 Algorithms for one mapping

Throughout this section, we will assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, C is a closed convex subset of H and $h : C \to C$ is a

L-Lipschitzian and α-strongly pseudo-contractive mapping, *i.e.*, there exist positive constants L and $\alpha \in [0,1)$ such that

$$\|h(x) - h(y)\| \le L\|x - y\|, \quad \forall x, y \in C, \tag{3.1}$$

$$\langle h(x) - h(y), x - y\rangle \le \alpha\|x - y\|^2, \quad \forall x, y \in C. \tag{3.2}$$

It is obvious that if h is a α-strong pseudo-contraction, then $I - h$ is a $(1 - \alpha)$-strongly monotone mapping, *i.e.*,

$$\langle (I - h)x - (I - h)y, x - y\rangle \ge (1 - \alpha)\|x - y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator.

Our first result is as follows.

Theorem 3.1 *Let $h : C \to C$ be a L-Lipschitzian and α-strongly pseudo-contractive mapping and let $T : C \to C$ be a nonexpansive mapping. If the sequence $(\lambda_n) \subset (0,1)$ satisfies the conditions*:

(i) $\lambda_n \to 0$ $(n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$;

then the sequence (x_n) generated by the algorithm

$$x_{n+1} = T\big(\lambda_n h(x_n) + (1 - \lambda_n)x_n\big), \quad n \ge 0, \tag{3.3}$$

where x_0 is selected in C arbitrarily, converges strongly to a fixed point of T, which also solves the variational inequality problem: *finding an element $x^* \in \operatorname{Fix}(T)$ such that*

$$\langle x^* - h(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \operatorname{Fix}(T). \tag{3.4}$$

Proof Noting that $I - h$ is a $(1 + L)$-Lipschitzian and $(1 - \alpha)$-strongly monotone mapping, so the variational inequality problem (3.4) has a unique solution, which is denoted by x^*. Now we try to prove that $x_n \to x^*$.

Firstly, we deduce from (3.1)-(3.3) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|T\big(\lambda_n h(x_n) + (1 - \lambda_n)x_n\big) - Tx^*\|^2 \\ &\le \|\lambda_n\big(h(x_n) - x^*\big) + (1 - \lambda_n)\big(x_n - x^*\big)\|^2 \\ &= \lambda_n^2\|h(x_n) - x^*\|^2 + 2\lambda_n(1 - \lambda_n)\langle h(x_n) - x^*, x_n - x^*\rangle + (1 - \lambda_n)^2\|x_n - x^*\|^2 \\ &\le 2\lambda_n^2\big[L^2\|x_n - x^*\|^2 + \|h(x^*) - x^*\|^2\big] \\ &\quad + 2\lambda_n(1 - \lambda_n)\big(\alpha\|x_n - x^*\|^2 + \langle h(x^*) - x^*, x_n - x^*\rangle\big) \\ &\quad + (1 - \lambda_n)^2\|x_n - x^*\|^2.\end{aligned} \tag{3.5}$$

Obviously

$$\begin{aligned}\langle h(x^*) - x^*, x_n - x^*\rangle &\le \|h(x^*) - x^*\| \cdot \|x_n - x^*\| \\ &\le \beta\|x_n - x^*\|^2 + \frac{1}{4\beta}\|h(x^*) - x^*\|^2,\end{aligned} \tag{3.6}$$

where β is a positive constant such that $\alpha + \beta < 1$. Thus, the combination of (3.5) and (3.6) leads to

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \left[1 - 2\lambda_n\left(1 - \lambda_n\left(\frac{1}{2} + L^2\right) - (1 - \lambda_n)(\alpha + \beta)\right)\right]\|x_n - x^*\|^2 \\ &\quad + 2\lambda_n^2\|h(x^*) - x^*\|^2 + 2\lambda_n(1 - \lambda_n)\frac{1}{4\beta}\|h(x^*) - x^*\|^2 \\ &\le \left[1 - 2\lambda_n\left(1 - \lambda_n\left(\frac{1}{2} + L^2\right) - (1 - \lambda_n)(\alpha + \beta)\right)\right]\|x_n - x^*\|^2 \\ &\quad + 2\lambda_n\left(1 + \frac{1}{4\beta}\right)\|h(x^*) - x^*\|^2. \end{aligned} \tag{3.7}$$

Noting the fact that $\lambda_n \to 0$ and

$$\lim_{n\to\infty}\left[1 - \lambda_n\left(\frac{1}{2} + L^2\right) - (1 - \lambda_n)(\alpha + \beta)\right] = 1 - (\alpha + \beta) > 0,$$

we assert that there exists some integer n_0 such that $2\lambda_n < 1$ and

$$1 - \lambda_n\left(\frac{1}{2} + L^2\right) - (1 - \lambda_n)(\alpha + \beta) > \frac{1}{2}\big(1 - (\alpha + \beta)\big) \tag{3.8}$$

hold for all $n \ge n_0$. So we see from (3.7) and (3.8) that for all $n \ge n_0$, the following relation holds:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \big(1 - \lambda_n\big(1 - (\alpha + \beta)\big)\big)\|x_n - x^*\|^2 \\ &\quad + \lambda_n\big(1 - (\alpha + \beta)\big)\frac{2}{1 - (\alpha + \beta)}\left(1 + \frac{1}{4\beta}\right)\|h(x^*) - x^*\|^2. \end{aligned} \tag{3.9}$$

Consequently

$$\|x_{n+1} - x^*\|^2 \le \max\left\{\|x_n - x^*\|^2, \frac{2}{1 - (\alpha + \beta)}\left(1 + \frac{1}{4\beta}\right)\|h(x^*) - x^*\|^2\right\}, \quad n \ge n_0,$$

and inductively

$$\|x_n - x^*\|^2 \le \max\left\{\|x_{n_0} - x^*\|^2, \frac{2}{1 - (\alpha + \beta)}\left(1 + \frac{1}{4\beta}\right)\|h(x^*) - x^*\|^2\right\}, \quad n \ge n_0.$$

This means that (x_n) is bounded, so is $(h(x_n))$.

From (3.3), we have

$$\begin{aligned} \|x_{n+1} - Tx_n\| &= \big\|\lambda_n\big(h(x_n) - x_n\big)\big\| \\ &\le \lambda_n\big(\big\|h(x_n)\big\| + \|x_n\|\big) \to 0 \quad (n \to \infty) \end{aligned} \tag{3.10}$$

due to the boundedness of (x_n) and $(h(x_n))$. Now we show that $\|x_{n+1} - x_n\| \to 0$. Setting $u_n = (1 - \lambda_n)x_n - (1 - \lambda_{n-1})x_{n-1}$, $v_n = \lambda_n h(x_n) - \lambda_{n-1}h(x_{n-1})$, we derive from Lemma 2.1, (3.1) and (3.2) that

$$\begin{aligned}\|u_n\|^2 &= \big\|(1-\lambda_n)x_n - (1-\lambda_{n-1})x_{n-1}\big\|^2 \\ &= \big\|(1-\lambda_n)(x_n - x_{n-1}) - (\lambda_n - \lambda_{n-1})x_{n-1}\big\|^2 \\ &\le (1-\lambda_n)^2\|x_n - x_{n-1}\|^2 - 2(\lambda_n - \lambda_{n-1})\big\langle x_{n-1}, (1-\lambda_n)x_n - (1-\lambda_{n-1})x_{n-1}\big\rangle \\ &\le (1-\lambda_n)^2\|x_n - x_{n-1}\|^2 + 2|\lambda_n - \lambda_{n-1}|\big(\|x_{n-1}\|^2 + \|x_{n-1}\|\cdot\|x_n\|\big), \end{aligned} \tag{3.11}$$

$$\begin{aligned}\|v_n\|^2 &= \big\|\lambda_n h(x_n) - \lambda_{n-1}h(x_{n-1})\big\|^2 \\ &= \big\|\lambda_n\big(h(x_n) - h(x_{n-1})\big) + (\lambda_n - \lambda_{n-1})h(x_{n-1})\big\|^2 \\ &\le \lambda_n^2\big\|h(x_n) - h(x_{n-1})\big\|^2 + 2(\lambda_n - \lambda_{n-1})\big\langle h(x_{n-1}), \lambda_n h(x_n) - \lambda_{n-1}h(x_{n-1})\big\rangle \\ &\le \lambda_n^2 L^2\|x_n - x_{n-1}\|^2 + 2|\lambda_n - \lambda_{n-1}|\big(\big\|h(x_{n-1})\big\|^2 \\ &\quad + \big\|h(x_{n-1})\big\|\cdot\big\|h(x_n)\big\|\big), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\langle u_n, v_n\rangle &= \big\langle (1-\lambda_n)(x_n - x_{n-1}) - (\lambda_n - \lambda_{n-1})x_{n-1}, \lambda_n\big(h(x_n) - h(x_{n-1})\big) \\ &\quad + (\lambda_n - \lambda_{n-1})h(x_{n-1})\big\rangle \\ &\le \lambda_n(1-\lambda_n)\alpha\|x_n - x_{n-1}\|^2 + |\lambda_n - \lambda_{n-1}|\|x_{n-1}\|\big(\big\|h(x_n)\big\| \\ &\quad + \big\|h(x_{n-1})\big\|\big) + |\lambda_n - \lambda_{n-1}|\big(\|x_n\|\cdot\big\|h(x_{n-1})\big\| \\ &\quad + \|x_{n-1}\|\cdot\big\|h(x_{n-1})\big\|\big) + |\lambda_n - \lambda_{n-1}|^2\|x_{n-1}\|\cdot\big\|h(x_{n-1})\big\|. \end{aligned} \tag{3.13}$$

Noting the boundedness of (x_n) and $(h(x_n))$, it follows from (3.11)-(3.13) that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\le \|u_n + v_n\|^2 \\ &\le \|u_n\|^2 + \|v_n\|^2 + 2\langle u_n, v_n\rangle \\ &\le [1-\gamma_n]\cdot\|x_n - x_{n-1}\|^2 + |\lambda_n - \lambda_{n-1}|M, \end{aligned}$$

where $\gamma_n = \lambda_n(2 - \lambda_n(1+L^2) - 2(1-\lambda_n)\alpha)$ and M is a positive constant independent on n. Observing that

$$\lim_{n\to\infty}\big[2 - \lambda_n\big(1+2L^2\big) - 2(1-\lambda_n)\alpha\big] = 2(1-\alpha) > 0,$$

we get from conditions (i) and (ii) that $\gamma_n \to 0$ and $\sum_{n=0}^{\infty}\gamma_n = \infty$ hold. Hence, this together with condition (iii) allows us to assert $\|x_{n+1} - x_n\| \to 0$ by using Lemma 2.2. From this together with (3.10) one concludes that $\|x_n - Tx_n\| \to 0$ and hence we obtain $\omega(x_n) \subset \mathrm{Fix}(T)$ by using Lemma 2.3.

Finally, we prove $x_n \to x^*$ $(n \to \infty)$. Again using (3.5), we also have

$$\begin{aligned}\big\|x_{n+1} - x^*\big\|^2 &\le \big[1 - \lambda_n\big(2 - \lambda_n\big(1+2L^2\big) - 2(1-\lambda_n)\alpha\big)\big]\big\|x_n - x^*\big\|^2 \\ &\quad + 2\lambda_n^2\big\|h\big(x^*\big) - x^*\big\|^2 + 2\lambda_n(1-\lambda_n)\big\langle h\big(x^*\big) - x^*, x_n - x^*\big\rangle \\ &\triangleq (1-\gamma_n)\big\|x_n - x^*\big\|^2 + \gamma_n\delta_n, \end{aligned} \tag{3.14}$$

where $\gamma_n = \lambda_n(2 - \lambda_n(1 + L^2) - 2(1 - \lambda_n)\alpha)$ and

$$\delta_n = \frac{2\lambda_n\|h(x^*) - x^*\|^2 + 2(1 - \lambda_n)\langle h(x^*) - x^*, x_n - x^*\rangle}{2 - \lambda_n(1 + L^2) - 2(1 - \lambda_n)\alpha}.$$

Take a subsequence (x_{n_k}) such that

$$\limsup_{n\to\infty} \langle h(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle h(x^*) - x^*, x_{n_k} - x^*\rangle.$$

Without loss of the generality, we assume that there exists some $\bar{x} \in \operatorname{Fix}(T)$ (noting that $\omega(x_n) \subset \operatorname{Fix}(T)$ holds) such that $x_{n_k} \rightharpoonup \bar{x}$ (otherwise, we may select some subsequence of (x_{n_k}) with this property). Hence, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle h(x^*) - x^*, x_n - x^*\rangle &= \lim_{k\to\infty} \langle h(x^*) - x^*, x_{n_k} - x^*\rangle \\ &= \langle h(x^*) - x^*, \bar{x} - x^*\rangle \le 0 \end{aligned} \tag{3.15}$$

noting that x^* is the unique solution of the variational inequality (3.4). Consequently, it is easy to verify that $\limsup_{n\to\infty} \delta_n \le 0$ by using (3.15). This together with $\gamma_n \to 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$ allows us to use Lemma 2.2 to (3.14) to obtain

$$\|x_n - x^*\| \to 0 \quad (n \to \infty).$$ □

Theorem 3.2 *Let $h : C \to C$ be a L-Lipschitzian and α-strongly pseudo-contractive mapping and let $T : C \to C$ be a nonexpansive mapping. Let $T_n = (1 - \beta_n)I + \beta_n T$, where $(\beta_n) \subset (0,1)$. Take $x_0 \in C$ arbitrarily and define a sequence (x_n) by the process*

$$x_{n+1} = T_n\big(\lambda_n h(x_n) + (1 - \lambda_n)x_n\big), \quad n \ge 0, \tag{3.16}$$

where $(\lambda_n) \subset (0,1)$. If (λ_n) and (β_n) satisfy the conditions:

(i) $\lambda_n \to 0$ $(n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) *there exist two constants β_* and β^* such that $0 < \beta_* \le \beta_n \le \beta^* < 1$ for all $n \ge 0$,*

then the sequence (x_n) converges strongly to a fixed point of T, which also solves the variational inequality problem: *finding a point $x^* \in \operatorname{Fix}(T)$ such that*

$$\langle x^* - h(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \operatorname{Fix}(T).$$

Proof Firstly, noting that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1 - \beta_n)\big\|\big(\lambda_n h(x_n) + (1 - \lambda_n)x_n\big) - x^*\big\| \\ &\quad + \beta_n\big\|T\big(\lambda_n h(x_n) + (1 - \lambda_n)x_n\big) - Tx^*\big\| \\ &\le \big\|\lambda_n h(x_n) + (1 - \lambda_n)x_n - x^*\big\|, \end{aligned} \tag{3.17}$$

we assert that (x_n) is bounded by an argument similar to the proof of Theorem 3.1. Moreover, in a way similar to getting (3.5), we have from (3.17)

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|\lambda_n(h(x_n) - x^*) + (1-\lambda_n)(x_n - x^*)\|^2 \\ &\le [1 - \lambda_n(2 - \lambda_n(1+2L^2) - 2(1-\lambda_n)\alpha)]\|x_n - x^*\|^2 \\ &\quad + 2\lambda_n^2\|h(x^*) - x^*\|^2 + 2\lambda_n(1-\lambda_n)\langle h(x^*) - x^*, x_n - x^*\rangle. \end{aligned} \tag{3.18}$$

Secondly, setting $z_n = \lambda_n h(x_n) + (1-\lambda_n)x_n$ and using Lemma 2.4, we have from (3.16) and (3.18)

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad = \|(1-\beta_n)(z_n - x^*) + \beta_n(Tz_n - x^*)\|^2 \\ &\quad = (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &\quad \le \|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &\quad \le \|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tz_n\|^2 \\ &\qquad + 2\lambda_n^2\|h(x^*) - x^*\|^2 + 2\lambda_n(1-\lambda_n)\|h(x^*) - x^*\| \cdot \|x_n - x^*\|. \end{aligned} \tag{3.19}$$

Setting

$$\begin{aligned} s_n &= \|x_n - x^*\|^2, \qquad \gamma_n = \lambda_n(2 - \lambda_n(1+2L^2) - 2(1-\lambda_n)\alpha), \\ \delta_n &= \frac{1}{2 - \lambda_n(1+2L^2) - 2(1-\lambda_n)\alpha}\bigl[2\lambda_n\|h(x^*) - x^*\|^2 \\ &\quad + 2(1-\lambda_n)\langle h(x^*) - x^*, x_n - x^*\rangle\bigr], \\ \eta_n &= \beta_n(1-\beta_n)\|z_n - Tz_n\|^2, \\ \alpha_n &= 2\lambda_n^2\|h(x^*) - x^*\|^2 + 2\lambda_n(1-\lambda_n)\|h(x^*) - x^*\| \cdot \|x_n - x^*\|, \end{aligned}$$

thus (3.18) and (3.19) can be rewritten as the form

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n s_n, \tag{3.20}$$

$$s_{n+1} \le s_n - \eta_n + \alpha_n. \tag{3.21}$$

Since $\gamma_n \to 0$, $\sum_{n=0}^\infty \gamma_n = \infty$ ($\lim_{n\to\infty}(2-\lambda_n(1+2L^2)-2(1-\lambda_n)\alpha) = 2(1-\alpha) > 0$) and $\alpha_n \to 0$ hold obviously, so in order to complete the proof by using Lemma 2.5, it suffices to verify that $\eta_{n_k} \to 0$ $(k \to \infty)$ implies that

$$\overline{\lim_{k\to\infty}} \delta_{n_k} \le 0$$

for any subsequence $(n_k) \subset (n)$.

Indeed, $\eta_{n_k} \to 0$ $(k \to \infty)$ implies that $\|z_{n_k} - Tz_{n_k}\| \to 0$ due to condition (iii). Furthermore, the relation

$$\begin{aligned} \|x_{n_k} - Tx_{n_k}\| &\le \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - Tz_{n_k}\| + \|Tz_{n_k} - Tx_{n_k}\| \\ &\le 2\|x_{n_k} - z_{n_k}\| + \|z_{n_k} - Tz_{n_k}\| \end{aligned} \tag{3.22}$$

together with the fact that

$$\|x_n - z_n\| \le \lambda_n \big[\big\| h(x_n) \big\| + \|x_n\| \big] \to 0$$

allows us to get $\|x_{n_k} - Tx_{n_k}\| \to 0$ $(k \to \infty)$. Using Lemma 2.3, we get $\omega(x_{n_k}) \subset \mathrm{Fix}(T)$. Thus we have

$$\overline{\lim_{k\to\infty}} \langle h(x^*) - x^*, x_{n_k} - x^* \rangle \le 0$$

and hence $\overline{\lim}_{k\to\infty} \delta_{n_k} \le 0$ holds. □

4 Algorithm for several mappings

In this section, we turn to considering an algorithm for finding a common fixed point of a family of finite nonexpansive mappings.

Let H be a real Hilbert space and let C be a closed convex subset of H. Let N be an integer such that $N \ge 2$ and let $T_i : C \to C$ $(i = 1, 2, \ldots, N)$ be a family of finite nonexpansive mappings. Set

$$T_i^n = (1 - \beta_i^n) I + \beta_i^n T_i, \quad n \ge 0, i = 1, 2, \ldots, N,$$

where $(\beta_i^n) \subset (0, 1)$ for all $n \ge 0$ and $i = 1, 2, \ldots, N$.

Our main result in this section is as follows.

Theorem 4.1 *Let $h : C \to C$ be a L-Lipschitzian and α-strongly pseudo-contractive mapping. Take a initial guess $x_0 \in C$ arbitrarily and define a sequence (x_n) by*

$$x_{n+1} = T_N^n T_{N-1}^n \cdots T_1^n \big(\lambda_n h(x_n) + (1 - \lambda_n) x_n \big), \quad n \ge 0, \tag{4.1}$$

where $(\lambda_n) \subset (0, 1)$ and T_i^n is given as above. If (λ_n) and (β_i^n) satisfy the conditions:

(i) $\lambda_n \to 0$ $(n \to \infty)$;
(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(iii) *there exist two constants β_* and β^* such that $0 < \beta_* \le \beta_i^n \le \beta^* < 1$ for all $n \ge 0$ and $i = 1, 2, \ldots, N$,*

then the sequence (x_n) generated by (4.1) converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$, which also solves the variational inequality problem: finding an element $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ such that

$$\langle x^* - h(x^*), x - x^* \rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(T_i). \tag{4.2}$$

Proof Without loss of the generality, we only give the proof for the case where $N = 2$.

It is clear that the variational inequality (4.2) has a unique solution, which is denoted by x^* in the sequel. An argument very similar to the proof of Theorem 3.2 allows us to verify that (x_n) is bounded (so is $(h(x_n))$) and the following relation holds:

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad \le \bigl[1 - \lambda_n\bigl(2 - \lambda_n\bigl(1 + 2L^2\bigr) - 2(1 - \lambda_n)\alpha\bigr)\bigr]\|x_n - x^*\|^2 \\ &\qquad + 2\lambda_n^2\bigl\|h\bigl(x^*\bigr) - x^*\bigr\|^2 + 2\lambda_n(1 - \lambda_n)\bigl\langle h\bigl(x^*\bigr) - x^*, x_n - x^*\bigr\rangle. \end{aligned} \tag{4.3}$$

On the other hand, setting $z_n = \lambda_n h(x_n) + (1 - \lambda_n)x_n$, we have, by using Lemma 2.4,

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\quad = \bigl\|\bigl(1 - \beta_2^n\bigr)\bigl(T_1^n z_n - x^*\bigr) + \beta_2^n\bigl(T_2T_1^n z_n - x^*\bigr)\bigr\|^2 \\ &\quad = \bigl(1 - \beta_2^n\bigr)\bigl\|T_1^n z_n - x^*\bigr\|^2 + \beta_2^n\bigl\|T_2T_1^n z_n - x^*\bigr\|^2 \\ &\qquad - \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_2T_1^n z_n - T_1^n z_n\bigr\|^2 \\ &\quad \le \bigl\|T_1^n z_n - x^*\bigr\|^2 - \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_2T_1^n z_n - T_1^n z_n\bigr\|^2 \\ &\quad = \bigl(1 - \beta_1^n\bigr)\bigl\|z_n - x^*\bigr\|^2 + \beta_1^n\bigl\|T_1 z_n - x^*\bigr\|^2 \\ &\qquad - \beta_1^n\bigl(1 - \beta_1^n\bigr)\|z_n - T_1 z_n\|^2 - \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_1^n z_n - T_2T_1^n z_n\bigr\|^2 \\ &\quad \le \bigl\|z_n - x^*\bigr\|^2 - \beta_1^n\bigl(1 - \beta_1^n\bigr)\|z_n - T_1 z_n\|^2 \\ &\qquad - \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_1^n z_n - T_2T_1^n z_n\bigr\|^2. \end{aligned} \tag{4.4}$$

Similar to (3.18), it is easy to verify that

$$\begin{aligned} \bigl\|z_n - x^*\bigr\|^2 \le{}& \bigl\|x_n - x^*\bigr\|^2 + 2\lambda_n^2\bigl\|h\bigl(x^*\bigr) - x^*\bigr\|^2 \\ &+ 2\lambda_n(1 - \lambda_n)\bigl\|h\bigl(x^*\bigr) - x^*\bigr\| \cdot \bigl\|x_n - x^*\bigr\|. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we derive that

$$\begin{aligned} \bigl\|x_{n+1} - x^*\bigr\|^2 \le{}& \bigl\|x_n - x^*\bigr\|^2 - \beta_1^n\bigl(1 - \beta_1^n\bigr)\|z_n - T_1 z_n\|^2 \\ &- \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_1^n z_n - T_2T_1^n z_n\bigr\|^2 + 2\lambda_n^2\bigl\|h\bigl(x^*\bigr) - x^*\bigr\|^2 \\ &+ 2\lambda_n(1 - \lambda_n)\bigl\|h\bigl(x^*\bigr) - x^*\bigr\| \cdot \bigl\|x_n - x^*\bigr\|. \end{aligned} \tag{4.6}$$

Setting

$$\begin{aligned} &s_n = \bigl\|x_n - x^*\bigr\|, \qquad \gamma_n = \lambda_n\bigl[2 - \lambda_n\bigl(2 - \lambda_n\bigl(1 + 2L^2\bigr) - 2(1 - \lambda_n)\alpha\bigr)\bigr], \\ &\delta_n = \frac{1}{1 - \lambda_n(2 - \lambda_n(1 + 2L^2) - 2(1 - \lambda_n)\alpha)}\bigl[2\lambda_n\bigl\|h\bigl(x^*\bigr) - x^*\bigr\|^2 \\ &\qquad + 2(1 - \lambda_n)\bigl\langle h\bigl(x^*\bigr) - x^*, x_n - x^*\bigr\rangle\bigr], \\ &\eta_n = \beta_1^n\bigl(1 - \beta_1^n\bigr)\|z_n - T_1 z_n\|^2 + \beta_2^n\bigl(1 - \beta_2^n\bigr)\bigl\|T_1^n z_n - T_2T_1^n z_n\bigr\|^2, \end{aligned}$$

$$\alpha_n = 2\lambda_n^2\|h(x^*) - x^*\|^2 + 2\lambda_n(1-\lambda_n)\|h(x^*) - x^*\| \cdot \|x_n - x^*\|,$$

thus (4.3) and (4.6) can be rewritten in the following forms, respectively:

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n s_n, \tag{4.7}$$

$$s_{n+1} \le s_n - \eta_n + \alpha_n. \tag{4.8}$$

By using Lemma 2.5, in order to complete the proof, it suffices to show that $\eta_{n_k} \to 0$ ($k \to \infty$) implies that

$$\overline{\lim_{k\to\infty}} \delta_{n_k} \le 0.$$

In fact, noting condition (iii), we get from $\eta_{n_k} \to 0$, $\|z_{n_k} - T_1 z_{n_k}\| \to 0$ and $\|T_1^{n_k} z_{n_k} - T_2 T_1^{n_k} z_{n_k}\| \to 0$ all hold. Consequently, $\|x_{n_k} - T_1 x_{n_k}\| \to 0$ follows from the inequality

$$\begin{aligned}\|x_{n_k} - T_1 x_{n_k}\| &\le \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - T_1 z_{n_k}\| + \|T_1 z_{n_k} - T_1 x_{n_k}\| \\ &\le 2\|x_{n_k} - z_{n_k}\| + \|z_{n_k} - T_1 z_{n_k}\|\end{aligned}$$

and the fact that $\|x_n - z_n\| \to 0$ ($n \to \infty$) and hence $\omega(x_{n_k}) \subset \operatorname{Fix}(T_1)$ holds. On the other hand, using $\|z_{n_k} - T_1 z_{n_k}\| \to 0$, $\|T_1^{n_k} z_{n_k} - T_2 T_1^{n_k} z_{n_k}\| \to 0$ and the relation

$$\begin{aligned}\|x_{n_k} - T_2 x_{n_k}\| &\le \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - T_1^{n_k} z_{n_k}\| + \|T_1^{n_k} z_{n_k} - T_2 T_1^{n_k} z_{n_k}\| \\ &\quad + \|T_2 T_1^{n_k} z_{n_k} - T_2 z_{n_k}\| + \|T_2 z_{n_k} - T_2 x_{n_k}\| \\ &\le 2\|x_{n_k} - z_{n_k}\| + 2\|z_{n_k} - T_1 z_{n_k}\| + \|T_1^{n_k} z_{n_k} - T_2 T_1^{n_k} z_{n_k}\|,\end{aligned}$$

we conclude that $\|x_{n_k} - T_2 x_{n_k}\| \to 0$ and this means that $\omega(x_{n_k}) \subset \operatorname{Fix}(T_2)$ also holds. Thus we have proved that

$$\omega(x_{n_k}) \subset \operatorname{Fix}(T_1) \cap \operatorname{Fix}(T_2). \tag{4.9}$$

Moreover, we have

$$\overline{\lim_{k\to\infty}} \langle h(x^*) - x^*, x_n - x^* \rangle \le 0,$$

and hence

$$\overline{\lim_{k\to\infty}} \delta_{n_k} \le 0. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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