Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty

A Khastan[1], Juan J Nieto[2,3*] and R Rodríguez-López[2]

*Correspondence: juanjose.nieto.roig@usc.es
[2]Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela, 15782, Spain
[3]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
We study the existence of solution for nonlinear fuzzy differential equations of fractional order involving the Riemann-Liouville derivative.

Keywords: fuzzy fractional differential equations; Riemann-Liouville fractional derivative; Schauder fixed point

1 Introduction

The Schauder Fixed-Point Theorem is one of the most celebrated results in Fixed-Point Theory and it states that *any compact convex nonempty subset of a normed space has the fixed-point property* (Schauder, 1930; Theorem 2.3.7 in [1]). It is also valid in locally convex spaces (Tychonoff, 1935; Theorem 2.3.8 in [1]). Recently, this Schauder fixed-point theorem has been generalized to semilinear spaces [2].

As explained in detail in many works in the field of fuzzy differential equations, uncertainty has to be considered in the formulation of a mathematical model for a better adequacy, due to the imprecision inherent to the information available or the behavior of the dynamical system itself (see, for instance, [3]). For this reason, the construction of models which try to be faithful to a certain real process involves, in many occasions, the consideration of fuzzy differential equations.

On the other hand, the subject of fractional calculus is not a recent topic, since many interesting questions concerning its main concepts and properties have been discussed since the end of the seventeenth century, with the contribution of mathematicians such as Leibnitz, Euler, Laplace, Lacroix, Fourier, Liouville or Riemann, among others (for details, see the introduction of [4] and other monographs on fractional calculus [5, 6]). The main references on fractional calculus also point out the power and usefulness of this topic to the modeling of phenomena in a wide range of scientific fields. The complexity of some processes in the physical world can be reproduced more faithfully with the help of fractional order models better than using classical integer order models, for instance, in electromagnetism, astrophysics, diffusion, material theory, chemistry, control theory, wave propagation, signal theory, electricity and thermodynamics [5]. Some theoretical aspects on the existence and uniqueness results for fractional differential equations have been considered by some authors [7–9].

Agarwal *et al.* have proposed the concept of the solution of fuzzy fractional differential equations in [10]. Arshad and Lupulescu [11] have deduced some existence and uniqueness results for fuzzy fractional differential equations under Riemann-Liouville derivative. Allahviranloo *et al.* have presented the explicit solutions of fuzzy fractional differential equations and some related results in [12, 13]. Some existence and uniqueness results for fuzzy fractional integral equations and fuzzy fractional integro-differential equations have been proposed in [14, 15].

In this paper, we consider nonlinear fuzzy fractional differential equations of the form

$$D^q u = f(t,u),$$

where $0 < q < 1$ and D^q is the Riemann-Liouville fractional derivative and $u(t)$ is a fuzzy real number for each $t \in (0,a]$, $a > 0$. We present some conditions to obtain a solution.

The paper is organized as follows. In Section 2, we recall the definitions of fuzzy fractional integral and derivative and related properties used in the paper. In Section 3, we present sufficient conditions to have at least a solution.

2 Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper, see for example [16].

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis, that is, maps $u : \mathbb{R} \to [0,1]$ satisfying the following properties:

(i) u is normal, *i.e.*, there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,
(ii) u is a convex fuzzy set (*i.e.* $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}$, $\forall t \in [0,1]$, $s, r \in \mathbb{R}$),
(iii) u is upper semicontinuous on $\mathbb{R}$,
(iv) $\mathrm{cl}\{s \in \mathbb{R} \mid u(s) > 0\}$ is compact where cl denotes the closure of a subset.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$ and $[u]^0 = \mathrm{cl}\{s \in \mathbb{R} \mid u(s) > 0\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval for all $0 \leq \alpha \leq 1$ and any $u \in \mathbb{R}_F$. The notation

$$[u]^\alpha = \left[\underline{u}^\alpha, \overline{u}^\alpha\right]$$

denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$ where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$. This is a consequence of Zadeh's Extension Principle [16].

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} d_H\left([u]^\alpha, [v]^\alpha\right) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}^\alpha - \underline{v}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right|\right\}.$$

The following properties are well known:

$$D(u+w,v+w)=D(u,v),\quad \forall u,v,w\in\mathbb{R}_F,$$

$$D(ku,kv)=|k|D(u,v),\quad \forall k\in\mathbb{R},u,v\in\mathbb{R}_F,$$

$$D(\lambda u,\mu u)=|\lambda-\mu|D(u,\tilde{0}),\quad \forall\lambda,\mu\geq 0,u\in\mathbb{R}_F,$$

$$D(u+v,w+e)\leq D(u,w)+D(v,e),\quad \forall u,v,w,e\in\mathbb{R}_F,$$

and $(\mathbb{R}_F,D)$ is a complete metric space.

The concept of a semi-linear space and similar concepts were already considered, for instance, in [17]. A semilinear metric space is a semilinear space S with a metric $d:S\times S\to\mathbb{R}_+$ which is translation invariant and positively homogeneous, that is,

- $d(a+c,b+c)=d(a,b)$,
- $d(\lambda a,\lambda b)=\lambda d(a,b)$, for all $\lambda\geq 0$,

for all $a,b,c\in S$ and $\lambda\geq 0$. In this case, we can define a norm on S by $\|x\|=d(x,\tilde{0})$, where $\tilde{0}$ is the zero element in S. If S is a semilinear metric space, then addition and scalar multiplication on S are continuous. If S is a complete metric space, then we say that S is a semilinear Banach space. For example, the set of fuzzy real numbers is not a vector space and hence it cannot be Banach space. The set of continuous functions from the real compact interval $[0,1]$ into the set of fuzzy real numbers is a semilinear Banach space. We say semilinear space S has cancellation property if $a+b=c+b$ implies $a=c$ for $a,b,c\in S$.

Let $a>0$. We denote by $C((0,a],\mathbb{R}_F)$ the space of all continuous fuzzy functions defined on $(0,a]$. Now, let $r\geq 0$. We define

$$C_r([0,a],\mathbb{R}_F)=\{u\in C((0,a],\mathbb{R}_F);u_r\in C([0,a],\mathbb{R}_F)\},$$

where $u_r(t)=t^r u(t)$, $t\in(0,a]$. Obviously, $C_r([0,a],\mathbb{R}_F)$ is a complete metric space with respect to the metric

$$h_r(u,v)=\max_{t\in[0,a]}t^r D(u(t),v(t)),\quad u,v\in C_r([0,a],\mathbb{R}_F).$$

We denote $h_r(u,\tilde{0})$ by $\|u\|_r$, which is not a norm in the classical sense, since $C_r([0,a],\mathbb{R}_F)$ is not a vector space. We point out that $C_0([0,a],\mathbb{R}_F)=C([0,a],\mathbb{R}_F)$. We define $\mathbb{R}_F^c$ as the space of fuzzy sets $u\in\mathbb{R}_F$ with the property that the function $\alpha\mapsto[u]^\alpha$ is continuous with respect to the Hausdorff metric on $[0,1]$. It is well known that $(\mathbb{R}_F^c,D)$ is a complete metric space [18]. If the functions take values in $\mathbb{R}_F^c$, we get the sets $C_r([0,1],\mathbb{R}_F^c)$, $r\geq 0$.

Definition 2.1 ([18]) A subset $A\subseteq\mathbb{R}_F^c$ is said to be compact-supported if there exists a compact set $K\subseteq\mathbb{R}$ such that $[y]^0\subseteq K$ for all $y\in A$.

Definition 2.2 ([18]) A subset $A\subseteq\mathbb{R}_F^c$ is said to be level-equicontinuous at $\alpha_0\in[0,1]$ if for all $\varepsilon>0$, there exists $\delta>0$ such that

$$|\alpha-\alpha_0|<\delta\quad\text{implies}\quad D([y]^\alpha,[y]^{\alpha_0})<\varepsilon,\quad\text{for all } y\in A.$$

A is level-equicontinuous on $[0,1]$ if A is level-equicontinuous at α for all $\alpha\in[0,1]$.

Theorem 2.3 ([18]) *Let A be a compact-supported subset of $\mathbb{R}_F^c$. Then the following assertions are equivalent*:

- *A is a relatively compact subset of* $(\mathbb{R}_F^c, D)$,
- *A is level-equicontinuous on* $[0,1]$.

In fact, if A is relatively compact in $(\mathbb{R}_F^c, D)$, then A is compact-supported and also level-equicontinuous on $[0,1]$. Conversely, if A is compact-supported in $\mathbb{R}_F^c$ and level-equicontinuous on $[0,1]$, then A is relatively compact in $(\mathbb{R}_F^c, D)$.

Definition 2.4 ([18]) A continuous function $f : [0; a] \times \mathbb{R}_F^c \to \mathbb{R}_F^c$ is said to be compact if for every subinterval $I \subseteq [0, a]$ and every bounded subset $A \subseteq \mathbb{R}_F^c$, then $f(I \times A)$ is relatively compact in $\mathbb{R}_F^c$.

Let $P_k(\mathbb{R})$ denote the family of all nonempty compact convex subsets of $\mathbb{R}$. $P_k(\mathbb{R})$ is endowed with the topology generated by the Hausdorff metric d_H.

Definition 2.5 A mapping $F : I \to \mathbb{R}_F$ is strongly measurable if, for all $\alpha \in [0,1]$, the set-valued mapping $F_\alpha : I \to P_k(\mathbb{R}^n)$ defined by the following:

$$F_\alpha(t) = \left[F(t)\right]^\alpha, \quad t \in I,$$

is Lebesgue measurable.

Definition 2.6 Let $F : I \to \mathbb{R}_F$. The integral of F over I, denoted by $\int_I F(t)\,dt$, is defined level-wise by the following expression:

$$\left[\int_I F(t)\,dt\right]^\alpha = \int_I F_\alpha(t)\,dt = \left\{\int_I f(t)\,dt \,\middle|\, f : I \to \mathbb{R} \text{ is a measurable selection for } F_\alpha\right\}$$

for all $0 < \alpha \leq 1$.

A function $F : I \to \mathbb{R}_F$ is called integrably bounded if there exists an integrable function $h : I \to \mathbb{R}_+$ such that $D(F_0(t), \tilde{0}) \leq h(t)$, for all $t \in I$. A strongly measurable and integrably bounded mapping $F : I \to \mathbb{R}_F$ is said to be integrable over I if $\int_I F(t)\,dt \in \mathbb{R}_F$.

Corollary 2.7 *If $F : I \to \mathbb{R}_F$ is continuous, then it is integrable.*

We denote by $L^1(I, \mathbb{R}_F)$ the space of Lebesgue integrable functions from I to $\mathbb{R}_F$.

Theorem 2.8 ([19]) *Let $F, G : I \to \mathbb{R}_F$ be integrable and $\lambda \in \mathbb{R}$. Then*

(i) $\int_I (F + G) = \int_I F + \int_I G$,
(ii) $\int_I \lambda F = \lambda \int_I F$,
(iii) *$D(F, G)$ is integrable on I*,
(iv) $D(\int_I F, \int_I G) \leq \int_I D(F, G)$.

Theorem 2.9 ([2], Schauder Fixed-Point Theorem for Semilinear Spaces) *Let B be a nonempty, closed, bounded and convex subset of a semilinear Banach space S having the cancellation property and suppose $P : B \to B$ is a compact operator. Then P has at least one fixed point in B.*

Definition 2.10 Let $u \in C((0,a],\mathbb{R}_F) \cap L^1((0,a],\mathbb{R}_F)$. The fuzzy fractional integral of order $q > 0$ of u is defined as

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\,ds, \quad t \in (0,a),$$

provided the integral in the right-hand side is defined for a.e. $t \in (0,a)$. For $q = 1$ we obtain $I^1 u(t) = \int_0^t u(s)\,ds$; that is, the classical integral operator.

Remark 2.11 ([11]) Let $q \in (0,1)$. If $u \in C_r([0,a],\mathbb{R}_F)$ with $r < q$, then $I^q u \in C((0,a],\mathbb{R}_F)$ and $I^q u(0^+) = \tilde{0}$. If $u \in C_q([0,a],\mathbb{R}_F)$, then $I^q u$ is bounded at $t = 0$, whereas if $u \in C_r([0,a],\mathbb{R}_F)$ with $q < r < 1$, then we may expect $I^q u$ to be unbounded at $t = 0$. This is similar to the crisp case [7].

Proposition 2.12 ([11]) *If $u \in C((0,a],\mathbb{R}_F) \cap L^1((0,a],\mathbb{R}_F)$ and $p, q > 0$, then*

$$I^p I^q u = I^{p+q} u.$$

Example 2.13 ([11]) Let $u : [0,a] \to \mathbb{R}_F$ be a constant fuzzy function, $u(t) = c \in \mathbb{R}_F$, for $t \in [0,a]$. Then

$$I^q u(t) = \frac{1}{\Gamma(q+1)} t^q c.$$

Example 2.14 ([11]) Let $u : (0,a] \to \mathbb{R}_F$ be a fuzzy function given by $u(t) = ct^r$, where $c \in \mathbb{R}_F$ and $r > -1$. Then

$$I^q u(t) = \frac{\Gamma(r+1)}{\Gamma(r+q+1)} c t^{q+r}.$$

Definition 2.15 ([2]) Let $u \in C((0,a],\mathbb{R}_F) \cap L^1((0,a],\mathbb{R}_F)$ and $q \in (0,1)$. If the fuzzy function $t \mapsto \int_0^t (t-s)^{-q} u(s)\,ds$ is Hukuhara differentiable on $(0,a]$, then we define the fuzzy fractional derivative of order q of u at t by

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} u(s)\,ds,$$

which defines a fuzzy number $D^q u(t) \in \mathbb{R}_F$.

Remark 2.16 Obviously $D^q u(t) = \frac{d}{dt} I^{1-q} u(t)$ for $t \in (0,a]$. Also we have

$$D^q(cu)(t) = cD^q(u)(t), \quad \forall c \in \mathbb{R}_F,$$

and

$$D^q(u+v)(t) = D^q(u)(t) + D^q(v)(t).$$

Proposition 2.17 ([11]) *If $u \in C((0,a],\mathbb{R}_F) \cap L^1((0,a],\mathbb{R}_F)$ and $0 < q < 1$, then*

$$D^q I^q u = u.$$

Example 2.18 ([11]) Let $u : (0, a] \to \mathbb{R}_F$ be a constant fuzzy function, $u(t) = c \in \mathbb{R}_F$ for $t \in (0, a]$. Then

$$D^q u(t) = \frac{t^{-q}}{\Gamma(1-q)} c.$$

Example 2.19 ([11]) Let $u : (0, a] \to \mathbb{R}_F$ be a fuzzy function given by $u(t) = ct^r$ where $c \in \mathbb{R}_F$ and $r > -1$, $r \neq q - 1$. Then

$$D^q u(t) = \frac{\Gamma(r+1)}{\Gamma(r-q+1)} ct^{r-q}.$$

We note that $D^q ct^{q-1} = \tilde{0}$.

According to Definition 2.15 and Example 2.19, we obtain the following lemma.

Lemma 2.20 *Let $u \in C((0, a], \mathbb{R}_F) \cap L^1((0, a], \mathbb{R}_F)$ and $0 < q < 1$. Then the solutions of the fuzzy fractional differential equation*

$$D^q u = 0$$

are $u(t) = ct^{q-1}$, $c \in \mathbb{R}_F$.

3 Fuzzy fractional differential equations

Consider the fuzzy fractional differential equation

$$D^q u = f(t, u), \tag{1}$$

where $0 < q < 1$ and $f : [0, a] \times \mathbb{R}_F \to \mathbb{R}_F$ is a continuous fuzzy function on $(0, a] \times \mathbb{R}_F$.

Definition 3.1 A fuzzy function $u \in C((0, a], \mathbb{R}_F) \cap L^1((0, a], \mathbb{R}_F)$ with continuous fractional derivative $D^q u$ on $(0, a]$ is a solution of the fuzzy fractional differential equation (1) if

$$D^q u(t) = f\bigl(t, u(t)\bigr), \quad \text{for all } t \in (0, a].$$

Remark 3.2 We may apply the results in Section 2 to consider a fuzzy integral equation which allows to obtain a solution to Eq. (1). Indeed, if $u \in C([0, a], \mathbb{R}_F)$ is a solution to the fuzzy integral equation

$$u(t) = I^q f\bigl(t, u(t)\bigr)$$

and $f(t, u(t)) \in C((0, a], \mathbb{R}_F) \cap L^1((0, a), \mathbb{R}_F)$, then u is also a solution to Eq. (1).

Lemma 3.3 *If $u : [0, a] \to \mathbb{R}_F$ is continuous, then u is bounded.*

Proof If u is continuous, the function $\underline{u}^\alpha : [0, 1] \to \mathbb{R}$ and $\overline{u}^\alpha : [0, 1] \to \mathbb{R}$ are continuous functions on $[0, 1]$, and bounded. Then $D(u, \tilde{0}) \leq \max\{|\underline{u}^\alpha|, |\overline{u}^\alpha|\}$ is bounded. □

In the sequel, if $u \in C([0,1],\mathbb{R}_F^c)$ and $0 \le r < q < 1$, we define the operator $\mathcal{A} : C([0,1],\mathbb{R}_F^c) \to C([0,1],\mathbb{R}_F^c)$ by

$$[\mathcal{A}u](t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u_{-r}(s)\,ds = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-r} u(s)\,ds.$$

Lemma 3.4 *The operator $\mathcal{A}$ is well-defined and continuous on $C([0,1],\mathbb{R}_F^c)$.*

Proof First we show that $\mathcal{A}$ is well defined, *i.e.*, for fixed $u \in C([0,1],\mathbb{R}_F^c)$, we check that $\mathcal{A}u \in C([0,1],\mathbb{R}_F^c)$. In fact we prove that $\mathcal{A}u$ is uniformly continuous on $[0,1]$. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and M such that

$$D\big(u(s), \tilde{0}\big) \le M, \quad \forall s \in [0,1].$$

Then

$$\begin{aligned}
&D\big(\mathcal{A}u(t_1), \mathcal{A}u(t_2)\big) \\
&\quad= \frac{1}{\Gamma(q)} D\biggl(\int_0^{t_1} (t_1-s)^{q-1} s^{-r} u(s)\,ds, \int_0^{t_2} (t_2-s)^{q-1} s^{-r} u(s)\,ds\biggr) \\
&\quad= \frac{1}{\Gamma(q)} D\biggl(\int_0^{t_1} (t_1-s)^{q-1} s^{-r} u(s)\,ds, \int_0^{t_1} (t_2-s)^{q-1} s^{-r} u(s)\,ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r} u(s)\,ds\biggr) \\
&\quad\le \frac{1}{\Gamma(q)} \biggl[D\biggl(\int_0^{t_1} (t_1-s)^{q-1} s^{-r} u(s)\,ds, \int_0^{t_1} (t_2-s)^{q-1} s^{-r} u(s)\,ds\biggr) \\
&\qquad + D\biggl(\int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r} u(s)\,ds, \tilde{0}\biggr)\biggr] \\
&\quad\le \frac{1}{\Gamma(q)} \biggl[\int_0^{t_1} D\big((t_1-s)^{q-1} s^{-r} u(s), (t_2-s)^{q-1} s^{-r} u(s)\big)\,ds \\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r} D\big(u(s), \tilde{0}\big)\,ds\biggr] \\
&\quad= \frac{1}{\Gamma(q)} \biggl[\int_0^{t_1} \big|(t_1-s)^{q-1} s^{-r} - (t_2-s)^{q-1} s^{-r}\big| D\big(u(s), \tilde{0}\big)\,ds \\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r} D\big(u(s), \tilde{0}\big)\,ds\biggr] \\
&\quad\le \frac{M}{\Gamma(q)} \biggl[\int_0^{t_1} \big|(t_1-s)^{q-1} s^{-r} - (t_2-s)^{q-1} s^{-r}\big|\,ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r}\,ds\biggr] \\
&\quad= \frac{M}{\Gamma(q)} \biggl[\int_0^{t_1} (t_1-s)^{q-1} s^{-r}\,ds - \int_0^{t_1} (t_2-s)^{q-1} s^{-r}\,ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r}\,ds\biggr] \\
&\quad= \frac{M}{\Gamma(q)} \biggl[\int_0^{t_1} (t_1-s)^{q-1} s^{-r}\,ds - \int_0^{t_2} (t_2-s)^{q-1} s^{-r}\,ds + 2\int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r}\,ds\biggr] \\
&\quad= M\biggl[\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}\big(t_1^{q-r} - t_2^{q-r}\big) + \frac{2}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} s^{-r}\,ds\biggr].
\end{aligned}$$

Therefore $D(\mathcal{A}u(t_1), \mathcal{A}u(t_2)) \to 0$ when $|t_1 - t_2| \to 0$.

Next, we prove continuity of $\mathcal{A}$. Let $v_n \to v$ as $n \to \infty$ in $C([0,1],\mathbb{R}_F^c)$, *i.e.*, $h_0(v_n, v) \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}
&h_0(\mathcal{A}v_n, \mathcal{A}v) \\
&\quad= \sup_{t\in[0,1]} D\big(\mathcal{A}v_n(t), \mathcal{A}v(t)\big) \\
&\quad= \sup_{t\in[0,1]} D\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}v_n(s)\,ds, \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}v(s)\,ds\right) \\
&\quad\le \frac{1}{\Gamma(q)}\sup_{t\in[0,1]}\int_0^t (t-s)^{q-1}s^{-r}D\big(v_n(s), v(s)\big)\,ds \\
&\quad\le \frac{1}{\Gamma(q)}h_0(v_n, v)\sup_{t\in[0,1]}\int_0^t (t-s)^{q-1}s^{-r}\,ds \\
&\quad= h_0(v_n, v)\sup_{t\in[0,1]}\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}t^{q-r} \\
&\quad= \frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}h_0(v_n, v).
\end{aligned}$$

Therefore $\mathcal{A}v_n \to \mathcal{A}v$ as $n \to \infty$ in $C([0,1],\mathbb{R}_F^c)$. □

Remark 3.5 If $G \subseteq C([0,1],\mathbb{R}_F^c)$ is bounded, then $\mathcal{A}(G)$ is bounded in $C([0,1],\mathbb{R}_F^c)$. Indeed, for $v \in G$ we have

$$D\big(\mathcal{A}v(t), \tilde{0}\big) \le \sup_{t\in[0,1]} D\big(v(t), \tilde{0}\big)\frac{\Gamma(1-r)}{\Gamma(1-r+q)}t^{q-r} \le \frac{\Gamma(1-r)}{\Gamma(1-r+q)}\sup_{t\in[0,1]} D\big(v(t), \tilde{0}\big).$$

Lemma 3.6 *If $G \subseteq C([0,1],\mathbb{R}_F^c)$ is bounded, then $\mathcal{A}(G)$ is equicontinuous in $C([0,1],\mathbb{R}_F^c)$.*

Proof Let $h_0(u, \tilde{0}) \le M$, for all $u \in G$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$. Then from the first part of the proof of the Lemma 3.4, we have, for all $u \in G$,

$$D\big(\mathcal{A}u(t_1), \mathcal{A}u(t_2)\big) \le M\left[\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}\big(t_1^{q-r} - t_2^{q-r}\big) + \frac{2}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}s^{-r}\,ds\right],$$

which tends to 0 as $|t_1 - t_2| \to 0$ uniformly in $u \in G$. Hence $\mathcal{A}(G)$ is equicontinuous in $C([0,1],\mathbb{R}_F^c)$. □

Remark 3.7 If $G \subseteq C([0,1],\mathbb{R}_F^c)$ is such that $\{v(s) \mid v \in G, s \in [0,1]\}$ is compact-supported in $\mathbb{R}_F^c$, then G is bounded. Indeed, there exists a compact set K in $\mathbb{R}$ such that $\{[v(s)]^0 \mid v \in G, s \in [0,1]\} \subseteq K$. On the other hand, for $v \in G$,

$$\begin{aligned}
h_0(v, \tilde{0}) &= \sup_{t\in[0,1]}\sup_{\alpha\in[0,1]} d_H\big([v(t)]^\alpha, \{0\}\big) \\
&= \sup_{t\in[0,1]}\sup_{\alpha\in[0,1]} d_H\big([\underline{v}^\alpha(t), \overline{v}^\alpha(t)], \{0\}\big) \\
&= \sup_{t\in[0,1]}\sup_{\alpha\in[0,1]} \max\big\{|\underline{v}^\alpha(t)|, |\overline{v}^\alpha(t)|\big\} \\
&\le \sup_{t\in[0,1]} \max\big\{|\underline{v}^0(t)|, |\overline{v}^0(t)|\big\}
\end{aligned}$$

$$\begin{aligned}&\le \sup_{t\in[0,1]} d_H\bigl(\bigl|[\nu(t)]^0\bigr|,\{0\}\bigr)\\&< +\infty.\end{aligned}$$

Then $G \subseteq C([0,1],\mathbb{R}_F^c)$ is bounded.

Lemma 3.8 *If $G \subseteq C([0,1],\mathbb{R}_F^c)$ is such that*

$$\bigl\{\nu(s) \mid \nu \in G, s \in [0,1]\bigr\},$$

is compact-supported and level-equicontinuous, then $\mathcal{A}(G)$ is relatively compact in $C([0,1], \mathbb{R}_F^c)$.

Proof By the Arzelà-Ascoli Theorem, we show that $\mathcal{A}(G)$ is an equicontinuous subset of $C([0,1],\mathbb{R}_F^c)$ and $\mathcal{A}(G)(t)$ is relatively compact in $\mathbb{R}_F^c$ for each $t \in [0,1]$. Since by Remark 3.7, G is bounded, using Lemma 3.6, it is sufficient to show $\mathcal{A}(G)(t)$ is relatively compact for each $t \in [0,1]$ in $\mathbb{R}_F^c$. By Theorem 2.3, it is equivalent to showing that $\mathcal{A}(G)(t)$ is a compact-supported subset of $\mathbb{R}_F^c$ and level-equicontinuous on $[0,1]$ for each $t \in [0,1]$. Since $\{\nu(s) \mid \nu \in G, s \in [0,1]\}$ is compact-supported, there exists a compact set $K \subseteq \mathbb{R}$ such that $[\nu(s)]^0 \subseteq K$ for all $s \in [0,1]$ and $\nu \in G$. Then, for all $\nu \in G$ and $t \in [0,1]$,

$$\begin{aligned}\bigl[\mathcal{A}(\nu)(t)\bigr]^0 &= \left[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}\nu(s)\,ds\right]^0\\&= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}\bigl[\nu(s)\bigr]^0\,ds\\&\subseteq \frac{K}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}\,ds\\&\subseteq \frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}K.\end{aligned}$$

Then $\mathcal{A}(G)(t)$ is compact-supported for each $t \in [0,1]$.

Now, to prove level-equicontinuity, take fixed $t \in [0,1]$ and $\varepsilon > 0$. If $w \in \mathcal{A}(G)(t)$, then $w = \mathcal{A}(\nu)(t)$, for some $\nu \in G$ so that

$$[w]^\alpha = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}\bigl[\nu(s)\bigr]^\alpha\,ds,\quad \alpha \in [0,1].$$

Therefore

$$d_H\bigl([w]^\alpha,[w]^\beta\bigr) \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{-r}d_H\bigl(\bigl[\nu(s)\bigr]^\alpha,\bigl[\nu(s)\bigr]^\beta\bigr)\,ds.$$

Since $\{\nu(s) \mid \nu \in G, s \in [0,1]\}$ is level-equicontinuous, then for given $\varepsilon\frac{\Gamma(-r+q+1)}{2\Gamma(-r+1)} > 0$, there exists $\delta > 0$ such that $|\alpha-\beta| < \delta$, then

$$d_H\bigl(\bigl[\nu(s)\bigr]^\alpha,\bigl[\nu(s)\bigr]^\beta\bigr) < \frac{\varepsilon\Gamma(-r+q+1)}{2\Gamma(-r+1)},\quad \nu \in G, s \in [0,1].$$

Hence

$$d_H([w]^\alpha,[w]^\beta) \le \frac{\varepsilon\Gamma(-r+q+1)}{2\Gamma(-r+1)}\frac{\Gamma(-r+1)}{\Gamma(-r+q+1)}t^{q-r} \le \frac{\varepsilon}{2}t^{q-r} < \varepsilon.$$

Then $\mathcal{A}(G)(t)$ is level-equicontinuous in $\mathbb{R}_F^c$ on $[0,1]$ for every $t \in [0,1]$. □

In the following, we consider

$$\mathcal{N} : \Omega \to C([0,a],\mathbb{R}_F^c),$$
$$\mathcal{N}u(t) = t^r f(t,u(t)), \quad t \in [0,a],$$

where

$$\Omega = \{u \in C([0,a],\mathbb{R}_F^c) \mid h_0(u,\tilde{0}) \le R\}.$$

The operator $\mathcal{N}$ is well-defined if $(t,u) \to t^r f(t,u)$ is a continuous function in $[0,a] \times \mathbb{R}_F^c$ with values in $\mathbb{R}_F^c$.

We define f_r as $f_r(t,u) = t^r f(t,u)$, $t \in [0,a]$. Now, let $S = \{x \in \mathbb{R}_F^c \mid D(x,\tilde{0}) \le R\}$. Then we have the following result.

Lemma 3.9 *Suppose that $f_r : [0,a] \times \mathbb{R}_F^c \to \mathbb{R}_F^c$ is uniformly continuous and bounded in $[0,a] \times S$. Then the operator $\mathcal{N}$ is continuous and bounded in $C([0,a],\mathbb{R}_F^c)$.*

Proof Let the sequence $u_n \to u$, as $n \to \infty$ in $C([0,a],\mathbb{R}_F^c)$ where $u_n, u \in \Omega$. Then for a given $\varepsilon > 0$ by the uniform continuity of f_r in $[0,a] \times S$, there exists $\delta > 0$ such that for $(t,x),(s,y) \in [0,a] \times S$,

$$H((t,x),(s,y)) = |t-s| + D(x,y) < \delta \quad \text{implies} \quad D(t^r f(t,x), s^r f(s,y)) < \varepsilon.$$

Now, given $\delta > 0$, since $u_n \to u$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that, for $n \ge n_0$, we have $h_0(u_n,u) < \delta$, *i.e.*, $\sup_{t\in[0,a]} D(u_n(t),u(t)) < \delta$. Then $H((t,u_n(t)),(t,u(t))) = D(u_n(t),u(t)) < \delta$, $\forall t \in [0,a]$, $\forall n \ge n_0$, so that $D(f_r(t,u_n(t)), f_r(t,u(t))) < \varepsilon$, $\forall t \in [0,a]$, $n \ge n_0$, and

$$\begin{aligned} h_0(\mathcal{N}u_n,\mathcal{N}u) &= \sup_{t\in[0,a]} D(\mathcal{N}u_n(t),\mathcal{N}u(t)) \\ &= \sup_{t\in[0,a]} D(f_r(t,u_n(t)), f_r(t,u(t))) \\ &\le \varepsilon. \end{aligned}$$

This proves that $\mathcal{N}u_n \to \mathcal{N}u$ in $C([0,a],\mathbb{R}_F^c)$. On the other hand, if B is bounded in Ω, then $h_0(u,\tilde{0}) \le M$, $\forall u \in B$, *i.e.*, $\sup_{t\in[0,a]} D(u(t),\tilde{0}) \le M$. Then $h_0(\mathcal{N}u,\tilde{0}) = \sup_{t\in[0,a]} D(\mathcal{N}u(t), \tilde{0}) = \sup_{t\in[0,a]} D(f_r(t,u(t)),\tilde{0})$, $\forall u \in B$. Since f_r is bounded in $[0,a] \times S$, then there exists a $K > 0$ such that $h_0(\mathcal{N}u,\tilde{0}) \le K$, $\forall u \in B$. Then $\mathcal{N}(B)$ is bounded. □

Lemma 3.10 *If $\{f_r(s,x) \mid (s,x) \in [0,a] \times S\}$ is compact-supported and level-equicontinuous, then*

$$\{u(s) \mid u \in N(\Omega), s \in [0,a]\}$$

is relatively compact.

Proof Since

$$\begin{aligned}\{f_r(s,u(s)) \mid u \in \Omega, s \in [0,a]\} &= \{(\mathcal{N}u)(s) \mid u \in \Omega, s \in [0,a]\} \\ &= \{u(s) \mid u \in \mathcal{N}(\Omega), s \in [0,a]\}\end{aligned}$$

is compact-supported and level-equicontinuous, it is relatively compact. □

Lemma 3.11 *Let $f_r(t,x)$ be a continuous mapping on $[0,a] \times S$. Then it is compact on $[0,1] \times S$ if and only if the set $\{t^r f(t,x) \mid t \in [0,a], x \in S\}$ is compact-supported and level-equicontinuous.*

Proof First, let $f_r(t,x)$ be continuous and compact on $[0,a] \times S$. Then by Theorem 2.3, $\{f_r(t,x) \mid t \in [0,a], x \in S\}$ is compact-supported and level-equicontinuous.

Now let $\{f_r(t,x) \mid t \in [0,a], x \in S\}$ is compact-supported and level-equicontinuous. Again by Theorem 2.3, it is relatively compact. Hence f_r is compact on $[0,a] \times S$. Since for any bounded set $B \subseteq [0,a] \times S$, the set $\{f_r(t,x) \mid (t,x) \in B\}$ is relatively compact. □

Definition 3.12 $\mathcal{T} : C([0,1],\mathbb{R}_F) \to C([0,1],\mathbb{R}_F)$ is a bounded operator if for every bounded B in $C([0,1],\mathbb{R}_F)$, $\mathcal{T}(B)$ is bounded in $C([0,1],\mathbb{R}_F)$.

In the following, we present a local existence theorem for the fuzzy fractional differential equation (1). For simplicity, in the rest of the paper, we shall often limit arguments to the choice $a = 1$.

Theorem 3.13 *Let $0 \leq r < q < 1$ and let $f : (0,1] \times \mathbb{R}_F^c \to \mathbb{R}_F^c$ be a given continuous fuzzy function in $(0,1] \times \mathbb{R}_F^c$. If $f_r : [0,1] \times \mathbb{R}_F^c \to \mathbb{R}_F^c$ is compact and uniformly continuous on $[0,1] \times \mathbb{R}_F^c$, then the fuzzy integral equation has at least one continuous solution defined on $[0,\delta]$, for a suitable $0 < \delta \leq 1$.*

Proof According to Remark 3.2, we need only consider the following fuzzy integral equation:

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds.$$

Define the set

$$\Omega = \{u \in C([0,1],\mathbb{R}_F^c) \mid h_0(u,\tilde{0}) \leq R\}.$$

It is easy to see that Ω is a closed, bounded and convex subset of the semilinear Banach space $C([0,1],\mathbb{R}_F^c)$. On the set Ω, we define the operator $\mathcal{T} : \Omega \to C([0,1],\mathbb{R}_F^c)$ by

$$(\mathcal{T}u)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds.$$

We claim that the operator $\mathcal{T}$ is continuous and compact. Indeed, the operator is the composition of two continuous and bounded operators $\mathcal{T} = \mathcal{A} \circ \mathcal{N}$, where

$$\mathcal{N}u(t) = t^r f(t,u(t)), \quad t \in [0,1],$$

and

$$\mathcal{A}v(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-r} v(s)\, ds, \quad v \in C\big([0,1], \mathbb{R}_F^c\big).$$

The operator $\mathcal{T}$ is well defined since it is the composition of $\mathcal{A}$ and $\mathcal{N}$. Since f_r is continuous and compact, by Lemma 3.11, $\{f_r(t,u) \mid t \in [0,1], u \in S\}$ is compact-supported and level-equicontinuous. Then by Lemma 3.10, $\{v(s) \mid v \in N(\Omega), s \in [0,1]\}$ is compact-supported and level-equicontinuous. Therefore, by Lemma 3.8, $\mathcal{A}(\mathcal{N}(\Omega))$ is relatively compact in $C([0,1], \mathbb{R}_F^c)$. Then operator $\mathcal{T}$ is compact on Ω.

Moreover, from Example 2.14, we have, for $0 \le t \le \delta \le 1$,

$$\begin{aligned} D\big(\mathcal{A}v(t), \chi_{\{0\}}\big) &\le \sup_{t \in [0,\delta]} D\big(v(t), \chi_{\{0\}}\big) \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{-r}\, ds \\ &\le \frac{\Gamma(1-r)}{\Gamma(1-r+q)} \delta^{q-r} \|v\|_0. \end{aligned}$$

Therefore, we have

$$\|\mathcal{A}v\|_0 \le \varepsilon \|v\|_0,$$

where we may assume $\varepsilon > 0$ to be as small as we want by shrinking $\delta > 0$. Now, fix B_ρ as a domain of the operator $\mathcal{T}$, where $B_\rho = \{v \in C([0,\delta], \mathbb{R}_F^c) : \|v\|_0 \le \rho\}$, which is a convex, bounded, and closed subset of the complete metric space $C([0,\delta], \mathbb{R}_F^c)$.

For $\delta > 0$ sufficiently small, we have

$$\mathcal{T}(B_\rho) \subseteq B_\rho.$$

Theorem 2.9 ensures that operator $\mathcal{T}$ has at least one fixed point. In consequence, Eq. (1) has at least one continuous solution u defined on $[0,\delta]$, where $\delta > 0$ and $\delta \le 1$. □

Corollary 3.14 *Under the conditions of Theorem* 3.13 *and assuming that* $f(\cdot, v(\cdot)) \in L^1((0,1), \mathbb{R}_F^c)$, *for every* $v \in C([0,1], \mathbb{R}_F^c)$, *then the fuzzy fractional differential equation* (1) *has at least a continuous solution defined on* $[0,\delta]$, *for a suitable* $0 < \delta \le 1$.

Proof If $u \in C([0,\delta], \mathbb{R}_F^c)$ is a solution to the fuzzy integral equation

$$u(t) = I^q f\big(t, u(t)\big),$$

using that $f : (0,\delta] \times \mathbb{R}_F^c \longrightarrow \mathbb{R}_F^c$ is continuous and $f(\cdot, v(\cdot)) \in L^1((0,\delta), \mathbb{R}_F^c)$, for every $v \in C([0,\delta], \mathbb{R}_F^c)$, then it is clear that $f(t, u(t)) \in C((0,\delta], \mathbb{R}_F) \cap L^1((0,\delta), \mathbb{R}_F)$ and u is a solution to Eq. (1) in $[0,\delta]$. □

Remark 3.15 If f is Lipchitz continuous in the second variable u, then in Theorem 3.13, one has uniqueness of the solution by using the classical Banach contraction fixed-point theorem. Note that Lipchitz continuity implies uniform continuity.

Theorem 3.16 *Let $f : (0,1] \times \mathbb{R}_F \longrightarrow \mathbb{R}_F$ be a given continuous fuzzy function in $(0,1] \times \mathbb{R}_F$. If f is a Lipschitz continuous function in the second variable on $[0,1] \times \mathbb{R}_F$, that is,*

$$D\big(f(t,u), f(t,v)\big) \leq KD(u,v), \quad t \in [0,1], u, v \in \mathbb{R}_F,$$

then the fuzzy fractional integral equation $u(t) = I^q f(t,u(t))$ has a unique solution defined on $[0,\delta]$, for a suitable $0 < \delta \leq 1$.

Proof Similar to proof of Theorem 3.13, we define $\mathcal{T} = \mathcal{A} \circ \mathcal{N}$. Then $\mathcal{T}$ is Lipschitz continuous and for $\delta > 0$ small, $\mathcal{T}$ is a contraction map. □

Corollary 3.17 *Under the conditions of Theorem* 3.16 *and assuming that $f(\cdot, v(\cdot)) \in L^1((0,1), \mathbb{R}_F)$, for every $v \in C([0,1], \mathbb{R}_F)$, the fuzzy fractional differential equation* (1) *has at least a continuous solution defined on $[0,\delta]$, for a suitable $0 < \delta \leq 1$.*

Remark 3.18 As indicated in results of [11], we cannot expect uniqueness for such solutions in general. Consider the equation

$$D^q u = u^r,$$

with $0 < r, q < 1$, which admits two solutions $u = \tilde{0}$ and

$$u(t) = kt^{\frac{q}{1-r}},$$

where

$$k = \left(\frac{\Gamma(\mu)}{\Gamma(\mu - q)}\right)^{\frac{1}{r-1}},$$

with $\mu = 1 + \frac{q}{1-r}$ as we see from Example 2.19. It is easy to check that $u(t) = [0,k] \cdot t^{\frac{q}{1-r}}$ is also a solution to this problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, AK, JJN, and RRL, contributed to each part of this study equally and read and approved the final version of the manuscript.

Author details
[1]Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran. [2]Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela, 15782, Spain. [3]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

Acknowledgements
This work has been partially supported by Ministerio de Economía y Competitividad (Spain), project MTM2010-15314, and co-financed by the European Community fund FEDER. This research was completed during the visit of the first author to the USC.

Received: 16 October 2013 Accepted: 5 January 2014 Published: 22 Jan 2014

References

1. Smart, DR: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
2. Agarwal, RP, Arshad, S, O'Regan, D, Lupulescu, V: A Schauder fixed point theorem in semilinear spaces and applications. Fixed Point Theory Appl. **2013**, 306 (2013). doi:10.1186/1687-1812-2013-306
3. Bede, B, Tenali, GB, Lakshmikantham, V: Perspectives of fuzzy initial value problems. Commun. Appl. Anal. **11**, 339-358 (2007)
4. Abbas, S, Benchohra, M, N'Guerekata, GM: Topics in Fractional Differential Equations. Developments in Mathematics, vol. 27. Springer, New York (2012)
5. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
6. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
7. Belmekki, M, Nieto, JJ, Rodríguez-López, R: Existence of periodic solutions for a nonlinear fractional differential equation. Bound. Value Probl. **2009**, Art. ID 324561 (2009)
8. Bonilla, B, Rivero, M, Rodríguez-Germà, L, Trujillo, JJ: Fractional differential equations as alternative models to nonlinear differential equations. Appl. Math. Comput. **187**, 79-88 (2007)
9. Delbosco, D, Rodino, L: Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. **204**, 609-625 (1996)
10. Agarwal, RP, Lakshmikantham, V, Nieto, JJ: On the concept of solution for fractional differential equations with uncertainty. Nonlinear Anal. **72**, 2859-2862 (2010)
11. Arshad, S, Lupulescu, V: On the fractional differential equations with uncertainty. Nonlinear Anal. **74**, 3685-3693 (2011)
12. Allahviranloo, T, Salahshour, S, Abbasbandy, S: Explicit solutions of fractional differential equations with uncertainty. Soft Comput. **16**, 297-302 (2012)
13. Salahshour, S, Allahviranloo, T, Abbasbandy, S, Baleanu, D: Existence and uniqueness results for fractional differential equations with uncertainty. Adv. Differ. Equ. **112**, 1-12 (2012)
14. Agarwal, RP, Arshad, S, O'Regan, D, Lupulescu, V: Fuzzy fractional integral equations under compactness type condition. Fract. Calc. Appl. Anal. **15**, 572-590 (2012)
15. Alikhani, R, Bahrami, F: Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations. Commun. Nonlinear Sci. Numer. Simul. **18**, 2007-2017 (2013)
16. Diamond, P, Kloeden, P: Metric Spaces of Fuzzy Sets. World Scientific, Singapore (1994)
17. Godini, G: A framework for best simultaneous approximation, normed almost linear spaces. J. Approx. Theory **43**, 338-358 (1985)
18. Román-Flores, H, Rojas-Medar, M: Embedding of level-continuous fuzzy sets on Banach spaces. Inf. Sci. **144**, 227-247 (2002)
19. Kaleva, O: Fuzzy differential inclusions. Fuzzy Sets Syst. **24**, 301-317 (1987)

10.1186/1687-1812-2014-21
Cite this article as: Khastan et al.: Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty. *Fixed Point Theory and Applications* 2014, 2014:21