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Some fixed point theorems concerning F-contraction in complete metric spaces

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Abstract
In this paper, we extend the result of Wardowski (Fixed Point Theory Appl. 2012:94, 2012) by applying some weaker conditions on the self map of a complete metric space and on the mapping F, concerning the contractions defined by Wardowski. With these weaker conditions, we prove a fixed point result for F-Suzuki contractions which generalizes the result of Wardowski.
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1 Introduction and preliminaries

Throughout this article, we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}_+$ the set of all positive real numbers, and by $\mathbb{N}$ the set of all natural numbers.

In 1922, Polish mathematician Banach [1] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. It is one of the fundamental results in fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [2–9] and references therein). Subsequently, in 1962, M Edelstein proved the following version of the Banach contraction principle.

Theorem 1.1 [10] *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that $d(Tx, Ty) < d(x, y)$ holds for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.*

In 2008, Suzuki [2] proved generalized versions of Edelstein's results in compact metric space as follows.

Theorem 1.2 [2] *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that for all $x, y \in X$ with $x \neq y$,*

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \Rightarrow \quad d(Tx, Ty) < d(x, y).$$

Then T has a unique fixed point in X.

In 2012, Wardowski [11] introduce a new type of contractions called F-contraction and prove a new fixed point theorem concerning F-contractions. In this way, Wardowski [11] generalized the Banach contraction principle in a different manner from the well-known results from the literature. Wardowski defined the F-contraction as follows.

Definition 1.3 Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall x,y \in X, \quad \big[d(Tx,Ty) > 0 \Rightarrow \tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big)\big], \tag{1}$$

where $F: \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, *i.e.* for all $x,y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions (F1)-(F3). For examples of the function F the reader is referred to [12] and [11].

Remark 1.4 From (F1) and (1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [11] stated a modified version of the Banach contraction principle as follows.

Theorem 1.5 [11] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Very recently, Secelean [12] proved the following lemma.

Lemma 1.6 [12] *Let $F: \mathbb{R}_+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then the following assertions hold*:

(a) *if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$, *then* $\lim_{n\to\infty} \alpha_n = 0$;

(b) *if* $\inf F = -\infty$, *and* $\lim_{n\to\infty} \alpha_n = 0$, *then* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

By proving Lemma 1.6, Secelean showed that the condition (F2) in Definition 1.3 can be replaced by an equivalent but a more simple condition,

(F2′) $\inf F = -\infty$

or, also, by

(F2″) there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Remark 1.7 Define $F_B: \mathbb{R}_+ \to \mathbb{R}$ by $F_B(\alpha) = \ln\alpha$, then $F_B \in \mathcal{F}$. Note that with $F = F_B$ the F-contraction reduces to a Banach contraction. Therefore, the Banach contractions are a particular case of F-contractions. Meanwhile there exist F-contractions which are not Banach contractions (see [11, 12]).

In this paper, we use the following condition instead of the condition (F3) in Definition 1.3:

(F3′) F is continuous on $(0,\infty)$.

We denote by $\mathfrak{F}$ the set of all functions satisfying the conditions (F1), (F2′), and (F3′).

Example 1.8 Let $F_1(\alpha) = \frac{-1}{\alpha}$, $F_2(\alpha) = \frac{-1}{\alpha} + \alpha$, $F_3(\alpha) = \frac{1}{1-e^{\alpha}}$, $F_4(\alpha) = \frac{1}{e^{\alpha}-e^{-\alpha}}$. Then $F_1, F_2, F_3, F_4 \in \mathfrak{F}$.

Remark 1.9 Note that the conditions (F3) and (F3′) are independent of each other. Indeed, for $p \geq 1$, $F(\alpha) = \frac{-1}{\alpha^p}$ satisfies the conditions (F1) and (F2) but it does not satisfy (F3), while it satisfies the condition (F3′). Therefore, $\mathfrak{F} \nsubseteq \mathcal{F}$. Again, for $a > 1$, $t \in (0, 1/a)$, $F(\alpha) = \frac{-1}{(\alpha+[\alpha])^t}$, where $[\alpha]$ denotes the integral part of α, satisfies the conditions (F1) and (F2) but it does not satisfy (F3′), while it satisfies the condition (F3) for any $k \in (1/a, 1)$. Therefore, $\mathcal{F} \nsubseteq \mathfrak{F}$. Also, if we take $F(\alpha) = \ln\alpha$, then $F \in \mathcal{F}$ and $F \in \mathfrak{F}$. Therefore, $\mathcal{F} \cap \mathfrak{F} \neq \emptyset$.

In view of Remark 1.9, it is meaningful to consider the result of Wardowski [11] with the mappings $F \in \mathfrak{F}$ instead $F \in \mathcal{F}$. Also, we define the F-Suzuki contraction as follows and we give a new version of Theorem 1.5.

Definition 1.10 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big),$$

where $F \in \mathfrak{F}$.

2 Main results

Theorem 2.1 *Let T be a self-mapping of a complete metric space X into itself. Suppose $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X, \quad \big[d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)\big].$$

Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.

Proof Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_1 = Tx_0, \qquad x_2 = Tx_1 = T^2x_0, \qquad \ldots, \qquad x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n \in \mathbb{N}. \tag{2}$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, the proof is complete. So, we assume that

$$0 < d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \tag{3}$$

For any $n \in \mathbb{N}$ we have

$$\tau + F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_{n-1}, x_n)\big),$$

i.e.,

$$F\big(d(Tx_{n-1}, Tx_n)\big) \leq F\big(d(x_{n-1}, x_n)\big) - \tau.$$

Repeating this process, we get

$$\begin{aligned} F\bigl(d(Tx_{n-1},Tx_n)\bigr) &\le F\bigl(d(x_{n-1},x_n)\bigr)-\tau \\ &= F\bigl(d(Tx_{n-2},Tx_{n-1})\bigr)-\tau \\ &\le F\bigl(d(x_{n-2},x_{n-1})\bigr)-2\tau \\ &= F\bigl(d(Tx_{n-3},Tx_{n-2})\bigr)-2\tau \\ &\le F\bigl(d(x_{n-3},x_{n-2})\bigr)-3\tau \\ &\vdots \\ &\le F\bigl(d(x_0,x_1)\bigr)-n\tau. \end{aligned} \tag{4}$$

From (4), we obtain $\lim_{n\to\infty} F(d(Tx_{n-1},Tx_n)) = -\infty$, which together with (F2′) and Lemma 1.6 gives $\lim_{n\to\infty} d(Tx_{n-1},Tx_n) = 0$, *i.e.*,

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{5}$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^\infty$ and $\{q(n)\}_{n=1}^\infty$ of natural numbers such that

$$p(n) > q(n) > n, \qquad d(x_{p(n)},x_{q(n)}) \ge \epsilon, \qquad d(x_{p(n)-1},x_{q(n)}) < \epsilon, \quad \forall n\in\mathbb{N}. \tag{6}$$

So, we have

$$\begin{aligned} \epsilon &\le d(x_{p(n)},x_{q(n)}) \le d(x_{p(n)},x_{p(n)-1}) + d(x_{p(n)-1},x_{q(n)}) \\ &\le d(x_{p(n)},x_{p(n)-1}) + \epsilon \\ &= d(x_{p(n)-1},Tx_{p(n)-1}) + \epsilon. \end{aligned}$$

It follows from (5) and the above inequality that

$$\lim_{n\to\infty} d(x_{p(n)},x_{q(n)}) = \epsilon. \tag{7}$$

On the other hand, from (5) there exists $N\in\mathbb{N}$, such that

$$d(x_{p(n)},Tx_{p(n)}) < \frac{\epsilon}{4} \quad \text{and} \quad d(x_{q(n)},Tx_{q(n)}) < \frac{\epsilon}{4}, \quad \forall n\ge N. \tag{8}$$

Next, we claim that

$$d(Tx_{p(n)},Tx_{q(n)}) = d(x_{p(n)+1},x_{q(n)+1}) > 0, \quad \forall n\ge N. \tag{9}$$

Arguing by contradiction, there exists $m\ge N$ such that

$$d(x_{p(m)+1},x_{q(m)+1}) = 0. \tag{10}$$

It follows from (6), (8), and (10) that

$$\begin{aligned}\epsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)})\\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)})\\ &= d(x_{p(m)}, Tx_{p(m)}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)}, Tx_{q(m)})\\ &< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} = \frac{\epsilon}{2}.\end{aligned}$$

This contradiction establishes the relation (9). Therefore, it follows from (9) and the assumption of the theorem that

$$\tau + F\big(d(Tx_{p(n)}, Tx_{q(n)})\big) \leq F\big(d(x_{p(n)}, x_{q(n)})\big), \quad \forall n \geq N. \tag{11}$$

From (F3′), (7), and (11), we get $\tau + F(\epsilon) \leq F(\epsilon)$. This contradiction shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x in X. Finally, the continuity of T yields

$$d(Tx, x) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n) = d\big(x^*, x^*\big) = 0.$$

Now, let us to show that T has at most one fixed point. Indeed, if $x, y \in X$ be two distinct fixed points of T, that is, $Tx = x \neq y = Ty$. Therefore,

$$d(Tx, Ty) = d(x, y) > 0,$$

then we get

$$F\big(d(x, y)\big) = F\big(d(Tx, Ty)\big) < \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big),$$

which is a contradiction. Therefore, the fixed point is unique. □

Theorem 2.2 *Let (X, d) be a complete metric space and $T : X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n=1}^{\infty}$ converges to x^*.*

Proof Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_1 = Tx_0, \qquad x_2 = Tx_1 = T^2 x_0, \qquad \ldots, \qquad x_{n+1} = Tx_n = T^{n+1} x_0, \quad \forall n \in \mathbb{N}. \tag{12}$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, the proof is complete. So, we assume that

$$0 < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{2} d(x_n, Tx_n) < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \tag{13}$$

For any $n \in \mathbb{N}$ we have

$$\tau + F\big(d\big(Tx_n, T^2x_n\big)\big) \le F\big(d(x_n, Tx_n)\big),$$

i.e.,

$$F\big(d(x_{n+1}, Tx_{n+1})\big) \le F\big(d(x_n, Tx_n)\big) - \tau.$$

Repeating this process, we get

$$\begin{aligned} F\big(d(x_n, Tx_n)\big) &\le F\big(d(x_{n-1}, Tx_{n-1})\big) - \tau \\ &\le F\big(d(x_{n-2}, Tx_{n-2})\big) - 2\tau \\ &\le F\big(d(x_{n-3}, Tx_{n-3})\big) - 3\tau \\ &\vdots \\ &\le F\big(d(x_0, Tx_0)\big) - n\tau. \end{aligned} \tag{14}$$

From (14), we obtain $\lim_{m\to\infty} F(d(x_n, Tx_n)) = -\infty$, which together with (F2′) and Lemma 1.6 gives

$$\lim_{m\to\infty} d(x_n, Tx_n) = 0. \tag{15}$$

Now, we claim that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \qquad d(x_{p(n)}, x_{q(n)}) \ge \epsilon, \qquad d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{16}$$

So, we have

$$\begin{aligned} \epsilon \le d(x_{p(n)}, x_{q(n)}) &\le d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\le d(x_{p(n)}, x_{p(n)-1}) + \epsilon \\ &= d(x_{p(n)-1}, Tx_{p(n)-1}) + \epsilon. \end{aligned}$$

It follows from (15) and the above inequality that

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{17}$$

From (15) and (17), we can choose a positive integer $N \in \mathbb{N}$ such that

$$\frac{1}{2} d(x_{p(n)}, Tx_{p(n)}) < \frac{1}{2}\epsilon < d(x_{p(n)}, x_{q(n)}), \quad \forall n \ge N.$$

So, from the assumption of the theorem, we get

$$\tau + F\big(d(Tx_{p(n)}, Tx_{q(n)})\big) \le F\big(d(x_{p(n)}, x_{q(n)})\big), \quad \forall n \ge N.$$

It follows from (12) that

$$\tau + F\bigl(d(x_{p(n)+1}, Tx_{q(n)+1})\bigr) \leq F\bigl(d(x_{p(n)}, x_{q(n)})\bigr), \quad \forall n \geq N. \tag{18}$$

From (F3′), (15), and (18), we get $\tau + F(\epsilon) \leq F(\epsilon)$. This contradiction shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point x^* in X. Therefore,

$$\lim_{n\to\infty} d\bigl(x_n, x^*\bigr) = 0. \tag{19}$$

Now, we claim that

$$\frac{1}{2}d(x_n, Tx_n) < d\bigl(x_n, x^*\bigr) \quad \text{or} \quad \frac{1}{2}d\bigl(Tx_n, T^2x_n\bigr) < d\bigl(Tx_n, x^*\bigr), \quad \forall n \in \mathbb{N}. \tag{20}$$

Again, assume that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_m, Tx_m) \geq d\bigl(x_m, x^*\bigr) \quad \text{and} \quad \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) \geq d\bigl(Tx_m, x^*\bigr). \tag{21}$$

Therefore,

$$2d\bigl(x_m, x^*\bigr) \leq d(x_m, Tx_m) \leq d\bigl(x_m, x^*\bigr) + d\bigl(x^*, Tx_m\bigr),$$

which implies that

$$d\bigl(x_m, x^*\bigr) \leq d\bigl(x^*, Tx_m\bigr). \tag{22}$$

It follows from (21) and (22) that

$$d\bigl(x_m, x^*\bigr) \leq d\bigl(x^*, Tx_m\bigr) \leq \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr). \tag{23}$$

Since $\frac{1}{2}d(x_m, Tx_m) < d(x_m, Tx_m)$, by the assumption of the theorem, we get

$$\tau + F\bigl(d\bigl(Tx_m, T^2x_m\bigr)\bigr) \leq F\bigl(d(x_m, Tx_m)\bigr).$$

Since $\tau > 0$, this implies that

$$F\bigl(d\bigl(Tx_m, T^2x_m\bigr)\bigr) < F\bigl(d(x_m, Tx_m)\bigr).$$

So, from (F1), we get

$$d\bigl(Tx_m, T^2x_m\bigr) < d(x_m, Tx_m). \tag{24}$$

It follows from (21), (23), and (24) that

$$\begin{aligned} d\bigl(Tx_m, T^2x_m\bigr) &< d(x_m, Tx_m) \\ &\leq d\bigl(x_m, x^*\bigr) + d\bigl(x^*, Tx_m\bigr) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) + \frac{1}{2}d\bigl(Tx_m, T^2x_m\bigr) \\ &= d\bigl(Tx_n, T^2x_n\bigr). \end{aligned}$$

This is a contradiction. Hence, (20) holds. So, from (20), for every $n \in \mathbb{N}$, either

$$\tau + F\bigl(d\bigl(Tx_n, Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(x_n, x^*\bigr)\bigr),$$

or

$$\tau + F\bigl(d\bigl(T^2x_n, Tx^*\bigr)\bigr) \leq F\bigl(d\bigl(Tx_n, x^*\bigr)\bigr) = F\bigl(d\bigl(x_{n+1}, x^*\bigr)\bigr)$$

holds. In the first case, from (19), (F2′), and Lemma 1.6, we obtain

$$\lim_{n\to\infty} F\bigl(d\bigl(Tx_n, Tx^*\bigr)\bigr) = -\infty.$$

It follows from (F2′) and Lemma 1.6 that $\lim_{n\to\infty} d(Tx_n, Tx^*) = 0$. Therefore,

$$d\bigl(x^*, Tx^*\bigr) = \lim_{n\to\infty} d\bigl(x_{n+1}, Tx^*\bigr) = \lim_{n\to\infty} d\bigl(Tx_n, Tx^*\bigr) = 0.$$

Also, in the second case, from (19), (F2′), and Lemma 1.6, we obtain

$$\lim_{n\to\infty} F\bigl(d\bigl(T^2x_n, Tx^*\bigr)\bigr) = -\infty.$$

It follows from (F2′) and Lemma 1.6 that $\lim_{n\to\infty} d(Tx_n, Tx^*) = 0$. Therefore,

$$d\bigl(x^*, Tx^*\bigr) = \lim_{n\to\infty} d\bigl(x_{n+2}, Tx^*\bigr) = \lim_{n\to\infty} d\bigl(T^2x_n, Tx^*\bigr) = 0.$$

Hence, x^* is a fixed point of T. Now let us show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ are two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$, then $d(x^*, y^*) > 0$. So, we have $0 = \frac{1}{2}d(x^*, Tx^*) < d(x^*, y^*)$ and from the assumption of the theorem, we obtain

$$F\bigl(d\bigl(x^*, y^*\bigr)\bigr) = F\bigl(d\bigl(Tx^*, Ty^*\bigr)\bigr) < \tau + F\bigl(d\bigl(Tx^*, Ty^*\bigr)\bigr) \leq F\bigl(d\bigl(x^*, y^*\bigr)\bigr),$$

which is a contradiction. Thus, the fixed point is unique. □

Example 2.3 Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} &S_1 = 1 \times 2, \qquad S_2 = 1 \times 2 + 2 \times 3, \qquad \ldots, \\ &S_n = 1 \times 2 + 2 \times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}, \qquad \ldots. \end{aligned}$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is complete metric space. Define the mapping $T : X \to X$ by $T(S_1) = S_1$ and $T(S_n) = S_{n-1}$ for every $n > 1$. Since

$$\lim_{n\to\infty} \frac{d(T(S_n), T(S_1))}{d(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - 2}{S_n - 2} = \frac{(n-1)n(n+1) - 6}{n(n+1)(n+2) - 6} = 1,$$

T is not a Banach contraction and a Suzuki contraction. On the other hand taking $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$, we obtain the result that T is an F-contraction with $\tau = 6$. To see this, let us consider the following calculation. First observe that

$$\frac{1}{2}d(S_n, TS_n) < d(S_n, S_m) \quad \Leftrightarrow \quad \big[(1 = n < m) \vee (1 \leq m < n) \vee (1 < n < m)\big].$$

For $1 = n < m$, we have

$$\begin{aligned} &\big|T(S_m) - T(S_1)\big| = |S_{m-1} - S_1| = 2 \times 3 + 3 \times 4 + \cdots + (m-1)m, \\ &|S_m - S_1| = 2 \times 3 + 3 \times 4 + \cdots + m(m+1). \end{aligned} \tag{25}$$

Since $m > 1$ and $\frac{-1}{2\times3+3\times4+\cdots+(m-1)m} < \frac{-1}{2\times3+3\times4+\cdots+m(m+1)}$, we have

$$\begin{aligned} &6 - \frac{1}{2 \times 3 + 3 \times 4 + \cdots + (m-1)m} + \big[2 \times 3 + 3 \times 4 + \cdots + (m-1)m\big] \\ &\quad < 6 - \frac{1}{2 \times 3 + 3 \times 4 + \cdots + m(m+1)} + \big[2 \times 3 + 3 \times 4 + \cdots + (m-1)m\big] \\ &\quad \leq -\frac{1}{2 + 3 + \cdots + m} + \big[2 \times 3 + 3 \times 4 + \cdots + (m-1)m\big] + m(m+1) \\ &\quad = -\frac{1}{2 + 3 + \cdots + m} + \big[2 \times 3 + 3 \times 4 + \cdots + (m-1)m + m(m+1)\big]. \end{aligned}$$

So, from (25), we get

$$6 - \frac{1}{|T(S_m) - T(S_1)|} + \big|T(S_m) - T(S_1)\big| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|.$$

For $1 \leq m < n$, similar to $1 = n < m$, we have

$$6 - \frac{1}{|T(S_m) - T(S_1)|} + \big|T(S_m) - T(S_1)\big| < -\frac{1}{|S_m - S_1|} + |S_m - S_1|.$$

For $1 < n < m$, we have

$$\begin{aligned} &\big|T(S_m) - T(S_n)\big| = n(n+1) + (n+1)(n+2) + \cdots + (m-1)m, \\ &|S_m - S_n| = (n+1)(n+2) + (n+2)(n+3) + \cdots + m(m+1). \end{aligned} \tag{26}$$

Since $m > n > 1$, we have

$$(m+1)m \geq (n+2)(n+1) = n(n+1) + 2(n+1) \geq n(n+1) + 6.$$

We know that $\frac{-1}{n(n+1)+(n+1)(n+2)+\cdots+(m-1)m} < \frac{-1}{(n+1)(n+2)+(n+2)(n+3)+\cdots+m(m+1)}$. Therefore

$$\begin{aligned} &6 - \frac{1}{n(n+1) + (n+1)(n+2) + \cdots + (m-1)m} \\ &\quad + \big[n(n+1) + (n+1)(n+2) + \cdots + (m-1)m\big] \end{aligned}$$

$$\begin{aligned}
&< 6 - \frac{1}{(n+1)(n+2) + (n+2)(n+3) + \cdots + m(m+1)} \\
&\quad + \big[n(n+1) + (n+1)(n+2) + \cdots + (m-1)m\big] \\
&= -\frac{1}{(n+1)(n+2) + (n+2)(n+3) \cdots + m(m+1)} \\
&\quad + 6 + n(n+1) + \big[(n+1)(n+2) + \cdots + (m-1)m\big] \\
&\leq -\frac{1}{(n+1)(n+2) + (n+2)(n+3) + \cdots + m(m+1)} \\
&\quad + m(m+1) + \big[(n+1)(n+2) + \cdots + (m-1)m\big] \\
&= -\frac{1}{(n+1)(n+2) + (n+2)(n+3) + \cdots + m(m+1)} \\
&\quad + \big[(n+1)(n+2) + \cdots + (m-1)m\big].
\end{aligned}$$

So from (26), we get

$$6 - \frac{1}{|T(S_m) - T(S_n)|} + \big|T(S_m) - T(S_n)\big| < -\frac{1}{|S_m - S_n|} + |S_m - S_n|.$$

Therefore $\tau + F(d(T(S_m), T(S_n))) \leq d(S_m, S_n)$ for all $m, n \in \mathbb{N}$. Hence T is an F-contraction and $T(S_1) = S_1$.

For $F_1(\alpha) = \ln(\alpha)$, $F_2(\alpha) = \ln(\alpha) + \alpha$, $F_3(\alpha) = \frac{-1}{\alpha} + \alpha$, and $F_4(\alpha) = \frac{-1}{\sqrt{\alpha + [\alpha]}} + \alpha$ in the above example, we compare the rate of convergence of the Banach contraction (F_1-contraction) and F-contractions for $F_2 \in \mathcal{F} \cap \mathfrak{F}$, $F_3 \in (\mathfrak{F} - \mathcal{F})$, and $F_4 \in (\mathcal{F} - \mathfrak{F})$ in Table 1.

Table 1 The generated iterations start from a point $x_0 = S_{30}$. C_F denotes $F(d(S_1, S_n)) - F(d(T(S_1), T(S_n)))$

n	x_n	C_{F_1}	C_{F_2}	C_{F_3}	C_{F_4}
3	7308	1.098612	13.09861	12.111111	12.12201
4	6552	0.727214	20.74721	20.02924	20.05196
5	5850	0.581922	30.58192	30.01161	30.02896
⋮	⋮	⋮	⋮	⋮	⋮
27	20	0.109231	756.1092	756.0000	756.0005
28	8	0.105388	812.1054	812.0000	812.0004
29	2	0.101807	870.1018	870.0000	870.0004
30	2	0.09846093	930.0983	930	930.0004
31	2	0.0923895	1056.092	**1056**	**1056**
32	2	0.08962648	1122.09	**1122**	**1122**
33	2	0.08702411	1190.087	**1190**	**1190**
⋮	⋮	⋮	⋮	⋮	⋮
314	2	0.00953902	98910.01	**98910**	**98910**
315	2	0.00950879	99540.01	**99540**	**99540**
316	2	0.00947875	**100172**	**100172**	**100172**
317	2	0.00944889	**100806**	**100806**	**100806**
318	2	0.00941922	**101442**	**101442**	**101442**
⋮	⋮	⋮	⋮	⋮	⋮
3×10^3	2	0.00099983	**9003000**	**9003000**	**9003000**
$n \to \infty$	$T(2) = 2$	tends to 0	$\geq \tau = 1$	$\geq \tau = 1$	$\geq \tau = 1$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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