Fixed Point Theory and Applications
a SpringerOpen Journal

**RESEARCH** **Open Access**

# Fixed points and orbits of non-convolution operators

Fernando León-Saavedra[1*] and Pilar Romero-de la Rosa[2]

*Correspondence:
fernando.leon@uca.es
[1]Department of Mathematics, University of Cádiz, Avda. de la Universidad s/n, Jerez de la Frontera, Cádiz 11405, Spain
Full list of author information is available at the end of the article

**Abstract**

A continuous linear operator $T$ on a Fréchet space $F$ is hypercyclic if there exists a vector $f \in F$ (which is called hypercyclic for $T$) such that the orbit $\{T^n f : n \in \mathbb{N}\}$ is dense in $F$. A subset $M$ of a vector space $F$ is spaceable if $M \cup \{0\}$ contains an infinite-dimensional closed vector space. In this paper note we study the orbits of the operators $T_{\lambda,b}f = f'(\lambda z + b)$ ($\lambda, b \in \mathbb{C}$) defined on the space of entire functions and introduced by Aron and Markose (J. Korean Math. Soc. 41(1):65-76, 2004). We complete the results in Aron and Markose (J. Korean Math. Soc. 41(1):65-76, 2004), characterizing when $T_{\lambda,b}$ is hypercyclic on $H(\mathbb{C})$. We characterize also when the set of hypercyclic vectors for $T_{\lambda,b}$ is spaceable. The fixed point of the map $z \to \lambda z + b$ (in the case $\lambda \neq 1$) plays a central role in the proofs.

**Keywords:** fixed point; Denjoy-Wolf theorem; non-convolution operator; hypercyclic operator; spaceability

## 1 Introduction

Let us denote by $F$ a complex infinite dimensional Fréchet space. A continuous linear operator $T$ defined on $F$ is said to be hypercyclic if there exists a vector $f \in F$ (called hypercyclic vector for $T$) such that the orbit ($\{T^n f : n \in \mathbb{N}\}$) is dense in $F$. We refer to the books [1, 2] and the references therein for further information on hypercyclic operators. From a modern terminology, a subset $M$ of a vector space $F$ is said to be spaceable if $M \cup \{0\}$ contains an infinite-dimensional closed vector space. The study of spaceability of (usually pathological) subsets is a natural question which has been studied extensively (see [1] Chapter 8 or the recent survey [3] and the references therein).

In 1991, Godefroy and Shapiro [4] showed that every continuous linear operator $L : H(\mathbb{C}) \to H(\mathbb{C})$ which commutes with translations (these operators are called convolution operators) and which is not a multiple of the identity is hypercyclic. This result unifies two classical results by Birkhoff and MacLane (see the survey [5]).

In [5], Aron and Markose introduced new examples of hypercyclic operators on $H(\mathbb{C})$ which are not convolution operators. Namely, $T_{\lambda,b}f = f'(\lambda z+b)$, $\lambda, b \in \mathbb{C}$. In the first section we show that if $\lambda \in \mathbb{D}$ and $b \in \mathbb{C}$ then $T_{\lambda,b}$ is not hypercyclic on $H(\mathbb{C})$. This result together with the results in [5] and [6] shows the following characterization: $T_{\lambda,b}$ is hypercyclic on $H(\mathbb{C})$ if and only if $|\lambda| \geq 1$. Thus, we complete the results of Aron and Markose [5] and Fernández and Hallack [6] characterizing when $T_{\lambda,b}$ ($\lambda, b \in \mathbb{C}$) is hypercyclic. Let us denote by $HC(T)$ the set of hypercyclic vectors for $T$. In Section 3 we characterize when $HC(T_{\lambda,b})$

is spaceable. Namely $HC(T_{\lambda,b})$ is spaceable if and only if $|\lambda| = 1$. During the proofs, it is essential to take into account the fixed point of the map $z \to \lambda z + b$ ($\lambda \neq 1$).

## 2 Characterizing the hypercyclicity of $T_{\lambda,b}$

The proof of this result follows the ideas of the proof of Proposition 14 in [5].

**Theorem 2.1** *For any $\lambda \in \mathbb{D}$ and $b \in \mathbb{C}$ and for any $f \in H(\mathbb{C})$, the sequence $T^n_{\lambda,b}f \to 0$ uniformly on compact subsets of $\mathbb{C}$. Therefore $T_{\lambda,b}$ is not hypercyclic on $H(\mathbb{C})$.*

*Proof* Set $\varphi(z) = \lambda z + b$, $\lambda \in \mathbb{D}$ and $b \in \mathbb{C}$. Since $\lambda \neq 1$, $\varphi(z)$ has a fixed point $z_0$. Indeed, $z_0 = \frac{b}{1-\lambda}$. We denote by $\varphi_n(z)$ the sequence of the iterates defined by

$$\varphi_n(z) = \varphi \circ \cdots \circ \varphi \quad (n \text{ times}),$$

an easy computation yields

$$\varphi_n(z) = \lambda^n z + \frac{1-\lambda^n}{1-\lambda} b.$$

Let us observe that the iterates of the operator $T_{\lambda,b}$ have the form

$$T^n_{\lambda,b} f(z) = \lambda^{\frac{n(n-1)}{2}} f^{(n)}\left(\lambda^n z + \frac{(1-\lambda^n)b}{1-\lambda}\right) = \lambda^{\frac{n(n-1)}{2}} f^{(n)}\big(\varphi_n(z)\big),$$

where $f^{(n)}$ denotes the $n$th derivative of $f$. It is well known that if $\lambda \in \mathbb{D}$ then $z_0$ is an attractive fixed point, that is, $\varphi_n(z)$ converges to the fixed point $z_0$ uniformly on compact subsets. Indeed, let $R > 0$. If $|z| \leq R$, then

$$\big|\varphi_n(z) - z_0\big| = \left|\lambda^n z + \frac{(1-\lambda^n)b}{1-\lambda} - \frac{b}{1-\lambda}\right| \leq |\lambda|^n R + \frac{|\lambda|^n}{|1-\lambda|}|b| \to 0$$

as $n \to \infty$. Thus, there exists $n_0$ such that if $|z| \leq R$ then $|\varphi_n(z) - z_0| < 1/2$ for all $n \geq n_0$.

If $n \geq n_0$ and $|z| \leq R$, we have by the Cauchy inequality

$$\big|f^{(n)}\big(\varphi_n(z)\big)\big| \leq Cn!2^n, \quad \text{where } C = \max\big\{\big|f(w)\big| : |w| \leq 1\big\}.$$

Now, it follows from Stirling's formula that $n! \leq en^{n+1/2}e^{-n}$. Hence, if $|z| \leq R$ and $n \geq n_0$, then

$$\big|T^n_{\lambda,b} f(z)\big| \leq Cn!2^n|\lambda|^{\frac{n(n-1)}{2}} \leq Cen^{1/2}\left(\frac{2n|\lambda|^{(n-1)/2}}{e}\right)^n,$$

and since $2n|\lambda|^{(n-1)/2} \to 0$ as $n \to \infty$, we conclude that $\max_{|z|\leq R} |T^n_{\lambda,b} f(z)| \to 0$, as $n \to \infty$, as desired. We point out that this is a refinement of the argument by Aron and Markose. One of the referees chased the constants and recovered the factor $n^{1/2}$ that was missing but that does not break the argument. □

Theorem 13 in [5] and Theorem 2.1 give the following characterization.

**Theorem 2.2** *For any $\lambda \in \mathbb{C}$ and $b \in \mathbb{C}$, the operator $T_{\lambda,b}$ is hypercyclic in $H(\mathbb{C})$ if and only if $|\lambda| \geq 1$.*

## 3 Spaceability of the set of hypercyclic vectors for $T_{\lambda,b}$

As stated in [3], there are few non-trivial examples of subsets $M$ which are lineable (that is, $M \cup \{0\}$ contains an infinite-dimensional vector space) and are not spaceable. The following result provides the following examples: for $|\lambda| > 1$, the set $HC(T_{\lambda,b})$ is lineable but it is not spaceable.

Shkarin [7] showed that for the derivative operator $D$, the set of hypercyclic vectors $HC(D)$ is spaceable.

**Theorem 3.1** *For any $\lambda \in \mathbb{C}$ and $b \in \mathbb{C}$, $HC(T_{\lambda,b})$ is spaceable if and only if $|\lambda| = 1$.*

*Proof* Firstly, let us suppose that $|\lambda| > 1$, and let us prove that $HC(T_{\lambda,b})$ does not contain a closed infinite dimensional subspace. Let $z_0$ be the fixed point of $\varphi(z) = \lambda z + b$. Then we consider a sequence of norms defining the topology of $H(\mathbb{C})$. Namely, for $n \in \mathbb{N}$ and $f \in H(\mathbb{C})$, we write

$$p_n(f) = \max_{|z-z_0|\leq|\lambda|^{n/4}} \bigl|f(z)\bigr|.$$

It is easy to see that the above sequence of semi-norms is increasing and defines the original topology on $H(\mathbb{C})$.

Given the sequence of increasing semi-norms $\{p_n\}$, according to Theorem 10.25 in [2], it is sufficient to find a sequence of subspaces $M_n \subset H(\mathbb{C})$ of finite codimension, positive numbers $C_n \to \infty$ and $N \geq 1$ satisfying the following:

(a) $p_N(f) > 0, \forall f \in HC(T_{\lambda,b})$.

(b) $p_N(T^n_{\lambda,b}f) \geq C_n p_n(f), \forall f \in M_n$.

Indeed, let us consider the subspaces

$$M_n = \bigl\{f \in H(\mathbb{C}) : f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0\bigr\},$$

which are clearly of finite codimension.

Notice that $\varphi_n(z) - z_0 = \lambda^n(z - z_0)$, so that $\varphi_n(z)$ maps the disk $D(z_0, 1) = \{|z - z_0| \leq 1\}$ onto $D(z_0, \|\lambda|^n)$. Hence,

$$\begin{aligned} p_0\bigl(T^n_{\lambda,b}f\bigr) &= \max_{|z-z_0|\leq 1} \bigl|T^n_{\lambda,b}f(z)\bigr| \\ &= |\lambda|^{\frac{n(n-1)}{2}} \max_{|z-z_0|\leq 1} \bigl|f^{(n)}\bigl(\varphi_n(z)\bigr)\bigr| \\ &= |\lambda|^{\frac{n(n-1)}{2}} \max_{|\varphi_n(z)-z_0|\leq|\lambda|^{n+1}} \bigl|f^{(n)}\bigl(\varphi_n(z)\bigr)\bigr| \\ &= |\lambda|^{\frac{n(n-1)}{2}} \max_{|w-z_0|\leq|\lambda|^n} \bigl|f^{(n)}(w)\bigr|. \end{aligned}$$

If $f \in M_1$ then $f(z_0) = 0$, so that $f(z) = \int_{[z_0,z]} f'(\xi)\,d\xi$. Therefore we have

$$\max_{|z-z_0|\leq R} \bigl|f(z)\bigr| \leq R \max_{|z-z_0|\leq R} \bigl|f'(z)\bigr|,$$

and it follows easily by induction that if $f \in M_n$ then

$$\max_{|z-z_0|\leq R} \bigl|f(z)\bigr| \leq R^n \max_{|z-z_0|\leq R} \bigl|f^{(n)}(z)\bigr|.$$

Thus,

$$\begin{aligned} p_0\left(T_{\lambda,b}^n f\right) &= |\lambda|^{\frac{n(n-1)}{2}} \max_{|w-z_0|\le|\lambda|^n}\left|f^{(n)}(w)\right| \\ &\ge |\lambda|^{\frac{n(n-1)}{2}} \max_{|w-z_0|\le|\lambda|^{n/4}}\left|f^{(n)}(w)\right| \\ &\ge |\lambda|^{\frac{n(n-1)}{2}} |\lambda|^{-n^2/4} \max_{|w-z_0|\le|\lambda|^{n/4}}\left|f(w)\right| \\ &= |\lambda|^{\frac{n^2-2n}{4}} p_n(f), \end{aligned}$$

and it follows that condition (b) is satisfied with $N = 0$ and $C_n = |\lambda|^{\frac{n^2-2n}{4}} \to \infty$ as $n \to \infty$, and therefore $HC(T_{\lambda,b})$ is not spaceable.

Now, let us suppose that $|\lambda| = 1$, and let us prove that $HC(T_{\lambda,b})$ is spaceable. Indeed, let us suppose first that $\lambda = 1$. If $b = 0$ then $T_{1,0} = D$, and it was proved by Shkarin [7] that $HC(D)$ is spaceable. If $b \neq 0$ then $T_{1,b} = De^{bD}$, so that $T_{1,b} = \psi(D)$, where $\psi(z) = ze^{bz}$ is an entire function of exponential type that is not a polynomial, and according to Example 10.12 in [2, p.275], the space $HC(T_{1,b})$ is spaceable.

Now let us consider the case $\lambda \in \partial\mathbb{D} \setminus \{1\}$. Set $z_0 = \frac{b}{1-\lambda}$ the fixed point of $\varphi(z) = \lambda z + b$. According to Theorem 10.2 in [2], since $T_{\lambda,b}$ satisfies the hypercyclicity criterion for the full sequence of natural numbers, it suffices to exhibit an infinite dimensional closed subspace $M_0$ of $H(\mathbb{C})$ on which suitable powers of $T_{\lambda,b}$ tend to 0. Now the proof mimics some ideas contained in Example 10.13 in [2]. Indeed, for any $n \ge 1$, there is some $C_n > 0$ such that

$$x^n \le 2^x \quad \text{for all } x \ge C_n. \tag{1}$$

Let us consider a strictly increasing sequence of positive integers $(n_k)_k$ satisfying $n_{k+1} \ge C_{n_k}$. If $j \ge k+1$, then $n_j \ge n_{k+1} \ge C_{n_k}$, therefore by (1) we have

$$n_j^{n_k} \le 2^{n_j} \quad \text{for } j \ge k+1. \tag{2}$$

Let us consider $M_0$ the closed subspace of $H(\mathbb{C})$ of all entire functions $f$ of the form

$$f(z) = \sum_{k=1}^{\infty} a_k (z-z_0)^{n_k - 1},$$

and let us prove that $T_{\lambda,b}^{n_k} f \to 0$ uniformly on compact subsets as $k \to \infty$.

We have

$$\left(T^{n_k} f\right)(z) = \lambda^{\frac{n_k(n_k-1)}{2}} \left(D^{n_k} f\right)\left(\varphi_{n_k}(z)\right).$$

Notice that $|\lambda| = 1$ and the map $\varphi_{n_k}$ takes the disc $D(z_0, R)$ onto itself, so that

$$\begin{aligned} \max_{|z-z_0|\le R}\left|\left(T^{n_k} f\right)(z)\right| &= \max_{|z-z_0|\le R}\left|\left(D^{n_k} f\right)\left(\varphi_{n_k}(z)\right)\right| \\ &= \max_{|w-z_0|\le R}\left|\left(D^{n_k} f\right)(w)\right|. \end{aligned}$$

Finally, we have

$$\begin{aligned}\max_{|w-z_0|\le R}\left|\left(D^{n_k}f\right)(w)\right| &= \max_{|w-z_0|\le R}\left|\sum_{j=k+1}^{\infty} a_j D^{n_k}(w-z_0)^{n_j-1}\right| \\ &\le \sum_{j=k+1}^{\infty}|a_j|(n_j-1)(n_j-2)\cdots(n_j-n_k)R^{n_j-n_k-1} \\ &\le \sum_{j=k+1}^{\infty}|a_j|n_j^{n_k}R^{n_j} \\ &\le \sum_{j=k+1}^{\infty}|a_j|(2R)^{n_j}\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

In the last step we used inequality (2). This completes the proof of Theorem 3.1. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Both authors contributed equally in this article. They read and approved the final manuscript.

**Author details**
[1]Department of Mathematics, University of Cádiz, Avda. de la Universidad s/n, Jerez de la Frontera, Cádiz 11405, Spain.
[2]Department of Mathematics, IES Sofía, Doctor Marañón, 13, Jerez de la Frontera, Cádiz 11407, Spain.

**Acknowledgements**
The research was supported by Junta de Andalucía FQM-257. The authors would like to thank the referee for reading our manuscript carefully and for giving such constructive comments, which helped improving the quality of the paper substantially.

Received: 1 July 2014 Accepted: 14 October 2014 Published: 29 October 2014

**References**
1. Bayart, F, Matheron, É: Dynamics of Linear Operators. Cambridge Tracts in Mathematics, vol. 179, p. xiv+337. Cambridge University Press, Cambridge (2009). doi:10.1017/CBO9780511581113
2. Grosse-Erdmann, K-G, Peris Manguillot, A: Linear Chaos. Universitext, p. xii+386. Springer, London (2011). doi:10.1007/978-1-4471-2170-1
3. Bernal-González, L, Pellegrino, D, Seoane-Sepúlveda, JB: Linear subsets of nonlinear sets in topological vector spaces. Bull. Am. Math. Soc. (N.S.) **51**(1), 71-130 (2014). doi:10.1090/S0273-0979-2013-01421-6
4. Godefroy, G, Shapiro, JH: Operators with dense, invariant, cyclic vector manifolds. J. Funct. Anal. **98**(2), 229-269 (1991). doi:10.1016/0022-1236(91)90078-J
5. Aron, R, Markose, D: On universal functions. Satellite Conference on Infinite Dimensional Function Theory J. Korean Math. Soc. **41**(1), 65-76 (2004). doi:10.4134/JKMS.2004.41.1.065
6. Fernández, G, Hallack, AA: Remarks on a result about hypercyclic non-convolution operators. J. Math. Anal. Appl. **309**(1), 52-55 (2005). doi:10.1016/j.jmaa.2004.12.006
7. Shkarin, S: On the set of hypercyclic vectors for the differentiation operator. Isr. J. Math. **180**, 271-283 (2010). doi:10.1007/s11856-010-0104-z

doi:10.1186/1687-1812-2014-221
**Cite this article as:** León-Saavedra and Romero-de la Rosa: **Fixed points and orbits of non-convolution operators.** *Fixed Point Theory and Applications* 2014 **2014**:221.