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Some fixed/coincidence point theorems under (ψ, φ)-contractivity conditions without an underlying metric structure

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Abstract

In this paper we prove a coincidence point result in a space which does not have to satisfy any of the classical axioms that define a metric space. Furthermore, the ambient space need not be ordered and does not have to be complete. Then, this result may be applied in a wide range of different settings (metric spaces, quasi-metric spaces, pseudo-metric spaces, semi-metric spaces, pseudo-quasi-metric spaces, partial metric spaces, G-spaces, *etc.*). Finally, we illustrate how this result clarifies and improves some well-known, recent results on this topic.

1 Introduction

Fixed point theory plays a crucial role in nonlinear functional analysis since, among other reasons, fixed point results are used to prove the existence (and also uniqueness) of solution when solving various types of equations. The *Banach contraction principle* is considered to be the pioneering result of the fixed point theory, and it is the most celebrated result in this field. The simplicity of its proof and the possibility of attaining the fixed point by using successive approximations let this theorem become a very useful tool in analysis and in applied mathematics. The great significance of Banach's principle, and the reason it is possibly one of the most frequently cited fixed point theorems in all of analysis, lies in the fact that its proof contains elements of fundamental importance to the theoretical and practical treatment of mathematical equations. After the appearance of this result in Banach's thesis in 1922, a great number of extensions (in many occasions, as well-known as the original result, such as those by Krasnoselskii and Zabreiko, Edelstein, Browder, Schauder, Göhde, Kirk, and Caristi; a comprehensive study can be found in [1]) have been proved in various different frameworks (see [2, 3] in partial metric spaces, [4–7] in G-metric spaces, [8, 9] in fuzzy metric spaces, [10, 11] in intuitionistic fuzzy metric spaces, [12, 13] in probabilistic metric spaces and [14, 15] in Menger spaces).

In recent times, one of the most attractive research topics in fixed point theory is to prove the existence of a fixed point in metric spaces endowed with partial orders. An initial result in this direction was given by Turinici [16] in 1986. Following this line of research, Ran and Reurings [17] (and later Nieto and Rodríguez-López [18]) used a partial order on the ambient metric space to introduce a slightly different contractivity condition, which must be only verified by comparable points. Thus, they reported two versions of the Banach contraction principle in partially ordered sets and applied them to the study of some applications to matrix equations. Their proofs involved combining the ideas of the iterative technique in the contractive mapping principle with those in the monotone technique. This approach led to a very recent branch of this field, with applications to matrix equations and ordinary differential equations. The literature on this topic has exponentially risen in recent years. To mention some advances on this topic, we highlight the following ones. Firstly, in order to guarantee the existence and uniqueness of a solution of periodic boundary value problems, Gnana-Bhaskar and Lakshmikantham [19] (and, subsequently, Lakshmikantham and Ćirić [20]) proved, in 2006, the existence and uniqueness of a *coupled fixed point* (a notion introduced by Guo and Lakshmikantham in [21]) in the setting of partially ordered metric spaces by introducing the notion of *mixed monotone property*. Later, the notions of *tripled fixed point*, *quadruple fixed point* and *multidimensional fixed point* were introduced by Berinde and Borcut [22], by Karapinar and Luong [23] and by Berzig and Samet [24] (see also [25]), respectively.

But the two main ingredients of all extensions are, basically, the same that we can find in the Banach contraction principle: a complete metric space and a self-mapping verifying a contractive condition. Although modern versions use, in many cases, different kind of mappings, the more intensively studied condition is based on the idea that the distance between the images of any two points (comparable or not) is upper bound by the product of a constant (small enough) and the distance between those points. The main aim of this manuscript is to provide a result powerful enough to guarantee that a nonlinear operator T has, at least, a fixed point, even when we consider that a measure mapping does not have to be an underlying metric structure on the ambient space X and the binary relationship is not necessarily a partial order on X. To do this, we present a result which can be applied in the following adverse conditions: the framework is a set X provided with a preorder and a measure mapping $d : X \times X \to \mathbb{R}$ that does not necessarily verify any of the four classical properties of a metric space (in fact, it need not be one of the following metric structures: a metric, a pseudo-metric, a quasi-metric, a pseudo-quasi-metric or a semi-metric). Furthermore, d has not to be symmetric and the triangular inequality must only be verified by a kind of comparable points. Even if d would verify some of the classical properties of a metric, (X, d) would not be a complete space. In this setting, none of the theorems proved until now can be applied to guarantee that a nonlinear operator (even if it is a contractive mapping) has, at least, a fixed point. We illustrate our results with a particular example. Finally, we show that they extend and improve some well-known fixed point theorems.

2 Preliminaries

Preliminaries and notation about coincidence points can also be found in [25]. Let n be a positive integer. Henceforth, X will denote a nonempty set and X^n will denote the product space $X \times X \times \overset{n}{\cdots} \times X$. Throughout this manuscript, m and k will denote nonnegative integers and $i, j, s \in \{1, 2, \dots, n\}$. Unless otherwise stated, 'for all m' will mean 'for all $m \geq 0$' and 'for all i' will mean 'for all $i \in \{1, 2, \dots, n\}$'. In the sequel, let $F : X^N \to X$ and $T, g : X \to X$ be three mappings. For brevity, $T(x)$ will be denoted by Tx.

Definition 2.1 A *binary relation on* X is a nonempty subset $\mathcal{R}$ of $X \times X$. For simplicity, we will write $x \preccurlyeq y$ if $(x, y) \in \mathcal{R}$, and we will say that $\preccurlyeq$ is the binary relation. We will write

$x \prec y$ when $x \preccurlyeq y$ and $x \neq y$, and we will write $y \succcurlyeq x$ when $x \preccurlyeq y$. We will say that x and y are $\preccurlyeq$-*comparable* if $x \preccurlyeq y$ or $y \preccurlyeq x$.

A binary relation $\preccurlyeq$ on X is *transitive* if $x \preccurlyeq z$ for all $x, y, z \in X$ such that $x \preccurlyeq y$ and $y \preccurlyeq z$. A preorder (or a quasi-order) $\preccurlyeq$ on X is a binary relation on X that is *reflexive* (*i.e.*, $x \preccurlyeq x$ for all $x \in X$) and *transitive*. In such a case, we say that $(X, \preccurlyeq)$ is a preordered space (or a preordered set). If a preorder $\preccurlyeq$ is also *antisymmetric* ($x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$), then $\preccurlyeq$ is called a partial order, and $(X, \preccurlyeq)$ is a partially ordered space.

All partial orders and equivalence relations are preorders, but preorders are more general. From now on, $(X, \preccurlyeq)$ will always denote a preordered space.

Definition 2.2 A *metric on* X is a mapping $d : X \times X \to [0, \infty[$ satisfying

$$(M_1) \quad d(x,x) = 0; \qquad (M_2) \quad d(x,y) = 0 \quad \Rightarrow \quad x = y;$$

$$(M_3) \quad d(y,x) = d(x,y); \qquad (M_4) \quad d(x,y) \leq d(z,x) + d(z,y)$$

for all $x, y, z \in X$. If d is a metric on X, we say that (X, d) is a *metric space*.

The function d is a *premetric* if it satisfies (M_1); a *pseudo-metric* if it satisfies (M_1), (M_3) and (M_4); a *quasi-metric* (or a *nonsymmetric metric*) if it satisfies (M_1), (M_2) and (M_4); a *quasi-pseudo-metric* if it satisfies (M_1) and (M_4); and a *semi-metric* if it satisfies (M_1), (M_2) and (M_3).

Remark 2.1 We point out that there exist different notions of *premetric* that are not universally accepted. For instance, Kasahara [26] used the term *premetric* to refer to a *quasi-pseudo-metric* defined on a subset of $X \times X$. However, Kim [27], in the same issue as Kasahara, preferred using the term *quasi-pseudo-metric* (see also Reilly *et al.* [28]). For our purposes and for the sake of clarity, we prefer using the previous definitions because we consider that it is a more modern nomenclature.

Definition 2.3 A *fixed point* of a self-mapping $T : X \to X$ is a point $x \in X$ such that $T(x) = x$. A *coincidence point* between two mappings $T, g : X \to Y$ is a point $x \in X$ such that $T(x) = g(x)$. A *common fixed point* of $T, g : X \to X$ is a point $x \in X$ such that $T(x) = g(x) = x$.

Remark 2.2 If $T, g : X \to X$ are commuting and $x_0 \in X$ is a coincidence point of T and g, then Tx_0 is also a coincidence point of T and g.

Definition 2.4 If $(X, \preccurlyeq)$ is a preordered space and $T, g : X \to X$ are two mappings, we will say that T is a $(g, \preccurlyeq)$-*nondecreasing mapping* if $Tx \preccurlyeq Ty$ for all $x, y \in X$ such that $gx \preccurlyeq gy$. If g is the identity mapping on X, T is *nondecreasing* (w.r.t. $\preccurlyeq$).

3 An illustrative example

Let $\mathbb{I}$ be the real interval $]-2, 1]$ and let $\mathbb{X} = \mathbb{I} \cup \{2, 3, 4, 5, 6, 7\}$ provided with the following binary relation:

$$\text{for } x, y \in \mathbb{X}, \quad x \preccurlyeq y \quad \Leftrightarrow \quad \begin{cases} \text{either} & x = y, \\ \text{or} & \{x, y\} = \{2, 3\}, \\ \text{or} & (x, y \in \mathbb{I} \text{ and } x \leq y). \end{cases}$$

Define $d : \mathbb{X} \times \mathbb{X} \to [0, \infty[$, for all $x, y \in \mathbb{X}$, by

$$d(x,y) = \begin{cases} 0, & \text{if } \{x,y\} = \{2,3\} \text{ or } \{x,y\} = \{4,5\}, \\ 0.5, & \text{if } x = y = 5, \\ 1, & \text{if } (x,y) = (6,7), \\ 2, & \text{if } (x,y) = (7,6), \\ |x-y|, & \text{otherwise.} \end{cases}$$

Let g the identity mapping on $\mathbb{X}$. Define $T : \mathbb{X} \to \mathbb{X}$ by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in \mathbb{I}, \\ 0, & \text{if } x \in \{2,3,6,7\}, \\ -1, & \text{if } x = 4, \\ 1, & \text{if } x = 5, \end{cases} \qquad \text{for all } x \in \mathbb{X}.$$

Then the following statements hold. Proofs can be found in Appendix 2.

1. The binary relation $\preccurlyeq$ is a preorder on $\mathbb{X}$, but it is not a partial order on $\mathbb{X}$.
2. The measure mapping d does not hold any of the four classical properties (M_1)-(M_4) that define a metric space. Indeed, it is not a metric on $\mathbb{X}$, neither a premetric nor any of the following: a pseudo-metric, a quasi-metric, a pseudo-quasi-metric, a semi-metric or a partial metric.
3. Even if d would verify some of the metric properties of Definition 2.2, $(\mathbb{X}, d)$ would not be a complete space.
4. T is not a d-contraction (that is, there is no $k \in [0,1)$ such that $d(Tx, Ty) \leq kd(x,y)$ for all $x, y \in X$) because $d(T4, T5) = d(-1,1) = 2$, but $d(4,5) = 1$.

Therefore, none of the theorems proved until now can be applied to the quadruple $(\mathbb{X}, \preccurlyeq, d, T)$ in order to guarantee that T has a fixed point.

4 Test functions

One of the most important ingredients of a contractivity condition is the kind of involved functions. Recently, many classes of families have been introduced, like altering distance functions, comparison functions, (c)-comparison functions, Geraghty functions, *etc.* In this section, we present the kind of functions we will use and we show how other classes can be seen as particular cases.

Definition 4.1 (Agarwal *et al.* [29]) We will denote by $\mathcal{F}$ the family of all pairs (ψ, φ), where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are functions, verifying the following three conditions.

$(\mathcal{F}_1)$ ψ is nondecreasing.
$(\mathcal{F}_2)$ If there exists $t_0 \in [0, \infty)$ such that $\varphi(t_0) = 0$, then $t_0 = 0$ and $\psi^{-1}(0) = \{0\}$.
$(\mathcal{F}_3)$ If $\{a_k\}, \{b_k\} \subset [0, \infty)$ are sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verifying $L < b_k$ and $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k, then $L = 0$.

Example 4.1 If $k \in [0,1)$ and we define $\psi_k(t) = t$ and $\varphi_k(t) = (1-k)t$ for all $t \geq 0$, then $(\psi_k, \varphi_k) \in \mathcal{F}$. Furthermore, $\psi_k(t) - \varphi_k(t) = kt$ for all $t \geq 0$.

Notice that axiom $(\mathcal{F}_2)$ does not necessarily imply the well known condition $\psi(t) = 0 \Leftrightarrow t = 0 \Leftrightarrow \varphi(t) = 0$. Furthermore, we do not impose any continuity condition neither on ψ

nor on φ. In order to prove that the family $\mathcal{F}$ is very general, next we will show a variety of pairs of functions in $\mathcal{F}$ that have been previously considered by other authors in the past.

A function $\phi : [0,\infty) \to [0,\infty)$ is *lower semi-continuous* if $\phi(t) \leq \liminf_{n\to\infty} \phi(t_n)$ for all sequence $\{t_n\} \subset [0,\infty)$ such that $\{t_n\} \to t$. Similarly, ϕ is *upper semi-continuous* if, in the same conditions, $\phi(t) \geq \limsup_{n\to\infty} \phi(t_n)$.

Definition 4.2 (Khan *et al.* [30]) An *altering distance function* is a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ if and only if $t = 0$.

Proposition 4.1 *If ϕ is an altering distance function and $\{a_m\} \subset [0,\infty)$ verifies $\{\phi(a_m)\} \to 0$, then $\{a_m\} \to 0$.*

The following lemma shows some examples of pairs in $\mathcal{F}$.

Lemma 4.1 (see [29]) *Let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be two functions such that ψ is an altering distance function.*

1. *If φ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$, then $(\psi, \varphi) \in \mathcal{F}$.*
2. *If φ is continuous and verifies $\varphi^{-1}(\{0\}) = \{0\}$, then $(\psi, \varphi) \in \mathcal{F}$.*
3. *If ψ and φ are altering distance functions, then $(\psi, \varphi) \in \mathcal{F}$.*

Notice that the condition $\varphi \leq \psi$ is not necessary.

Proof We prove item (1). Conditions $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ are obvious. Next, assume that $\{a_k\}, \{b_k\} \subset [0,\infty)$ are sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verify $L < b_k$ and $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k. Therefore, $\psi(b_k) \leq (\psi - \varphi)(a_k) = \psi(a_k) - \varphi(a_k) \leq \psi(a_k)$. Hence $0 \leq \varphi(a_k) \leq \psi(a_k) - \psi(b_k)$ for all k. Letting $k \to \infty$ and taking into account that ψ is continuous, we deduce that $\lim_{k\to\infty} \varphi(a_k) = 0$. As $\{a_k\} \to L$ and φ is lower semi-continuous, we deduce that $\varphi(L) \leq \liminf_{t\to L} \varphi(t) \leq \lim_{k\to\infty} \varphi(a_k) = 0$. Hence $L = 0$. The other two items immediately follow from item 1. □

Example 4.2 (see [29])

1. If $a, b > 0$ and we define $\psi(t) = at$ and $\varphi(t) = bt$ for all $t \geq 0$, then $(\psi, \varphi) \in \mathcal{F}$. The case $a \geq b$ is usually included in other papers, but the case $a < b$ is new.
2. If $\psi(t) = \varphi(t) = t + 1$ for all $t \geq 0$, then $(\psi, \varphi) \in \mathcal{F}$. Notice that, in this case, $(\mathcal{F}_3)$ holds because it is impossible to find such kind of sequences since $1 \leq 1 + b_k = \psi(b_k) \leq (\psi - \varphi)(a_k) = 0$. In this case, the condition $\psi(t) = 0 \Leftrightarrow t = 0$ does not hold.

Corollary 4.1 *Let $\psi, \phi : [0,\infty) \to [0,\infty)$ be two functions such that ψ is an altering distance function and ϕ is upper semi-continuous verifying $\phi^{-1}(\{0\}) = \{0\}$ and $\phi(t) < \psi(t)$ for all $t > 0$. Then $(\psi, \varphi) \in \mathcal{F}$, where $\varphi = \psi - \phi$.*

Proof It follows from item 1 of Lemma 4.1 because $\varphi = \psi - \phi$ is lower semi-continuous and verifies $\varphi^{-1}(\{0\}) = \{0\}$. □

A function $\alpha : [0,\infty) \to [0,1)$ is a *Geraghty function* if the condition $\{\alpha(t_n)\} \to 1$ implies that $\{t_n\} \to 0$.

Lemma 4.2 (Agarwal *et al.* [29]) *If α is a Geraghty function and we define $\psi(t) = t$ and $\varphi(t) = (1 - \alpha(t))t$ for all $t \geq 0$, then $(\psi, \varphi) \in \mathcal{F}$.*

In [31], the authors used a contractivity condition as follows:

$$\psi\bigl(d(Tx, Ty)\bigr) \leq \beta\bigl(\psi\bigl(d(x, y)\bigr)\bigr) \cdot \psi\bigl(d(x, y)\bigr),$$

where ψ is an altering distance function and β is a Geraghty function.

Lemma 4.3 *If ψ is an altering distance function and β is a Geraghty function, then $(\psi, (1 - \beta \circ \psi) \cdot \psi) \in \mathcal{F}$.*

Proof Let $\varphi = (1 - \beta \circ \psi) \cdot \psi$, that is, $\varphi(t) = (1 - \beta(\psi(t)))\psi(t)$ for all $t \geq 0$. Notice that the image of $\beta \circ \psi$ is contained in the image of β, which is in $[0, 1)$. Therefore, $\beta(\psi(s))\psi(s) \leq \psi(s)$ for all $s \geq 0$ (if $\psi(s) = 0$, both members are equal, and if $\psi(s) > 0$, then $\beta(\psi(s)) \cdot \psi(s) < \phi(s)$ since $\beta(\psi(s)) < 1$).

($\mathcal{F}_1$) Since ψ is an altering distance function, then it is nondecreasing.

($\mathcal{F}_2$) Assume that there exists $t_0 \in [0, \infty)$ such that $\varphi(t_0) = 0$. Then $(1 - \beta(\psi(t_0)))\psi(t_0) = 0$. Since $1 - \beta(\psi(t_0)) > 0$, then $\psi(t_0) = 0$, which means that $t_0 = 0$. In such a case, $\psi^{-1}(0) = \{0\}$ because it is an altering distance function.

($\mathcal{F}_3$) Let $\{a_k\}, \{b_k\} \subset [0, \infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verify $L < b_k$ and $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k. Since ψ is continuous, $\lim_{k\to\infty} \psi(a_k) = \lim_{k\to\infty} \psi(b_k) = \psi(L)$. Moreover,

$$\psi(b_k) \leq \psi(a_k) - \varphi(a_k) = \psi(a_k) - \bigl[\psi(a_k) - \beta\bigl(\psi(a_k)\bigr)\psi(a_k)\bigr] = \beta\bigl(\psi(a_k)\bigr)\psi(a_k).$$

Let us show that $L = 0$ reasoning by contradiction. Suppose that $L > 0$. Since ψ is nondecreasing,

$$0 < \psi(L) \leq \psi(b_k) \leq \beta\bigl(\psi(a_k)\bigr)\psi(a_k) \leq \psi(a_k).$$

In particular, $\psi(a_k) \neq 0$ and

$$\frac{\psi(b_k)}{\psi(a_k)} \leq \beta\bigl(\psi(a_k)\bigr) \leq 1 \quad \text{for all } k.$$

Letting $n \to \infty$, we deduce that $\{\beta(\psi(a_k))\} \to 1$. Since β is a Geraghty function, $\psi(L) = \lim_{n\to\infty} \psi(a_k) = 0$, which contradicts that $\psi(L) > 0$ because $L > 0$ and ψ is an altering distance function. $\square$

In [32], the authors used a contractivity condition as follows:

$$\begin{aligned} &\psi\bigl(d(Tx, Ty)\bigr) \leq \psi\bigl(N(x, y)\bigr) - \varphi\bigl(N(x, y)\bigr), \\ &\quad \text{where } N(x, y) = \max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right), \end{aligned} \tag{1}$$

ψ is continuous and φ verifies that $\{\varphi(t_n)\} \to 0$ implies that $\{t_n\} \to 0$.

Lemma 4.4 *Let $\psi, \varphi : [0,\infty) \to [0,\infty)$ be two functions such that ψ is an altering distance function and φ verifies the following condition*:

$$\textit{if } \{t_n\} \subset [0,\infty) \textit{ and } \{\varphi(t_n)\} \to 0, \textit{ then } \{t_n\} \to 0. \tag{2}$$

Then $(\psi,\varphi) \in \mathcal{F}$.

Proof $(\mathcal{F}_1)$ Since ψ is an altering distance function, then it is nondecreasing.

$(\mathcal{F}_2)$ Assume that there exists $t_0 \in [0,\infty)$ such that $\varphi(t_0) = 0$. Letting $t_n = t_0$ for all $n \geq 1$ and applying (2), we deduce that $t_0 = 0$. In such a case, $\psi^{-1}(0) = \{0\}$ because it is an altering distance function.

$(\mathcal{F}_3)$ Let $\{a_k\}, \{b_k\} \subset [0,\infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verify $L < b_k$ and $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k. Hence $0 \leq \varphi(a_k) \leq \psi(a_k) - \psi(b_k)$ for all k. Letting $k \to \infty$ and taking into account that ψ is continuous, we deduce that $\lim_{k\to\infty} \varphi(a_k) = 0$. By condition (2), $L = \lim_{k\to\infty} a_k = 0$. □

A *comparison function* is a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

Lemma 4.5 *If ϕ is a continuous comparison function, and we define $\psi(t) = t$ and $\varphi(t) = t - \phi(t)$ for all $t \geq 0$, then $(\psi,\varphi) \in \mathcal{F}$.*

Proof It is clear that every comparison function ϕ verifies $\phi(t) < t$ for all $t > 0$. In such a case, if ϕ is continuous, then $\phi(0) = 0$, so $\phi(t) \leq t$ for all $t \geq 0$. Moreover, if $\phi(t) = t$, then $t = 0$.

$(\mathcal{F}_1)$ Since ψ is an altering distance function, then it is nondecreasing.

$(\mathcal{F}_2)$ Assume that there exists $t_0 \in [0,\infty)$ such that $\varphi(t_0) = 0$. Then $\phi(t_0) = t_0$, so $t_0 = 0$. In such a case, $\psi^{-1}(0) = \{0\}$ because it is an altering distance function.

$(\mathcal{F}_3)$ Let $\{a_k\}, \{b_k\} \subset [0,\infty)$ be sequences such that $\{a_k\} \to L$, $\{b_k\} \to L$ and verify $L < b_k$ and $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k. Therefore

$$b_k = \psi(b_k) \leq (\psi - \varphi)(a_k) = \phi(a_k) < a_k.$$

Letting $k \to \infty$, we deduce that $\lim_{k\to\infty} \phi(a_k) = L$. Therefore, as ϕ is continuous, $\phi(L) = \lim_{k\to\infty} \phi(a_k) = L$, which is only possible when $L = 0$. □

In [33], Berzig *et al.* introduced the notion of *pair of generalized altering distance functions*, which is a pair (ψ,ϕ), where $\psi,\phi : [0,\infty) \to [0,\infty)$, verifying the following conditions:

(a1) ψ is continuous;
(a2) ψ is nondecreasing;
(a3) $\lim_{n\to\infty} \phi(t_n) = 0 \Longrightarrow \lim_{n\to\infty} t_n = 0$.

The condition (a3) was introduced by Popescu in [34] and Moradi and Farajzadeh in [35]. Notice that the above conditions do not determine the values $\psi(0)$ and $\phi(0)$. If $\phi < \psi$ in $(0,\infty)$, then $(\psi, \varphi = \psi - \phi) \in \mathcal{F}$. This is the case of Lemma 4.4.

After we have shown many different contexts in which some pairs of $\mathcal{F}$ appear, we present some of their useful properties.

Lemma 4.6 (Agarwal *et al.* [29]) *Let* $(\psi,\varphi)\in\mathcal{F}$.

1. *If* $t,s\in[0,\infty)$ *and* $\psi(t)\leq(\psi-\varphi)(s)$, *then either* $t<s$ *or* $t=s=0$. *In any case,* $t\leq s$.
2. *If* $t\in[0,\infty)$ *and* $\psi(t)\leq(\psi-\varphi)(t)$, *then* $t=0$.
3. *If* $\{a_k\},\{b_k\}\subset[0,\infty)$ *are such that* $\psi(a_k)\leq(\psi-\varphi)(b_k)$ *for all* k *and* $\{b_k\}\to 0$, *then* $\{a_k\}\to 0$.
4. *If* $\{a_k\}\subset[0,\infty)$ *and* $\psi(a_{k+1})\leq(\psi-\varphi)(a_k)$ *for all* k, *then* $\{a_k\}\to 0$.

Remark 4.1 In [36], Berzig introduced the class of *shifting distance functions*, which are pairs of functions $\psi,\phi:[0,\infty)\to\mathbb{R}$ verifying the following two conditions:

(i) for $u,v\in[0,\infty)$, if $\psi(u)\leq\phi(v)$, then $u\leq v$;

(ii) for $\{u_n\},\{v_n\}\subset[0,\infty)$ with $\lim_{n\to\infty}u_n=\lim_{n\to\infty}v_n=\omega$, if $\psi(u_n)\leq\phi(v_n)$ for all $n\in\mathbb{N}$, then $\omega=0$.

Pairs of functions in $\mathcal{F}$ are intimately related with the class $\mathcal{F}_{\mathrm{shi}}$ of pairs of shifting distance functions, but they are different. On the one hand, pairs in $\mathcal{F}_{\mathrm{shi}}$ can take values in $\mathbb{R}$, but pairs in $\mathcal{F}$ take values in $[0,\infty)$. On the other hand, if a pair (ψ,φ) verifies $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, then the pair $(\psi,\phi=\psi-\varphi)$ satisfies (i). Furthermore, if $(\psi,\phi=\psi-\varphi)$ satisfies (ii), then (ψ,φ) satisfies $(\mathcal{F}_3)$.

5 A fixed point theorem without an underlying metric structure

The main aim of this section is to show sufficient conditions in order to ensure that T and g (given in Section 3) have a coincidence point. To set the framework, throughout this section, let $(X,\preccurlyeq)$ be a preordered space, and let $d:X\times X\to\mathbb{R}$ and $T,g:X\to X$ be three mappings. The following definitions are usually considered when X has a metric structure. However, we do not suppose, *a priori*, any condition on the mapping d. Indeed, we will only be able to prove that d takes nonnegative values as a consequence of a particular version of the triangular inequality. However, in general, we do not consider necessary to assume this sign constraint.

Definition 5.1 We will say that a sequence $\{x_m\}\subseteq X$:

- *d-converges to* $x_0\in X$ (and we will write $\{x_m\}\xrightarrow{d}x_0$ or simply $\{x_m\}\to x_0$) if for all $\varepsilon>0$ there exists $m_0\in\mathbb{N}$ such that $d(x_m,x_0)\leq\varepsilon$ for all $m\geq m_0$;
- is *d-Cauchy* if for all $\varepsilon>0$ there exists $m_0\in\mathbb{N}$ such that $d(x_m,x_{m'})\leq\varepsilon$ for all $m'\geq m\geq m_0$.

We will say that (X,d) is *complete* if every d-Cauchy sequence in X is d-convergent in X.

With respect to the previous notions, the following remarks must be done.

Remark 5.1

- When the distance measure d is not symmetric (that is, it does not verify axiom (M_3)), the definition of convergence or Cauchyness of sequences usually depends on the side, because $d(x_n,x_m)$ and $d(x_m,x_n)$ can be different. This is the case, for instance, of quasi-metric spaces. In such cases, the previous definitions correspond to the idea of *right-convergence* and *right-Cauchyness* (see Jleli and Samet [37]) because the more advanced term (which is nearer to the limit) is placed at the right argument of d. Similarly, it can be defined the notions of *left-convergence* (using $d(x_0,x_m)$) and *left-Cauchyness* (using $d(x_{m'},x_m)$) of sequences. Notice that some of the concepts we will present can also be introduced by the right side or by the left side. However, in order not to complicate the notation, we prefer avoiding the term *right-* in all definitions and theorems.

- In [38, Section 3], the authors did a complete study (completion, topology and powerdomains) of spaces verifying axioms (M_1) and (M_4), which they called *generalized metric spaces*. They solved the previous discussion using the terms *forward convergent sequences* (for right-convergent sequences) and *backward convergent sequences* (for left-convergent sequences). However, as we will not assume (M_1) nor (M_4), we also prefer avoiding these prefixes.
- Notice that if d does not verify (M_2), then the limit of a sequence, if there exists, might not be unique.
- And if d only takes nonpositive values, then all sequences converge to all points.

Definition 5.2 We will say that a subset $A \subseteq X$ is $(d, \preccurlyeq)$-*nondecreasing-closed* if any d-limit of any $\preccurlyeq$-nondecreasing sequence of points of A is also in A.

$$\left[\{x_m\} \subseteq A, \{x_m\} \xrightarrow{d} x \in X, x_m \preccurlyeq x_{m+1}, \forall m\right] \quad \Rightarrow \quad x \in A.$$

Similarly can be defined the concepts of $(d, \preccurlyeq)$-*nonincreasing-closed set* and $(d, \preccurlyeq)$-*monotone-closed set*, and, more generally, a d-*closed set*, when any d-limit of any convergent sequence of points of A is also in A.

Definition 5.3 A mapping $T : X \to X$ is $(d, \preccurlyeq)$-*nondecreasing-continuous at* $x_0 \in X$ if we have that $\{Tx_m\}$ d-converges to Tx_0 for all $\preccurlyeq$-nondecreasing sequence $\{x_m\}$ d-convergent to x_0.

In a similar way, the concepts of $(d, \preccurlyeq)$-*nonincreasing-continuous mapping* and $(d, \preccurlyeq)$-*monotone-continuous mapping* may be considered and, more generally, a d-*continuous mapping*, when $\{Tx_m\}$ d-converges to Tx_0 for all sequence $\{x_m\}$ d-convergent to x_0.

Definition 5.4 We will say that a point $x \in X$ is a d-*precoincidence point of* T *and* g if $d(Tx, gx) = d(gx, Tx) = 0$.

In the following results, we will assume some of the following conditions.

(a) $T(X) \subseteq g(X)$.
(b) T is $(g, \preccurlyeq)$-nondecreasing.
(c) There exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$.
(d) There exists $(\psi, \varphi) \in \mathcal{F}$ such that

$$\psi\big(d(Tx, Ty)\big) \leq (\psi - \varphi)\big(d(gx, gy)\big) \quad \text{for all } x, y \in X \text{ for which } gx \preccurlyeq gy. \tag{3}$$

(e) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ such that $x \preccurlyeq y \preccurlyeq z$.
(f) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$ such that $x \preccurlyeq y \preccurlyeq z$.
(g) Every $\preccurlyeq$-nondecreasing, d-Cauchy sequence in X is d-convergent in X.
(h) If $\{x_m\}$ is a nondecreasing sequence and $\{x_m\}$ d-converges to $x \in X$, then $x_m \preccurlyeq x$ for all m.
(i) $d(x, z) \leq d(y, x) + d(y, z)$ for all $x, y, z \in X$ such that $y \preccurlyeq x$ and $y \preccurlyeq z$.

(j) Every d-precoincidence point of T and g is a coincidence point of T and g (that is, if $d(Tx, gx) = d(gx, Tx) = 0$, then $Tx = gx$).

As we have shown in Section 3, notice that a mapping d verifying (a)-(i) does not have to verify any of the conditions that define a metric on X.

Remark 5.2 *A priori*, d can take negative values in $\mathbb{R}$. However, condition (f) lets us prove some constraints about the sign of d. Indeed, if we take $x = y = z$ in condition (f), we deduce that $d(x,x) \geq 0$ for all $x \in X$. Furthermore, if $y = x$ in (f), it follows that $d(x,z) \geq 0$ for all $x, z \in X$ such that $x \preccurlyeq z$. This does not mean that d is nonnegative because d could take negative values when $z \prec x$ or x and z are not $\preccurlyeq$-comparable.

Remark 5.3 As we shall show in the proofs, the mapping d could only be considered on the set $\Omega = \{(x,y) \in X^2 : x \preccurlyeq y\}$, that is, we will only use $d|_\Omega : \Omega \to \mathbb{R}$. In this case, the previous remark shows that, as usual, $d(\Omega) \subseteq [0, \infty[$.

Theorem 5.1 *Let* $(X, \preccurlyeq)$ *be a preordered space and let* $T, g : X \to X$ *be two mappings verifying* (a)-(c). *Then there exists a sequence* $\{x_m\}_{m\geq 0}$ *such that* $gx_{m+1} = Tx_m \preccurlyeq Tx_{m+1} = gx_{m+2}$ *for all* $m \geq 0$. *In particular,*

$$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \cdots \preccurlyeq gx_{m-1} \preccurlyeq gx_m \preccurlyeq gx_{m+1} \preccurlyeq \cdots .$$

Proof Since $Tx_0 \in T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preccurlyeq Tx_0 = gx_1$. Since T is $(g, \preccurlyeq)$-nondecreasing, $Tx_0 \preccurlyeq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preccurlyeq Tx_1 = gx_2$. Since T is $(g, \preccurlyeq)$-nondecreasing, $Tx_1 \preccurlyeq Tx_2$. Repeating this argument, there exists a sequence $\{x_m\}_{m\geq 0}$ such that $gx_{m+1} = Tx_m \preccurlyeq Tx_{m+1} = gx_{m+2}$ for all $m \geq 0$. □

Theorem 5.2 *Let* $(X, \preccurlyeq)$ *be a preordered space and let* $d : X \times X \to \mathbb{R}$ *and* $T, g : X \to X$ *be three mappings verifying* (a)-(e). *Then any sequence* $\{gx_m\}_{m\geq 0}$ *such that* $Tx_m = gx_{m+1}$ *for all* $m \geq 0$, *is* d-*Cauchy* ($\{x_m\}$ *is given as in Theorem* 5.1).

Proof Since $gx_{m+1} \preccurlyeq gx_{m+2}$ for all $m \geq 0$, it follows from (d) that, for all $m \geq 1$,

$$\psi\bigl(d(gx_{m+1}, gx_{m+2})\bigr) = \psi\bigl(d(Tx_m, Tx_{m+1})\bigr) \leq (\psi - \varphi)\bigl(d(gx_m, gx_{m+1})\bigr) \leq \psi\bigl(d(gx_m, gx_{m+1})\bigr).$$

By item 4 of Lemma 4.6, the sequence $\{d(gx_{m+1}, gx_{m+2})\}$ d-converges to zero. Using the same reasoning, since $gx_{m+1} \preccurlyeq gx_{m+1}$ for all $m \geq 0$, it follows that $\{d(gx_{m+1}, gx_{m+1})\}$ also d-converges to zero. Therefore

$$\bigl\{d(gx_m, gx_m)\bigr\}_{m\geq 0} \to 0 \quad \text{and} \quad \bigl\{d(gx_m, gx_{m+1})\bigr\}_{m\geq 0} \to 0. \tag{4}$$

Let us show that $\{gx_m\}$ is d-Cauchy reasoning by contradiction. Suppose that $\{gx_m\}$ is not d-Cauchy. Then there exist $\varepsilon_0 > 0$ and partial subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ verifying

$$k < n(k) < m(k) < n(k+1) \quad \text{and} \quad d(gx_{n(k)}, gx_{m(k)-1}) \leq \varepsilon_0 < d(gx_{n(k)}, gx_{m(k)}) \quad \text{for all } k \geq 1 \tag{5}$$

($m(k)$ is the least integer number, greater than $n(k)$, such that $d(gx_{n(k)},gx_{m(k)}) > \varepsilon_0$). Since $n(k) \leq m(k) - 1 < m(k)$, we have $gx_{n(k)} \preccurlyeq gx_{m(k)-1} \preccurlyeq gx_{m(k)}$. By (e),

$$\varepsilon_0 < d(gx_{n(k)},gx_{m(k)}) \leq d(gx_{n(k)},gx_{m(k)-1}) + d(gx_{m(k)-1},gx_{m(k)}) \leq \varepsilon_0 + d(gx_{m(k)-1},gx_{m(k)}).$$

Therefore, the sequence $\{b_k = d(gx_{n(k)},gx_{m(k)})\}_{k\geq 1}$ satisfies

$$\lim_{k\to\infty} b_k = \varepsilon_0 \quad \text{and} \quad \varepsilon_0 < b_k \quad \text{for all } k. \tag{6}$$

Now, let us apply condition (e) and (5) to $gx_{n(k)-1} \preccurlyeq gx_{n(k)} \preccurlyeq gx_{m(k)-1}$, and we deduce, for all k,

$$d(gx_{n(k)-1},gx_{m(k)-1}) \leq d(gx_{n(k)-1},gx_{n(k)}) + d(gx_{n(k)},gx_{m(k)-1}) \leq d(gx_{n(k)-1},gx_{n(k)}) + \varepsilon_0. \tag{7}$$

By (d), we also have, for all k,

$$\psi\bigl(d(gx_{n(k)},gx_{m(k)})\bigr) = \psi\bigl(d(Tx_{n(k)-1},Tx_{m(k)-1})\bigr) \leq (\psi - \varphi)\bigl(d(gx_{n(k)-1},gx_{m(k)-1})\bigr). \tag{8}$$

By item 1 of Lemma 4.6, it follows that

$$d(gx_{n(k)},gx_{m(k)}) \leq d(gx_{n(k)-1},gx_{m(k)-1}) \quad \text{for all } k.$$

Joining this inequality and (7), we deduce that, for all k,

$$d(gx_{n(k)},gx_{m(k)}) \leq d(gx_{n(k)-1},gx_{m(k)-1}) \leq d(gx_{n(k)-1},gx_{n(k)}) + \varepsilon_0.$$

Letting $k \to \infty$ and using (4) and (6), we deduce that the sequence $\{a_k = d(gx_{n(k)-1}, gx_{m(k)-1})\}_{k\geq 1}$ also verifies $\{a_k\} \to \varepsilon_0$, and by (8), we have that $\psi(b_k) \leq (\psi - \varphi)(a_k)$ for all k. Since $(\psi,\varphi) \in \mathcal{F}$, axiom $(\mathcal{F}_3)$ guarantees that $\varepsilon_0 = 0$, which is a contradiction with the fact that $\varepsilon_0 > 0$. This contradiction shows that $\{gx_m\}$ is a d-Cauchy sequence. □

After the previous technical results, we give the main results of this manuscript.

Theorem 5.3 *Let* $(X,\preccurlyeq)$ *be a preordered space and let* $d : X \times X \to \mathbb{R}$ *and* $T, g : X \to X$ *be three mappings which fulfil conditions* (a)-(h). *Assume that the following condition holds.*

(p) $g(X)$ *is* $(d,\preccurlyeq)$*-nondecreasing-closed.*

Then there exists $z \in X$ *such that the sequence* $\{gx_m\}$ (*defined in Theorem* 5.2) *d-converges to gz and to Tz. Furthermore, if* (i) *holds, then z is a d-precoincidence point of T and g.*

Notice that, in the previous result, g and T need not be continuous.

Proof Theorem 5.2 guarantees that $\{gx_m\}$ is d-Cauchy. Since it is $\preccurlyeq$-nondecreasing, condition (g) implies that there exists $y \in X$ such that $\{gx_m\}$ d-converges to y (that is, $\{d(gx_m,y)\} \to 0$). Moreover, since $g(X)$ is $(d,\preccurlyeq)$-nondecreasing-closed, $y \in g(X)$, so there exists $z \in X$ such that $y = gz$. Applying (h), $gx_{m+1} \preccurlyeq y = gz$ for all m and, hence,

$$\psi\bigl(d(gx_{m+2},Tz)\bigr) = \psi\bigl(d(Tx_{m+1},Tz)\bigr) \leq (\psi - \varphi)\bigl(d(gx_{m+1},gz)\bigr) = (\psi - \varphi)\bigl(d(gx_{m+1},y)\bigr).$$

Since $\{d(gx_m, y)\} \to 0$, item 3 of Lemma 4.6 guarantees that $\{d(gx_{m+2}, Tz)\} \to 0$, that is, $\{gx_{m+2}\}$ d-converges to Tz.

Now suppose that (i) holds. By (h), $gx_{m+2} \preccurlyeq Tz$ for all m and (i) implies that

$$\begin{aligned} &d(gz, Tz) \leq d(gx_{m+2}, Tz) + d(gx_{m+2}, gz) \quad \text{and} \\ &d(Tz, gz) \leq d(gx_{m+2}, gz) + d(gx_{m+2}, Tz). \end{aligned} \tag{9}$$

Therefore $d(gz, Tz) = d(Tz, gz) = 0$. □

Theorem 5.4 *Let* $(X, \preccurlyeq)$ *be a preordered space and let* $d : X \times X \to \mathbb{R}$ *and* $T, g : X \to X$ *be three mappings which fulfil conditions* (a)-(h). *Suppose also*:

(p′) *T and g are $(d, \preccurlyeq)$-nondecreasing-continuous and commuting and, at least, one of the following conditions holds*:

- (p_1') *T is a $\preccurlyeq$-nondecreasing mapping.*
- (p_2') *g is a $\preccurlyeq$-nondecreasing mapping.*
- (p_3') *If $z, \omega \in X$ and $\{z_m\} \subseteq X$ is a $\preccurlyeq$-nondecreasing sequence such that $\{gz_m\}$ d-converges to z and to ω at the same time, then $d(z, \omega) = d(\omega, z) = 0$.*

Then there exists $y \in X$ such that the sequence $\{gy_m\}$ (defined in Theorem 5.2*) d-converges to gy and to Ty at the same time. Furthermore, if* (i) *holds, then y is a d-precoincidence point of T and g.*

In any case, T and g have, at least, a d-precoincidence point.

Proof Theorem 5.2 guarantees that $\{gx_{m+1}\}$ is d-Cauchy. Since it is $\preccurlyeq$-nondecreasing, condition (g) implies that there exists $y \in X$ such that $\{gx_m\}$ d-converges to y. Thus, taking into account that g and T are $(d, \preccurlyeq)$-nondecreasing-continuous, $\{ggx_m\}$ d-converges to gy and $\{Tgx_m\}$ d-converges to Ty. Furthermore, since T and g are commuting, $Tgx_{m+1} = gTx_{m+1} = ggx_{m+1}$ for all m, which means that $\{ggx_m\}$ d-converges, at the same time, to gy and to Ty.

Next, assume that (i) holds, and we claim that y is a d-precoincidence point of T and g. Firstly, if T (or g) is a $\preccurlyeq$-nondecreasing mapping, then the sequence $\{ggx_m\} = \{Tgx_{m-1}\}$ is $\preccurlyeq$-nondecreasing. Since it d-converges to gy and to Ty, property (h) implies that $ggx_m \preccurlyeq gy$ and $ggx_m \preccurlyeq Ty$ for all m. Reasoning as in (9), we conclude that y is a d-precoincidence point of T and g. Secondly, if T and g are not necessarily $\preccurlyeq$-nondecreasing mappings, we could apply (p_3') to the sequence $\{gx_m\}$ in order to deduce that $d(gy, Ty) = d(Ty, gy) = 0$, that is, y is a d-precoincidence point of T and g. □

We summarize and improve all previous results in the following theorem.

Theorem 5.5 *Let* $(X, \preccurlyeq)$ *be a preordered space and let* $d : X \times X \to \mathbb{R}$ *and* $T, g : X \to X$ *be three mappings verifying the following properties.*

(a) *$T(X) \subseteq g(X)$.*
(b) *T is $(g, \preccurlyeq)$-nondecreasing.*
(c) *There exists $x_0 \in X$ such that $gx_0 \preccurlyeq Tx_0$.*

(d) *There exist* $(\psi,\varphi)\in\mathcal{F}$ *such that*

$$\psi\big(d(Tx,Ty)\big)\leq(\psi-\varphi)\big(d(gx,gy)\big)\quad \textit{for all } x,y\in X \textit{ for which } gx\preccurlyeq gy.$$

(e) $d(x,z)\leq d(x,y)+d(y,z)$ *for all* $x,y,z\in X$ *such that* $x\preccurlyeq y\preccurlyeq z$.
(f) $d(x,y)\leq d(x,z)+d(y,z)$ *for all* $x,y,z\in X$ *such that* $x\preccurlyeq y\preccurlyeq z$.
(g) *Every* $\preccurlyeq$*-nondecreasing,* d*-Cauchy sequence in* X *is* d*-convergent in* X.
(h) *If* $\{x_m\}$ *is a* $\preccurlyeq$*-nondecreasing sequence and* $\{x_m\}$ d*-converges to* $x\in X$, *then* $x_m\preccurlyeq x$ *for all* m.
(i) $d(x,z)\leq d(y,x)+d(y,z)$ *for all* $x,y,z\in X$ *such that* $y\preccurlyeq x$ *and* $y\preccurlyeq z$.
(j) *Every* d*-precoincidence point of* T *and* g *is a coincidence point of* T *and* g.

Assume also either

(p) $g(X)$ *is* $(d,\preccurlyeq)$*-nondecreasing-closed, or*
(p′) T *and* g *are* $(d,\preccurlyeq)$*-nondecreasing-continuous and commuting and, at least, one of the following conditions holds*:

 (p'_1) T *is a* $\preccurlyeq$*-nondecreasing mapping.*
 (p'_2) g *is a* $\preccurlyeq$*-nondecreasing mapping.*
 (p'_3) *If* $z,\omega\in X$ *and* $\{z_m\}\subseteq X$ *is a* $\preccurlyeq$*-nondecreasing sequence such that* $\{gz_m\}$ d*-converges to* z *and to* ω *at the same time, then* $d(z,\omega)=d(\omega,z)=0$.

Then T *and* g *have, at least, a coincidence point.*

Notice that condition (j) does not mean that $d(x,y)=0$ implies $x=y$.

Remark 5.4 If g is the identity mapping on the ambient space, then the quadruple $(\mathbb{X},\preccurlyeq,d,T)$ introduced in Section 3 verifies all conditions (a)-(j) and (p)-(p′) (see more details in Appendix 2).

Remark 5.5 Obviously, similar results can be stated changing the following hypothesis.

($\widetilde{b}$) T is $(g,\preccurlyeq)$-nonincreasing.
($\widetilde{c}$) There exists $x_0\in X$ such that $gx_0\succcurlyeq Tx_0$.
($\widetilde{g}$) Every nonincreasing, d-Cauchy sequence in X is d-convergent in X.
($\widetilde{h}$) If $\{x_m\}$ is a nonincreasing sequence and $\{x_m\}$ d-converges to $x\in X$, then $x_m\succcurlyeq x$ for all m.
($\widetilde{p}$) $g(X)$ is $(d,\preccurlyeq)$-nonincreasing-closed.
($\widetilde{p}'$) T and g are $(d,\preccurlyeq)$-nonincreasing-continuous and commuting and, at least, one of the following conditions holds:

 ($\widetilde{p}'_1$) T is a $\preccurlyeq$-nonincreasing mapping.
 ($\widetilde{p}'_2$) g is a $\preccurlyeq$-nonincreasing mapping.
 ($\widetilde{p}'_3$) If $z,\omega\in X$ and $\{z_m\}\subseteq X$ is a $\preccurlyeq$-nonincreasing sequence such that $\{gz_m\}$ d-converges to z and to ω at the same time, then $d(z,\omega)=d(\omega,z)=0$.

The unicity of the coincidence point cannot be guaranteed unless additional conditions are imposed. A result in this direction is the following.

Theorem 5.6 *Under the hypothesis of Theorem* 5.5, *let* $x, y \in X$ *be two coincidence points of* T *and* g *verifying that there exists* $u \in X$ *such that* $gu \preccurlyeq gx$ *and* $gu \preccurlyeq gy$. *Then* $d(Tx, Ty) = d(Ty, Tx) = d(gx, gy) = d(gy, gx) = 0$.

Proof Define $u_0 = u$. Since $Tu_0 \in T(X) \subseteq g(X)$, there exists $u_1 \in X$ such that $gu_1 = Tu_0$. Repeating this process, there exists a sequence $\{u_m\}_{m\geq 0}$ such that $gu_{m+1} = Tu_m$ for all $m \geq 0$. We claim that $\{gu_m\} \xrightarrow{d} gx$ and $\{gu_m\} \xrightarrow{d} gy$. Firstly, we reason using x, but the same argument is valid for y.

Indeed, notice that $gu_0 = gu \preccurlyeq gx$. As T is $(g, \preccurlyeq)$-nondecreasing, then $Tu_0 \preccurlyeq Tx = gx$, which means that $gu_1 \preccurlyeq gx$. Again, $gu_1 \preccurlyeq gx$ implies $Tu_1 \preccurlyeq Tx = gx$, which means that $gu_2 \preccurlyeq gx$. By induction, it is possible to prove that $gu_m \preccurlyeq gx$ for all $m \geq 0$. Using condition (d), it follows that $\psi(d(gu_{m+1}, gx)) = \psi(d(Tu_m, Tx)) \leq (\psi - \varphi)(d(gu_m, gx))$ for all $m \geq 0$. Thus, by item 4 of Lemma 4.6, $\{d(gu_m, gx)\} \to 0$. The same argument proves that $gu_m \preccurlyeq gy$ for all $m \geq 0$ and $\{d(gu_m, gy)\} \to 0$. As a consequence, by (i), $d(Tx, Ty) = d(gx, gy) \leq d(gu_m, gx) + d(gu_m, gy)$ for all m, which lets us conclude that $d(Tx, Ty) = d(gx, gy) = 0$. □

Corollary 5.1 *Under the hypothesis of Theorem* 5.5, *assume the following conditions*.

- *For all coincidence points* $x, y \in X$ *of* T *and* g, *there exists* $u \in X$ *such that* $gu \preccurlyeq gx$ *and* $gu \preccurlyeq gy$.
- g *is injective on the set of all coincidence points of* T *and* g.
- *If* $z, \omega \in T(X)$ *verify* $d(z, \omega) = d(\omega, z) = 0$, *then* $z = \omega$.

Then T *and* g *have a unique coincidence point*. *Furthermore*, *if* T *and* g *are commuting*, *it is a common fixed point of* T *and* g.

Proof Let $x, y \in X$ be two coincidence points of T and g. Then $gx = Tx \in T(X)$ and $gy = Ty \in T(X)$. By Theorem 5.6, $d(gx, gy) = d(gy, gx) = 0$. Therefore $gx = gy$. As g is injective on the set of all coincidence points of T and g, we conclude that $x = y$.

Now let $x \in X$ be a coincidence point of T and g, and let $z = Tx$. By Remark 2.2, z is also a coincidence point of T and g. Then $x = z = Tx = gx$, so x is a common fixed point of T and g. □

Taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$ for all $t \geq 0$ in the previous results, we obtain the following particular case.

Corollary 5.2 *Theorems* 5.1, 5.2, 5.3, 5.4, 5.5, 5.6 *and Corollary* 5.1 *also hold if we replace condition* (d) *by the following one*.

(d′) *There exists* $k \in [0, 1)$ *such that*$d(Tx, Ty) \leq k\, d(gx, gy)$*for all* $x, y \in X$ *for which* $gx \preccurlyeq gy$.

6 Consequences

This section is devoted to show how to apply Theorem 5.5 in many different contexts, and how to deduce unidimensional, coupled, tripled, quadruple and multidimensional fixed point theorems (for completeness, they are included in Appendix 1).

6.1 Fixed/coincidence point theorems in partially ordered metric spaces

In this subsection we show that different results in partially ordered metric spaces, including unidimensional, coupled, tripled, quadruple and multidimensional fixed point theorems, can be seen as simple consequences of Theorem 5.5.

Corollary 6.1 *Theorems* A.1, A.2 *and* A.3 *follow from Theorem* 5.5.

However, Theorems A.1 and A.2 cannot be applied to the example of Section 3.

Corollary 6.2 *Theorem* A.4 *follows from Theorem* 5.5.

Proof Let $Y = X^2$ provided with the partial order $(x,y) \sqsubseteq (u,v)$ if and only if $x \succcurlyeq u$ and $y \preccurlyeq v$, and the metric $\delta : Y \times Y \to \mathbb{R}_0^+$ given by $\delta((x,y),(u,v)) = \max(d(x,u),d(y,v))$ for all $(x,y),(u,v) \in Y$. Define $T_F : Y \to Y$ by $T_F(x,y) = (F(x,y),F(y,x))$ for all $(x,y) \in Y$, and let G be the identity mapping on Y. Let $X_0 = (x_0,y_0) \in Y$. Then the hypothesis of Theorem A.4 implies the hypothesis of Theorem 5.5 (for instance, T_F is $(\sqsubseteq, G)$-nondecreasing because F has the mixed $\preccurlyeq$-monotone property). The contractivity condition holds since, if $(x,y) \sqsubseteq (u,v)$,

$$\begin{aligned}
&\delta\bigl(T_F(x,y),T_F(u,v)\bigr)\\
&\quad= \delta\bigl(\bigl(F(x,y),F(y,x)\bigr),\bigl(F(u,v),F(v,u)\bigr)\bigr)\\
&\quad= \max\bigl(d\bigl(F(x,y),F(u,v)\bigr),d\bigl(F(y,x),F(v,u)\bigr)\bigr)\\
&\quad= \max\bigl(d\bigl(F(x,y),F(u,v)\bigr),d\bigl(F(v,u),F(y,x)\bigr)\bigr)\\
&\quad\le \max\left(\frac{k}{2}\bigl(d(x,u)+d(y,v)\bigr),\frac{k}{2}\bigl(d(v,y)+d(u,x)\bigr)\right) \le \frac{k}{2}2\max\bigl(d(x,u),d(y,v)\bigr)\\
&\quad= k\delta\bigl((x,y),(u,v)\bigr).
\end{aligned}$$

Theorem 5.5 assures us that T_F and G have a coincidence point, that is, F has a coupled fixed point. □

Tripled, quadruple and multidimensional theorems can be proved similarly using X^3, X^4 and X^n, obtaining the following result.

Corollary 6.3 *Theorems* A.5, A.6 *and* A.7 *follow from Theorem* 5.5.

We remark that the techniques used in this paper might be applied in order to prove other coupled, tripled, quadruple, n-tupled fixed point theorems in the framework of various abstract spaces, *e.g.*, partial metric spaces, cone metric spaces, fuzzy metric spaces, b-metric spaces, *etc.*

6.2 Fixed/coincidence point theorems in quasi-metric spaces

Recall that a mapping $q : X \times X \to [0,\infty)$ is a *quasi-metric on* X if it satisfies (M_1), (M_2) and (M_4), that is, if it verifies, for all $x,y,z \in X$:

(q_1) $q(x,y) = 0$ if and only if, $x = y$,
(q_2) $q(x,y) \le q(x,z) + q(z,y)$.

In such a case, the pair (X,q) is called a *quasi-metric space*. Some preliminaries about convergence, Cauchy sequences and completeness in quasi-metric spaces can be found in [29, 37].

Theorem 6.1 *Let (X,q) be a complete quasi-metric space and let $T,g : X \to X$ be given mappings. Suppose that $T(X) \subseteq g(X)$ and that there exists $(\psi,\varphi) \in \mathcal{F}$ such that*

$$\psi\big(q(Tx,Ty)\big) \le \psi\big(q(gx,gy)\big) - \varphi\big(q(gx,gy)\big) \quad \textit{for all } x,y \in X.$$

Then T and g have a unique coincidence point.

Proof Assume that $\preccurlyeq$ is the preorder in X given by $x \preccurlyeq y$ for all $x,y \in X$, that is, all points are $\preccurlyeq$-comparable. Then all conditions of Theorem 5.5 hold. Moreover, as $g(X)$ is $(d,\preccurlyeq)$-nondecreasing-closed, we deduce that T and g have, at least, a coincidence point. Furthermore, if u and v are two distinct coincidence points of T and g, then

$$\psi\big(q(gu,gv)\big) = \psi\big(q(Tu,Tv)\big) \le \psi\big(q(gu,gv)\big) - \varphi\big(q(gy,gv)\big) < \psi\big(q(gu,gv)\big),$$

which is a contradiction. □

If g is the identity mapping on X, we have the following statement.

Corollary 6.4 *Let (X,q) be a complete quasi-metric space and let $T : X \to X$ be a given mapping. Suppose that there exists $(\psi,\varphi) \in \mathcal{F}$ such that*

$$\psi\big(q(Tx,Ty)\big) \le \psi\big(q(x,y)\big) - \varphi\big(q(x,y)\big) \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point.

If $\psi(t) = t$ for all $t \ge 0$, we have the following particular case.

Corollary 6.5 (Jleli and Samet [37], Theorem 3.2) *Let (X,q) be a complete quasi-metric space and let $T : X \to X$ be a mapping satisfying*

$$q(Tx,Ty) \le q(x,y) - \varphi\big(q(x,y)\big) \quad \textit{for all } x,y \in X,$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous with $\varphi^{-1}(0) = \{0\}$. Then T has a unique fixed point.

6.3 Fixed/coincidence point theorems in *G*-metric spaces

Following [7, 39], recall that a *generalized metric on X* (or, more specifically, a *G-metric* on X) is a mapping $G : X \times X \times X \to [0,\infty)$ verifying the following properties.

Definition 6.1

(G_1) $G(x,y,z) = 0$ if $x = y = z$;
(G_2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;
(G_3) $G(x,x,y) \le G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;
(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);
(G_5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ (rectangle inequality) for all $x,y,z,a \in X$.

In such a case, the pair (X,G) is called a *G-metric space*. Some preliminaries about convergence, Cauchy sequences and completeness in quasi-metric spaces can be found in [7, 29, 37, 39].

Lemma 6.1 (see, *e.g.*, [29, 37]) *Let (X, G) be a G-metric space and let us define $q_G, q'_G : X^2 \to [0, \infty)$ by*

$$q_G(x,y) = G(x,y,y) \quad \text{and} \quad q'_G(x,y) = G(x,x,y) \quad \text{for all } x,y \in X.$$

Then the following properties hold.

1. *q_G and q'_G are quasi-metrics on X. Moreover,*

$$q'_G(x,y) \leq 2q_G(x,y) \leq 4q'_G(x,y) \quad \text{for all } x,y \in X. \tag{10}$$

2. *In (X, q_G) and in (X, q'_G), a sequence is right-convergent (respectively, left-convergent) if and only if it is convergent. In such a case, its right-limit, its left-limit and its limit coincide.*
3. *In (X, q_G) and in (X, q'_G), a sequence is right-Cauchy (respectively, left-Cauchy) if and only if it is Cauchy.*
4. *In (X, q_G) and in (X, q'_G), every right-convergent (respectively, left-convergent) sequence has a unique right-limit (respectively, left-limit).*
5. *If $\{x_n\} \subseteq X$ and $x \in X$, then $\{x_n\} \xrightarrow{G} x \Longleftrightarrow \{x_n\} \xrightarrow{q_G} x \Longleftrightarrow \{x_n\} \xrightarrow{q'_G} x$.*
6. *If $\{x_n\} \subseteq X$, then $\{x_n\}$ is G-Cauchy $\Longleftrightarrow$ $\{x_n\}$ is q_G-Cauchy $\Longleftrightarrow$ $\{x_n\}$ is q'_G-Cauchy.*
7. *(X, G) is complete $\Longleftrightarrow$ (X, q_G) is complete $\Longleftrightarrow$ (X, q'_G) is complete.*

Theorem 6.2 *Let (X, G) be a complete G-metric space and let $T, g : X \to X$ be given mappings. Suppose that $T(X) \subseteq g(X)$ and that there exists $(\psi, \varphi) \in \mathcal{F}$ such that*

$$\psi\big(G(Tx, Ty, Ty)\big) \leq \psi\big(G(gx, gy, gy)\big) - \varphi\big(G(gx, gy, gy)\big) \quad \text{for all } x,y \in X.$$

Then T and g have a unique coincidence point.

Proof It follows from Theorem 6.1 applied to the quasi-metric $q_G(x,y) = G(x,y,y)$ for all $x,y \in X$ (as in Lemma 6.1). □

To conclude the paper, we include two appendices: in the first one, we recall some celebrated theorems that can be seem as particular cases of our main results; in the second one, we prove the statements announced in Section 3 and why our results can be applied.

Appendix 1: Some recent results we generalize

The following statements are well-known fixed point theorems in partially ordered metric spaces.

Theorem A.1 (Ran and Reurings [17]) *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) *(X, d) is complete.*
(b) *T is nondecreasing (w.r.t. $\preccurlyeq$).*
(c) *T is continuous.*
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$.*

(e) *There exists a constant* $k \in (0,1)$ *such that* $d(Tx,Ty) \leq kd(x,y)$ *for all* $x,y \in X$ *with* $x \succcurlyeq y$.

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X^2$ *there exists* $z \in X$ *such that* $x \preccurlyeq z$ *and* $y \preccurlyeq z$, *we obtain uniqueness of the fixed point.*

Nieto and Rodríguez-López [18] slightly modified the hypothesis of the previous result obtaining the following theorem.

Theorem A.2 (Nieto and Rodríguez-López [18]) *Let* $(X,\preccurlyeq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(a) (X,d) *is complete.*
(b) T *is nondecreasing* (*w.r.t.* $\preccurlyeq$).
(c) *If a nondecreasing sequence* $\{x_m\}$ *in* X *converges to some point* $x \in X$, *then* $x_m \preccurlyeq x$ *for all* m.
(d) *There exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$.
(e) *There exists a constant* $k \in (0,1)$ *such that* $d(Tx,Ty) \leq kd(x,y)$ *for all* $x,y \in X$ *with* $x \succcurlyeq y$.

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X^2$ *there exists* $z \in X$ *such that* $x \preccurlyeq z$ *and* $y \preccurlyeq z$, *we obtain uniqueness of the fixed point.*

Theorem A.3 (Harjani and Sadarangani [40]) *Let* $(X,\preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X,d) *is a complete metric space. Let* $T : X \to X$ *be a nondecreasing mapping such that*

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \psi\bigl(d(x,y)\bigr) - \phi\bigl(d(x,y)\bigr) \quad \text{for all } x \succcurlyeq y,$$

where ψ *and* ϕ *are altering distance functions. Also assume that, at least, one of the following conditions holds.*

(i) T *is continuous, or*
(ii) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preccurlyeq x$ *for all* n.

If there exists $x_0 \in X$ *with* $x_0 \preccurlyeq Tx_0$, *then* T *has a fixed point.*

Theorem A.4 (Bhaskar and Lakshmikantham [19]) *Let* $(X,\preccurlyeq)$ *be a partially ordered set endowed with a metric* d. *Let* $F : X \times X \to X$ *be a given mapping. Suppose that the following conditions hold*:

(i) (X,d) *is complete*;
(ii) F *has the mixed monotone property*;
(iii) F *is continuous or* X *has the following properties*:

  (X_1) *if a nondecreasing sequence* $\{x_n\}$ *in* X *converges to some point* $x \in X$, *then* $x_n \preccurlyeq x$ *for all* n,
  (X_2) *if a decreasing sequence* $\{y_n\}$ *in* X *converges to some point* $y \in X$, *then* $y_n \succcurlyeq y$ *for all* n;

(iv) *there exist* $x_0, y_0 \in X$ *such that* $x_0 \preccurlyeq F(x_0,y_0)$ *and* $y_0 \succcurlyeq F(y_0,x_0)$;

(v) *there exists a constant* $k \in (0,1)$ *such that for all* $(x,y),(u,v) \in X \times X$ *with* $x \succcurlyeq u$ *and* $y \preccurlyeq v$,

$$d\big(F(x,y),F(u,v)\big) \leq \frac{k}{2}\big[d(x,u)+d(y,v)\big].$$

Then F has a coupled fixed point $(x^*,y^*) \in X \times X$. *Moreover, if for all* $(x,y),(u,v) \in X \times X$ *there exists* $(z_1,z_2) \in X \times X$ *such that* $(x,y) \preccurlyeq_2 (z_1,z_2)$ *and* $(u,v) \preccurlyeq_2 (z_1,z_2)$, *we have uniqueness of the coupled fixed point and* $x^* = y^*$.

Theorem A.5 (Berinde and Borcut [22]) *Let* $(X,\preccurlyeq)$ *be a partially ordered set and suppose that there is a metric d on X such that* (X,d) *is a complete metric space. Let* $F : X \times X \times X \to X$ *be a mapping having the mixed g-monotone property. Assume that there exist constants* $j,k,\ell \in [0,1)$ *with* $j+k+\ell < 1$ *such that*

$$d\big(F(x,y,z),F(u,v,w)\big) \leq jd(x,u)+kd(y,v)+\ell d(z,w)$$

for all $x,y,z,u,v,w \in X$ *with* $x \preccurlyeq u$, $y \succcurlyeq v$, $z \preccurlyeq w$. *Suppose that either F is continuous or* $(X,d,\preccurlyeq)$ *has the following properties*:

(a) *if a nondecreasing sequence* $\{x_m\} \to x$, *then* $x_m \preccurlyeq x$ *for all m*;

(b) *if a nondecreasing sequence* $\{y_m\} \to y$, *then* $y_m \preccurlyeq y$ *for all m*.

If there exist $x_0,y_0,z_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0,y_0,z_0), \qquad y_0 \succcurlyeq F(y_0,x_0,y_0) \quad \textit{and} \quad z_0 \preccurlyeq F(z_0,y_0,x_0),$$

then there exist $x,y,z \in X$ *such that*

$$x = F(x,y,z), \qquad y = F(y,x,y) \quad \textit{and} \quad z = F(z,y,x).$$

A quadruple version was obtained by Karapınar and Luong in [23].

Theorem A.6 (Karapınar and Luong [23]) *Let* $(X,\preccurlyeq)$ *be a partially ordered set and* (X,d) *be a complete metric space. Let* $F : X \times X \times X \times X \to X$ *be a mapping having the mixed monotone property. Assume that there exists a constant* $k \in [0,1)$ *such that*

$$d\big(F(x,y,z,w),F(u,v,r,t)\big) \leq \frac{k}{4}\big[d(x,u)+d(y,v)+d(z,r)+d(w,t)\big]$$

for all $x,y,z,u,v,w \in X$ *with* $x \succcurlyeq u$, $y \preccurlyeq v$, $z \succcurlyeq r$ *and* $w \preccurlyeq t$. *Suppose that there exist* $x_0,y_0,z_0,w_0 \in X$ *such that*

$$x_0 \preccurlyeq F(x_0,y_0,z_0,w_0), \qquad y_0 \succcurlyeq F(y_0,z_0,w_0,x_0), \qquad z_0 \preccurlyeq F(z_0,w_0,x_0,y_0) \quad \textit{and}$$
$$w_0 \succcurlyeq F(w_0,x_0,y_0,z_0).$$

Suppose that either F is continuous or $(X,d,\preccurlyeq)$ *has the following properties*:

(a) *if a nondecreasing sequence* $\{x_m\} \to x$, *then* $x_m \preccurlyeq x$ *for all m*;

(b) *if a nondecreasing sequence* $\{y_m\} \to y$, *then* $y_m \preccurlyeq y$ *for all m*.

Then there exist $x, y, z, w \in X$ *such that*

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z \quad \text{and} \quad F(w,x,y,z) = w.$$

Later, Berzig and Samet extended the previous result to the multidimensional case in the following way.

Theorem A.7 (Berzig and Samet [24]) *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there is a metric* d *on* X *such that* (X,d) *is a complete metric space. For* N, m *positive integers,* $N \geq 2$, $1 \leq m < N$, *let* $F : X^N \to X$ *be a continuous mapping having the m-mixed monotone property. Assume that there exist the constants* $\delta_i \in [0,1)$ *with* $\sum_{i=1}^N \delta_i < 1$ *for which*

$$d\big(F(U),F(V)\big) \leq \sum_{i=1}^{N} \delta_i d(x_i,y_i)$$

for all $U = (x_1,\ldots,x_N)$, $V = (y_1,\ldots,y_N) \in X^N$ *such that*

$$x_1 \preccurlyeq y_1, \qquad \ldots, \qquad x_m \preccurlyeq y_m, \qquad x_{m+1} \succcurlyeq y_{m+1}, \qquad \ldots, \qquad x_N \succcurlyeq y_N.$$

If there exists $U^{(0)} = (x_1^{(0)},\ldots,x_N^{(0)}) \in X^N$ *such that*

$$\begin{aligned}
&x_1^{(0)} \preccurlyeq F\big(x^{(0)}\big[\varphi_1(1:m)\big], x^{(0)}\big[\psi_1(m+1:N)\big]\big),\\
&\vdots\\
&x_m^{(0)} \preccurlyeq F\big(x^{(0)}\big[\varphi_m(1:m)\big], x^{(0)}\big[\psi_m(m+1:N)\big]\big),\\
&x_{m+1}^{(0)} \succcurlyeq F\big(x^{(0)}\big[\varphi_{m+1}(1:m)\big], x^{(0)}\big[\psi_{m+1}(m+1:N)\big]\big),\\
&\vdots\\
&x_N^{(0)} \succcurlyeq F\big(x^{(0)}\big[\varphi_N(1:m)\big], x^{(0)}\big[\psi_N(m+1:N)\big]\big),
\end{aligned}$$

where $\varphi_1,\ldots,\varphi_m : \{1,\ldots,m\} \to \{1,\ldots,m\}$, $\psi_1,\ldots,\psi_m : \{m+1,\ldots,N\} \to \{m+1,\ldots,N\}$, $\varphi_{m+1},\ldots,\varphi_N : \{1,\ldots,m\} \to \{m+1,\ldots,N\}$, *and* $\psi_{m+1},\ldots,\psi_N : \{m+1,\ldots,N\} \to \{1,\ldots,m\}$, *then there exists* $(x_1,x_2,\ldots,x_N) \in X^N$ *satisfying*

$$\begin{aligned}
&x_1 = F\big(x\big[\varphi_1(1:m)\big], x\big[\psi_1(m+1:N)\big]\big),\\
&\vdots\\
&x_m = F\big(x\big[\varphi_m(1:m)\big], x\big[\psi_m(m+1:N)\big]\big),\\
&x_{m+1} = F\big(x\big[\varphi_{m+1}(1:m)\big], x\big[\psi_{m+1}(m+1:N)\big]\big),\\
&\vdots\\
&x_N = F\big(x\big[\varphi_N(1:m)\big], x\big[\psi_N(m+1:N)\big]\big).
\end{aligned}$$

Theorem A.8 (Dutta and Choudhury [41]) *Let (X,d) be a complete metric space and let $T:X\to X$ be a self-mapping satisfying the inequality*

$$\psi\bigl(d(Tx,Ty)\bigr)\leq\psi\bigl(d(x,y)\bigr)-\varphi\bigl(d(x,y)\bigr)$$

for all $x,y\in X$, where $\psi,\varphi:[0,\infty[\to\infty$ are both continuous and monotone nondecreasing functions with $\psi(t)=\varphi(t)=0$ if and only if $t=0$. Then T has a unique fixed point.

Theorem A.9 (Luong and Thuan [42]) *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0,y_0\in X$ with $x_0\preccurlyeq F(x_0,y_0)$ and $y_0\succcurlyeq F(y_0,x_0)$. Suppose that there exist $\psi\in\Psi$ and $\varphi\in\Phi$ such that*

$$\psi\bigl(d\bigl(F(x,y),F(u,v)\bigr)\bigr)\leq\frac{1}{2}\psi\bigl(d(x,u)+d(y,v)\bigr)-\varphi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

for all $x,y,u,v\geq X$ with $x\succcurlyeq u$ and $y\preccurlyeq v$. Suppose either

(a) *F is continuous, or*
(b) *X has the following properties:*
  (i) *if a nondecreasing sequence $\{x_m\}\to x$, then $x_m\preccurlyeq x$ for all m,*
  (ii) *if a nonincreasing sequence $\{y_m\}\to y$, then $y_m\succcurlyeq y$ for all m.*

Then there exist $x,y\in X$ such that $x=F(x,y)$ and $y=F(y,x)$, that is, F has a coupled fixed point in X.

Appendix 2: Proof of statements of Section 3 and Remark 5.4

Recall that $\mathbb{I}=]-2,1]$, $\mathbb{X}=\mathbb{I}\cup\{2,3,4,5,6,7\}$,

$$x\preccurlyeq y\quad\Leftrightarrow\quad\begin{cases}\text{either} & x=y,\\ \text{or} & \{x,y\}=\{2,3\},\\ \text{or} & (x,y\in\mathbb{I}\text{ and }x\leq y),\end{cases}$$

and $d:\mathbb{X}\times\mathbb{X}\to[0,\infty[$ and $T:\mathbb{X}\to\mathbb{X}$ are defined by

$$d(x,y)=\begin{cases}0, & \text{if }\{x,y\}=\{2,3\}\text{ or }\{x,y\}=\{4,5\},\\ 0.5, & \text{if }x=y=5,\\ 1, & \text{if }(x,y)=(6,7),\\ 2, & \text{if }(x,y)=(7,6),\\ |x-y|, & \text{otherwise.}\end{cases}\qquad Tx=\begin{cases}\frac{x}{2}, & \text{if }x\in\mathbb{I},\\ 0, & \text{if }x\in\{2,3,6,7\},\\ -1, & \text{if }x=4,\\ 1, & \text{if }x=5.\end{cases}$$

We firstly prove some properties of this space.

(P_1) The binary relation $\preccurlyeq$ *is a preorder on* $\mathbb{X}$ (reflexive and transitive), but *it is not a partial order* since $2\preccurlyeq 3$, $3\preccurlyeq 2$ and $2\neq 3$. Furthermore, if $x,y\in\mathbb{X}$ and $x\preccurlyeq y$, then

$$x\in\mathbb{I}\quad\Leftrightarrow\quad y\in\mathbb{I};\qquad x\in\{2,3\}\quad\Leftrightarrow\quad y\in\{2,3\};\qquad x\in\{4,5,6,7\}\quad\Rightarrow\quad y=x.$$

If $x\preccurlyeq y\preccurlyeq z$, then either $x,y,z\in\mathbb{I}$ or $x,y,z\in\{2,3\}$ or $x=y=z\in\{4,5,6,7\}$. The same is true if $y\preccurlyeq x$ and $y\preccurlyeq z$.

(P_2) *d does not verify* (M_1) because $d(5,5) = 0.5$.

(P_3) *d does not verify* (M_2) since $d(2,3) = 0$ but $2 \neq 3$.

(P_4) *d does not verify* (M_3) since $d(6,7) = 1$ and $d(7,6) = 2$.

(P_5) *d does not verify* (M_4): if $x = 2$, $y = 4$ and $z = 3$, then $d(x,y) = 2$ and $d(z,x) + d(z,y) = 1$.

(P_6) *$(\mathbb{X}, d)$ is not complete.* Since $\mathbb{I} =]-2,1] \subset \mathbb{X}$ and $d|_{\mathbb{I}\times\mathbb{I}}$ is the Euclidean metric on $\mathbb{I}$, then the sequence $\{x_m = -2 + 1/m\}_{m\in\mathbb{N}}$ is d-Cauchy but it is not d-convergent in $\mathbb{X}$.

(P_7) *If $\{x_m\}$ is a $\preccurlyeq$-nondecreasing sequence in $\mathbb{X}$, then one, and only one, of the following cases holds.*

- $\triangleright$ *$x_m \in \mathbb{I}$ for all m.*
- $\triangleright$ *$x_m \in \{2,3\}$ for all m.*
- $\triangleright$ *There exists $z_0 \in \{4,5,6,7\}$such that $x_m = z_0$ for all m* (that is, x_m is a constant sequence).

It follows from that fact that points of $\mathbb{I}$ (respectively, $\{2,3\}$, $\{4,5,6,7\}$) are only $\preccurlyeq$-related with points of $\mathbb{I}$ (respectively, $\{2,3\}$, themselves), and $x_1 \preccurlyeq x_m$ for all $m \in \mathbb{N}$.

(P_8) *If $\{x_m\}$ is a $\preccurlyeq$-nondecreasing, d-convergent sequence in $\mathbb{X}$, then one, and only one, of the following cases holds.*

- $\triangleright$ *$x_m \in \mathbb{I}$ for all m. In this case, its d-limit is also in $\mathbb{I}$.*
- $\triangleright$ *$x_m \in \{2,3\}$ for all m. In this case, its d-limit is also in $\{2,3\}$.*
- $\triangleright$ *There exists $z_0 \in \{4,6,7\}$ such that $x_m = z_0$ for all m. In this case, its d-limit is z_0.*

It follows from (P_7). Notice that $z_0 \neq 5$ since $d(5,5) = 0.5 > 0$.

(P_9) *T is not a d-contraction* (that is, there is no $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in \mathbb{X}$) because $d(T4,T5) = d(-1,1) = 2$ but $d(4,5) = 1$.

Now we prove assertions (a)-(j), (p) and (p′) taking into account that g is the identity mapping on $\mathbb{X}$.

(a) $T(\mathbb{X}) \subseteq \mathbb{X}$. It is obvious.

(b) *T is $\preccurlyeq$-nondecreasing.* Let $x,y \in \mathbb{X}$ be such that $x \preccurlyeq y$ and $x \neq y$. By (P_1), if $x \in \mathbb{I}$, then $y \in \mathbb{I}$ and $x \leq y$. Then $Tx = x/2 \leq y/2 = Ty$, being $Tx, Ty \in \mathbb{I}$, so $Tx \preccurlyeq Ty$. If $x \in \{2,3\}$, then $y \in \{2,3\}$, so $Tx = Ty$. The case $x \in \{4,5,6,7\}$ is impossible since $y \neq x$.

(c) *There exists $x_0 \in \mathbb{X}$ such that $x_0 \preccurlyeq Tx_0$.* If $x_0 = -1$, then $x_0 = -1 \preccurlyeq -1/2 = Tx_0$.

(d) *There exists $k \in [0,1[$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$ for which $x \preccurlyeq y$.* Let $k = 1/2$ and assume that $x,y \in X$ are such that $x \preccurlyeq y$. By (P_1), if $x \in \mathbb{I}$, then $y, Tx, Ty \in \mathbb{I}$ and $d(Tx,Ty) = |x/2 - y/2| = |x - y|/2 = (1/2)d(x,y)$. If $x \in \{2,3\}$, then $y \in \{2,3\}$ and $d(Tx,Ty) = 0 = d(x,y)$. If $x \in \{4,5,6,7\}$, then $y = x$ and $Tx = Ty \in \{-1,0,1\} \subset \mathbb{I}$. Therefore $d(Tx,Ty) = 0$.

(e) *$d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathbb{X}$ such that $x \preccurlyeq y \preccurlyeq z$.* By (P_1), there are only three cases. If $x,y,z \in \mathbb{I}$, the triangular inequality holds because it is true in $\mathbb{R}$ provided with the Euclidean metric. If $x,y,z \in \{2,3\}$, then all distances are zero. Finally, if $x = y = z \in \{4,5,6,7\}$, all distances are either zero (if $x = y = z \in \{4,6,7\}$) or 0.5 (if $x = y = z = 5$).

(f) *$d(x,y) \leq d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{X}$ such that $x \preccurlyeq y \preccurlyeq z$.* It is similar to the previous one using (P_1).

(g) *Every d-Cauchy, $\preccurlyeq$-nondecreasing sequence in $\mathbb{X}$ is d-convergent.* It follows from (P_7).

(h) *If $\{x_m\}$ is a $\preccurlyeq$-nondecreasing sequence and $\{x_m\}$ d-converges to $x \in \mathbb{X}$, then $x_m \preccurlyeq x$ for all m.* It also follows from (P_7).

(i) *$d(x,z) \leq d(y,x) + d(y,z)$ for all $x, y, z \in \mathbb{X}$ such that $y \preccurlyeq x$ and $y \preccurlyeq z$.* It is similar to (e) using (P_1).

(j) *Every d-precoincidence point of T and g is a coincidence point of T and g.* Assume that $d(Tx, gx) = d(gx, Tx) = 0$. Since $Tx \in T(\mathbb{X}) \subset \mathbb{I}$, it is impossible $Tx, gx \in \{2, 3\}$ or $Tx, gx \in \{4, 5\}$ (the only cases, far from $\mathbb{I}$, in which the distance between them can be zero). Then $d(Tx, gx) = 0$ implies that $Tx, gx \in \mathbb{I}$ and $0 = d(Tx, gx) = |Tx - gx|$, so $Tx = gx$.

(p) *$g(\mathbb{X}) = \mathbb{X}$ is $\preccurlyeq$-nondecreasing d-closed.* It is immediate: indeed, $g(\mathbb{X}) = \mathbb{X}$ is d-closed.

(p′) *T and g are $(d, \preccurlyeq)$-nondecreasing-continuous and commuting, and g is $\preccurlyeq$-nondecreasing.* It is only necessary to prove that T is $(d, \preccurlyeq)$-nondecreasing-continuous. Actually, it follows from (P_8) since there are only three cases: if $x_m \in \mathbb{I}$ for all m, its d-limit x_0 is also in $\mathbb{I}$, and $\{Tx_m = -x_m/2\}$ d-converges to $Tx_0 = x_0/2$; if $x_m \in \{2, 3\}$ for all m, then $x_0 \in \{2, 3\}$ and $Tx_m = 0 = Tx_0$ for all m; finally, if there exists $z_0 \in \{4, 6, 7\}$ such that $x_m = z_0$ for all m, then $x_0 = z_0 = x_m$ for all m.

We point out that T is not d-continuous: if $x_m = 4$ and $x_0 = 5$, then $\{x_m\} \to x_0$ but $\{Tx_m = T4 = -1\}$ does not d-converge to $T5 = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References
1. Brooks, RM, Schmitt, K: The contraction mapping principle and some applications. Electron. J. Differ. Equ., Monograph 09 (2009). Available in http://ejde.math.txstate.edu or http://ejde.math.unt.edu
2. Aydi, H: Some coupled fixed point results on partial metric spaces. Int. J. Math. Math. Sci. **2011**, Article ID 647091 (2011)
3. Shatanawi, W, Samet, B, Abbas, M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Model. **55**(3-4), 680-687 (2012)
4. Agarwal, RP, Karapınar, E: Remarks on some coupled fixed point theorems in *G*-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 2 (2013)
5. Agarwal, RP, Kadelburg, Z, Radenović, S: On coupled fixed point results in asymmetric *G*-metric spaces. J. Inequal. Appl. **2013**, Article ID 528 (2013)
6. Kadelburg, Z, Nashine, HK, Radenović, S: Common coupled fixed point results in partially ordered *G*-metric spaces. Bull. Math. Anal. Appl. **4**, 51-63 (2012)
7. Mustafa, Z, Sims, B: Fixed point theorems for contractive mappings in complete *G*-metric spaces. Fixed Point Theory Appl. **2009**, Article ID 917175 (2009)
8. Fang, JX: On fixed point theorem in fuzzy metric spaces. Fuzzy Sets Syst. **46**, 107-113 (1992)
9. Gregori, V, Sapena, A: On fixed point theorem in fuzzy metric spaces. Fuzzy Sets Syst. **125**, 245-252 (2002)
10. Alaca, C, Turkoglu, D, Yildiz, C: Fixed points in intuitionistic fuzzy metric spaces. Chaos Solitons Fractals **29**, 1073-1078 (2006)
11. Cho, YJ, Roldán, A, Martínez-Moreno, J, Roldán, C: Coupled coincidence point theorems in (intuitionistic) fuzzy normed spaces. J. Inequal. Appl. **2013**, Article ID 104 (2013)
12. Hadžić, O, Pap, E: The Fixed Point Theory in Probabilistic Metric Spaces. Kluwer Academic, Dordrecht (2001)
13. Pap, E, Hadžić, O, Mesiar, R: A fixed point theorem in probabilistic metric spaces and an application. J. Math. Anal. Appl. **202**, 433-449 (1996)
14. Fang, JX: Fixed point theorems of local contraction mappings on Menger spaces. Appl. Math. Mech. **12**, 363-372 (1991)
15. Fang, JX: Common fixed point theorems of compatible and weakly compatible maps in Menger spaces. Nonlinear Anal. **71**(5-6), 1833-1843 (2009)

16. Turinici, M: Abstract comparison principles and multivariable Gronwall-Bellman inequalities. J. Math. Anal. Appl. **117**, 100-127 (1986)
17. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
18. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorem in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
19. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**(7), 1379-1393 (2006)
20. Lakshmikantham, V, Ćirić, LJ: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**(12), 4341-4349 (2009)
21. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
22. Borcut, M, Berinde, V: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 5929-5936 (2012)
23. Karapınar, E, Luong, NV: Quadruple fixed point theorems for nonlinear contractions. Comput. Math. Appl. **64**, 1839-1848 (2012)
24. Berzig, M, Samet, B: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63**, 1319-1334 (2012)
25. Roldán, A, Martínez-Moreno, J, Roldán, C: Multidimensional fixed point theorems in partially ordered complete metric spaces. J. Math. Anal. Appl. **396**, 536-545 (2012)
26. Kasahara, S: A remark on the contraction principle. Proc. Jpn. Acad. **44**, 21-26 (1968)
27. Kim, Y-W: Pseudo quasi metric spaces. Proc. Jpn. Acad. **44**, 1009-1012 (1968)
28. Reilly, IL, Subrahmanjam, PV, Vamanamurthy, MK: Cauchy sequences in quasi-pseudo-metric spaces. Monatshefte Math. **93**, 127-140 (1982)
29. Agarwal, RP, Karapınar, E, Roldán-López-de-Hierro, AF: Fixed point theorems in quasi-metric spaces and applications. J. Nonlinear Convex Anal. (accepted)
30. Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**(1), 1-9 (1984)
31. Kadelburg, Z, Radenović, S: A note on some recent best proximity point results for nonself mappings. Gulf J. Math. **1**, 36-41 (2013)
32. Karapınar, E, Moradi, S: Fixed point theory for cyclic generalized $(\varphi - \phi)$-contraction mappings. Ann. Univ. Ferrara **59**, 117-125 (2013)
33. Berzig, M, Karapınar, E, Roldán, A: Discussion on generalized-$(\alpha\psi, \beta\varphi)$-contractive mappings via generalized altering distance function and related fixed point theorems. Abstr. Appl. Anal. **2014**, Article ID 259768 (2014)
34. Popescu, O: Fixed points for (ψ, φ)-weak contractions. Appl. Math. Lett. **24**(1), 1-4 (2011)
35. Moradi, S, Farajzadeh, A: On the fixed point of (ψ, φ)-weak and generalized (ψ, φ)-weak contraction mappings. Appl. Math. Lett. **25**(10), 1257-1262 (2012)
36. Berzig, M: Generalization of the Banach contraction principle. Preprint arXiv:1310.0995 (2013)
37. Jleli, M, Samet, B: Remarks on *G*-metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, Article ID 210 (2012)
38. Bonsangue, MM, van Breugel, F, Rutten, JJMM: Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding. Theor. Comput. Sci. **193**, 1-51 (1998)
39. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**, 289-297 (2006)
40. Harjani, J, Sadarangani, K: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72**, 1188-1197 (2010)
41. Dutta, PN, Choudhury, BS: A generalization of contraction principle in metric spaces. Fixed Point Theory Appl. **2008**, Article ID 406368 (2008)
42. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)

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