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The existence and convergence of best proximity points for generalized proximal contraction mappings

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Abstract
In 2011, Sadiq Basha (Nonlinear Anal. 74:5844-5850, 2011) studied and established best proximity point theorems for proximal contractions of the first and the second kinds which are more general than the fixed point theorems of self-contractions. The purpose of this paper is to extend the notion of proximal contraction mappings of the first and the second kinds. We also establish the existence and convergence of best proximity point theorems for these classes and give an example to validate our main results.
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1 Introduction

Let X be an arbitrary nonempty set. A fixed point for a self-mapping $T : X \to X$ is a point $x \in X$ such that $Tx = x$. The applications of fixed point theory are very important in diverse disciplines of mathematics, statistics, chemistry, biology, computer science, engineering, and economics. One of the very popular tools of fixed point theory is the Banach contraction principle, which first appeared in 1922. It states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping (*i.e.*, $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, where $\alpha \in [0, 1)$), then T has a unique fixed point. It has been generalized in different ways by mathematicians over the years (see [1–4]). However, almost all such results relate to the existence of a fixed point for self-mappings.

One of the most interesting studies is the extension of Banach's contraction principle to the case of non-self-mappings. In fact, given nonempty closed subsets A and B of a complete metric space (X, d), a contraction non-self-mapping $T : A \to B$ does not necessarily have a fixed point.

Eventually, it is quite natural to seek an element x such that $d(x, Tx)$ is minimum, which implies that x and Tx are in close proximity to each other. As a matter of fact, $d(x, Tx)$ is at least $d(A, B)$, and best proximity point theorems accentuate the preceding viewpoint further to guarantee the existence of an element x such that $d(x, Tx)$ assumes the least possible value $d(A, B)$, thereby accomplishing the highest possible closeness between x and Tx. A point x in A for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T.

Whenever non-self-mapping T has no fixed point, a best proximity point represents an optimal approximate solution to the equation $Tx = x$, for the error involved, $d(x, Tx)$, attains the global minimal value $d(A, B)$ for any best proximity point x. One finds that best proximity point theorems are natural generalizations of the contraction principle to the case of non-self-mappings because a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self-mapping.

In 1969, a best approximation theorem was introduced by Fan [5]. Afterward, several authors have derived extensions of Fan's theorem in many directions (see, *e.g.*, [6–9]). Other works concerning the existence of a best proximity point theorems for single-valued and set-valued mappings have been established in [10–29].

Recently, Sadiq Basha in [30] gave necessary and sufficient conditions to the claim of the existence of a best proximity point for proximal contractions of the first kind and the second kind which are non-self-mapping analogs of contraction self-mappings and also established some best proximity and convergence theorems. However, the main result of Sadiq Basha [30] is an essential tool to claim the existence of a best proximity point and a sequence that converges to a best proximity point for some non-self-mappings. It is most interesting to find another auxiliary tool for the claim of the existence of a best proximity point and a sequence that converges to this point.

In this work, we introduce a new class of non-self-mappings. Indeed, the classes of proximal contractions of the first kind and the second kind are proper subclasses of these classes. We prove the existence and convergence as regards best proximity point theorems for these classes and also give some illustrative examples of our main results. Our results generalize, extend, and unify several well-known comparable results in the literature and these results can be applied to a much wider class of problems.

2 Preliminaries

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbf{R}$ and $\mathbf{N}$, respectively. We also suppose that A and B are nonempty subsets of a metric space (X, d) and use the following notations:

$$\begin{aligned} d(A,B) &:= \inf\{d(x,y) : x \in A \text{ and } y \in B\}, \\ A_0 &:= \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}. \end{aligned}$$

We observe that if $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Also, if A_0 or B_0 is nonempty, then both A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$.

Definition 1 ([30]) A mapping $S : A \to B$ is said to be a *proximal contraction of the first kind* if there exists $\alpha \in [0, 1)$ such that

$$d(a, Sx) = d(b, Sy) = d(A, B) \quad \Longrightarrow \quad d(a, b) \leq \alpha d(x, y)$$

for all $a, b, x, y \in A$.

Clearly, a self-mapping that is a proximal contraction of the first kind is precisely a contraction. However, a non-self proximal contraction is not necessarily a contraction.

Definition 2 ([30]) A mapping $S : A \to B$ is said to be a *proximal contraction of the second kind* if there exists $\alpha \in [0,1)$ such that

$$d(a,Sx) = d(b,Sy) = d(A,B) \quad \Longrightarrow \quad d(Sa,Sb) \leq \alpha d(Sx,Sy)$$

for all $a,b,x,y \in A$.

The necessary condition for a self-mapping S to be a proximal contraction of the second kind is that

$$d(SSx,SSy) \leq \alpha d(Sx,Sy)$$

for all x,y in the domain of S. Therefore, every contraction self-mapping is a proximal contraction of the second kind but the converse is not true (see the example below).

Example 1 Consider $\mathbf{R}$ endowed with the Euclidean metric. Let the self-mapping $S : [0,1] \to [0,1]$ be defined as

$$S(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{otherwise.} \end{cases}$$

Then S is a proximal contraction of the second kind but not a contraction.

The above example shows that a self-mapping that is a proximal contraction of the second kind is not necessarily continuous.

Definition 3 ([30]) Let $S : A \to B$ and $T : B \to A$ be two mappings. The pair (S,T) is said to be a *proximal cyclic contraction pair* if there exists $\alpha \in [0,1)$ such that

$$d(a,Sx) = d(b,Ty) = d(A,B) \quad \Longrightarrow \quad d(a,b) \leq \alpha d(x,y) + (1-\alpha)d(A,B)$$

for all $a,b,x,y \in A$.

Definition 4 ([30]) Let $S : A \to B$ be a mapping and $g : A \to A$ be an isometry. The mapping S is said to *preserve the isometric distance with respect to* g if

$$d(Sgx,Sgy) = d(Sx,Sy)$$

for all $x,y \in A$.

Definition 5 ([30]) A point $x \in A$ is said to be a *best proximity point* of the mapping $S : A \to B$ if it satisfies the condition that

$$d(x,Sx) = d(A,B).$$

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 6 ([30]) A is said to be *approximatively compact with respect to B* if every sequence $\{x_n\}$ in A that satisfies the condition that $d(y, x_n) \to d(y, A)$ for some $y \in B$ has a convergent subsequence.

We observe that every set is approximatively compact with respect to itself, and that every compact set is approximatively compact. Moreover, A_0 and B_0 are nonempty sets if A is compact and B is approximatively compact with respect to A.

3 Main results

For mappings $S : A \to B$ and $g : A \to A \cup B$, we let $\Xi_S(g)$ be a collection of mappings $\xi_S : A \to [0, 1)$ which satisfies the following condition:

$$d(gx, Sy) = d(A, B) \quad \Longrightarrow \quad \xi_S(x) \leq \xi_S(y)$$

for $x, y \in A$.

Definition 7 A mapping $S : A \to B$ is said to be a *generalized proximal contraction of the first kind with respect to* $g : A \to A \cup B$ if there exists a mapping $\xi_S \in \Xi_S(g)$ such that

$$d(a, Sx) = d(b, Sy) = d(A, B) \quad \Longrightarrow \quad d(a, b) \leq \xi_S(x) d(x, y)$$

for all $a, b, x, y \in A$.

It is easy to see that a generalized proximal contraction of the first kind with respect to the mapping g reduces to proximal contraction of the first kind if we set $\xi_S(x) = \alpha$ for all $x \in A$ where $\alpha \in [0, 1)$. But the converse is not true (see the example below).

Example 2 Consider the metric space $\mathbf{R}^n$ with Euclidean metric, where $n \in \mathbf{N}$. Let

$$A = \{(y, 0, \pi, \pi, \ldots, \pi) \in \mathbf{R}^n : -1 < y < 1\}$$

and

$$B = \{(y, 1, \pi, \pi, \ldots, \pi) \in \mathbf{R}^n : -1 < y < 1\}.$$

Define two mappings $S : A \to B$ and $g : A \to A \cup B$ as follows:

$$S(y, 0, \pi, \pi, \ldots, \pi) = \left(\frac{y^2}{2}, 1, \pi, \pi, \ldots, \pi\right)$$

and

$$g(y, 0, \pi, \pi, \ldots, \pi) = (-y, 0, \pi, \pi, \ldots, \pi).$$

Then it is easy to see that $d(A, B) = 1$.

It is easy to show that there is no $\alpha \in [0,1)$ that satisfies

$$d(a,Sx) = d(b,Sy) = d(A,B) \quad \Longrightarrow \quad d(a,b) \leq \alpha d(x,y)$$

for all $a,b,x,y \in A$. Therefore, S is not a proximal contraction of the first kind.

Next, we show that S is a generalized proximal contraction of the first kind with respect to g. Consider a function $\xi_S : A \to [0,1)$ defined by

$$\xi_S(y,0,\pi,\pi,\ldots,\pi) = \frac{|y|+1}{2}.$$

It is easy to see that $\xi_S \in \Xi_S(g)$.

If $(y_1,0,\pi,\pi,\ldots,\pi),(y_2,0,\pi,\pi,\ldots,\pi) \in A$ such that

$$d\bigl(a,S(y_1,0,\pi,\pi,\ldots,\pi)\bigr) = d(A,B) = 1$$

and

$$d\bigl(b,S(y_2,0,\pi,\pi,\ldots,\pi)\bigr) = d(A,B) = 1$$

for all $a,b \in A$, then we have

$$a = \left(\frac{y_1^2}{2},0,\pi,\pi,\ldots,\pi\right), \qquad b = \left(\frac{y_2^2}{2},0,\pi,\pi,\ldots,\pi\right).$$

Therefore, it follows that

$$\begin{aligned} d(a,b) &= d\left(\left(\frac{y_1^2}{2},0,\pi,\pi,\ldots,\pi\right),\left(\frac{y_2^2}{2},0,\pi,\pi,\ldots,\pi\right)\right) \\ &= \left|\frac{y_1^2}{2} - \frac{y_2^2}{2}\right| \\ &= \left(\frac{|y_1+y_2|}{2}\right)|y_1-y_2| \\ &\leq \left(\frac{|y_1|+|y_2|}{2}\right)|y_1-y_2| \\ &\leq \left(\frac{|y_1|+1}{2}\right)|y_1-y_2| \\ &= \xi_S(y_1,0,\pi,\pi,\ldots,\pi)d\bigl((y_1,0,\pi,\pi,\ldots,\pi),(y_2,0,\pi,\pi,\ldots,\pi)\bigr). \end{aligned}$$

This implies that the non-self-mapping S is a generalized proximal contraction of the first kind with respect to g with the function ξ_S.

Definition 8 A mapping $S : A \to B$ is said to be a *generalized proximal contraction of the second kind with respect to* $g : A \to A \cup B$ if there exists a mapping $\xi_S \in \Xi_S(g)$ such that

$$d(a,Sx) = d(b,Sy) = d(A,B) \quad \Longrightarrow \quad d(Sa,Sb) \leq \xi_S(x)d(Sx,Sy)$$

for all $a,b,x,y \in A$.

Remark 1 The class of generalized proximal contractions of the second kind with respect to g is more general than the class of proximal contractions of the second kind (Definition 2).

Next, we give the result for generalized proximal contractions of the first kind.

Theorem 1 *Let (X,d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 or B_0 is nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions:*

(a) *S is a generalized proximal contractions of the first kind with respect to $g|_A$ and T is a generalized proximal contractions of the first kind with respect to $g|_B$.*
(b) *g is an isometry.*
(c) *The pair (S,T) is a proximal cyclic contraction.*
(d) *$S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$.*
(e) *$A_0 \subseteq g(A_0)$ and $B_0 \subseteq g(B_0)$.*

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x,y) = d(A,B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_n, Sx_{n-1}) = d(A,B),$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by

$$d(gy_n, Ty_{n-1}) = d(A,B),$$

converges to the element y.

Furthermore, a sequence $\{u_n\}$ in A converges to x if $\{\xi_S(x_n) : n \in \mathbf{N}\}$ is bounded with constant $M < 1$ and there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A,B)$.

Proof Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, it is ascertained that there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Sx_0) = d(A,B). \tag{1}$$

Again, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Sx_1) = d(A,B). \tag{2}$$

This process can be continued. Therefore, we can construct the sequence $\{x_n\}$ in A_0 such that

$$d(gx_n, Sx_{n-1}) = d(A,B) \tag{3}$$

for all $n \in \mathbf{N}$.

It follows from S being a generalized proximal contraction of the first kind with respect to $g|_A$ that

$$d(gx_{n+1}, gx_n) \leq \xi_S(x_n) d(x_n, x_{n-1}) \tag{4}$$

for all $n \in \mathbf{N}$. Since g is an isometry, we have

$$d(x_{n+1}, x_n) \leq \xi_S(x_n) d(x_n, x_{n-1}) \tag{5}$$

for all $n \in \mathbf{N}$. From (3) and the notion of a generalized proximal contraction of the first kind with respect to $g|_A$, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \xi_S(x_n) d(x_n, x_{n-1}) \\ &\leq \xi_S(x_{n-1}) d(x_n, x_{n-1}) \\ &\leq \xi_S(x_{n-2}) d(x_n, x_{n-1}) \\ &\ \ \vdots \\ &\leq \xi_S(x_0) d(x_n, x_{n-1}) \end{aligned} \tag{6}$$

for all $n \in \mathbf{N}$. By repeating (6), we get

$$d(x_{n+1}, x_n) \leq \zeta^n d(x_1, x_0) \tag{7}$$

for all $n \in \mathbf{N}$, where $\zeta = \xi_S(x_0) \in [0,1)$. For positive integers m and n with $n > m$, it follows from (7) that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \zeta^{n-1} d(x_1, x_0) + \zeta^{n-2} d(x_1, x_0) + \cdots + \zeta^m d(x_1, x_0) \\ &\leq \left(\frac{\zeta^m}{1-\zeta}\right) d(x_1, x_0), \end{aligned} \tag{8}$$

which implies that $d(x_n, x_m) \to 0$ as $m, n \to \infty$ and then $\{x_n\}$ is a Cauchy sequence in A. By the completeness of A, the sequence $\{x_n\}$ converges to some $x \in A$.

Similarly, in view of the fact that $T(B_0) \subseteq A_0$ and $A_0 \subseteq g(A_0)$, we can conclude that, for fixed $y_0 \in B_0$, there is a sequence $\{y_n\}$ in B such that

$$d(gy_n, Ty_{n-1}) = d(A, B) \tag{9}$$

for all $n \in \mathbf{N}$. Since T is a generalized proximal contraction of the first kind with respect to $g|_B$, we have

$$d(gy_{n+1}, gy_n) \leq \xi_T(y_n) d(y_n, y_{n-1}). \tag{10}$$

From g being an isometry, we get

$$d(y_{n+1}, y_n) \leq \xi_T(y_n) d(y_n, y_{n-1}) \tag{11}$$

for all $n \in \mathbf{N}$. By virtue of (9) and T being a generalized proximal contraction of the first kind with respect to $g|_B$, we get

$$\begin{aligned} d(x_{n+1}, y_n) &\le \xi_T(y_n) d(y_n, y_{n-1}) \\ &\le \xi_T(y_{n-1}) d(y_n, y_{n-1}) \\ &\le \xi_T(y_{n-2}) d(y_n, y_{n-1}) \\ &\ \ \vdots \\ &\le \xi_T(y_0) d(y_n, y_{n-1}) \end{aligned} \tag{12}$$

for all $n \in \mathbf{N}$. By repeating (12), we get

$$d(y_{n+1}, y_n) \le \eta^n d(y_1, y_0) \tag{13}$$

for all $n \in \mathbf{N}$, where $\eta = \xi_T(y_0) \in [0,1)$. For positive integers m and n with $n > m$, it follows from (13) that

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m) \\ &\le \eta^{n-1} d(y_1, y_0) + \eta^{n-2} d(y_1, y_0) + \cdots + \eta^m d(y_1, y_0) \\ &\le \left(\frac{\eta^m}{1-\eta}\right) d(y_1, y_0), \end{aligned} \tag{14}$$

which implies that $d(y_n, y_m) \to 0$ as $m, n \to \infty$ and then $\{y_n\}$ is a Cauchy sequence in B. By the completeness of B, the sequence $\{y_n\}$ converges to some $y \in B$.

Since the pair (S, T) is a proximal cyclic contraction, we have

$$d(x_{n+1}, y_{n+1}) = d(gx_{n+1}, gy_{n+1}) \le \alpha d(x_n, y_n) + (1-\alpha) d(A, B). \tag{15}$$

We take the limit in (15) as $n \to \infty$, and it follows that

$$d(x, y) = d(A, B), \tag{16}$$

which implies that $x \in A_0$ and $y \in B_0$. It follows from $S(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$ that there are $p \in A$ and $q \in B$ such that

$$d(p, Sx) = d(A, B) \tag{17}$$

and

$$d(q, Ty) = d(A, B). \tag{18}$$

From (3), (17), and the notion of a generalized proximal contraction of the first kind of S, we get

$$d(p, gx_n) \le \xi_S(x) d(x, x_{n-1}) \tag{19}$$

for all $n \in \mathbf{N}$. Letting $n \to \infty$, we conclude that $p = gx$. Therefore

$$d(gx, Sx) = d(A, B). \tag{20}$$

Similarly, we can show that $q = gy$ and then

$$d(gy, Ty) = d(A, B). \tag{21}$$

From (16), (20), and (21), we get

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

For the uniqueness, let us suppose that there exist $x^* \in A$ and $y^* \in B$ such that

$$d\bigl(x^*, y^*\bigr) = d\bigl(gx^*, Sx^*\bigr) = d\bigl(gy^*, Ty^*\bigr) = d(A, B).$$

Since g is an isometry and S and T are generalized proximal contractions of the first kind with respect to $g|_A$ and $g|_B$, respectively, it follows that

$$d\bigl(x, x^*\bigr) = d\bigl(gx, gx^*\bigr) \leq \xi_S(x) d\bigl(x, x^*\bigr)$$

and

$$d\bigl(y, y^*\bigr) = d\bigl(gy, gy^*\bigr) \leq \xi_T(y) d\bigl(y, y^*\bigr).$$

It follows from $\xi_S(x)$ and $\xi_T(y)$ being contained in $[0, 1)$ that $x = x^*$ and $y = y^*$.

On the other hand, let $\{u_n\}$ be a sequence in A and $\{\epsilon_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \text{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$. Since g is an isometry and S is a generalized proximal contraction of the first kind with respect to $g|_A$, we have

$$d(x_{n+1}, z_{n+1}) = d(gx_{n+1}, gz_{n+1}) \leq \xi_S(x_n) d(x_n, u_n) \tag{22}$$

and hence

$$d(x_{n+1}, z_{n+1}) \leq M d(x_n, u_n). \tag{23}$$

Given $\epsilon > 0$, we choose a positive integer N such that $\epsilon_n \leq \epsilon$ for all $n \geq N$. For each $n \geq N$, we get

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\leq d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\leq M d(x_n, u_n) + \epsilon_n, \end{aligned} \tag{24}$$

which implies that $d(x_{n+1},u_{n+1}) \leq M^n d(x_1,x_0) + \sum_{i=1}^{n} M^{n-i}\epsilon_i$. Therefore, for each $n \geq N$, we have

$$\begin{aligned} d(u_{n+1},x) &\leq d(u_{n+1},x_{n+1}) + d(x_{n+1},x) \\ &\leq M^n d(x_1,x_0) + \sum_{i=1}^{n} M^{n-i}\epsilon_i + d(x_{n+1},x) \\ &\leq M^n d(x_1,x_0) + M^{n-N}\sum_{i=1}^{N} M^{N-i}\epsilon_i + \epsilon \sum_{i=N+1}^{n} M^{n-i} + d(x_{n+1},x). \end{aligned}$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} d(u_{n+1},x) \leq \frac{\epsilon}{1-M}$. It follows from $\epsilon > 0$ being arbitrary that $\{u_n\}$ is convergent and it converges to x. This completes the proof of the theorem. □

Now, we give an example to illustrate Theorem 1.

Example 3 Consider the complete metric space $\mathbf{R}^2$ with Euclidean metric. Let

$$A = \{(0,y) : -1 \leq y \leq 1\} \quad \text{and} \quad B = \{(1,y) : -1 \leq y \leq 1\}.$$

Define three mappings $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ as follows:

$$S(0,y) = \left(1, \frac{y^2}{4}\right), \qquad T(1,y) = \left(0, \frac{y^2}{4}\right), \qquad g(x,y) = (x,-y).$$

Then it is easy to see that $d(A,B) = 1$, $A_0 = A$, $B_0 = B$, and the mapping g is an isometry.

Next, we claim that S is a generalized proximal contraction of the first kind with respect to $g|_A$ and T is a generalized proximal contraction of the first kind with respect to $g|_B$.

Consider a function $\xi_S : A \to [0,1)$ defined by

$$\xi_S(0,y) = \frac{|y|+1}{4}.$$

Then $\xi_S \in \Xi_S(g|_A)$. If $(0,y_1),(0,y_2) \in A$ such that

$$d(a,S(0,y_1)) = d(A,B) = 1 \quad \text{and} \quad d(b,S(0,y_2)) = d(A,B) = 1$$

for all $a,b \in A$, then we have

$$a = \left(0, \frac{y_1^2}{4}\right), \qquad b = \left(0, \frac{y_2^2}{4}\right).$$

Therefore, it follows that

$$\begin{aligned} d(a,b) &= d\left(\left(0, \frac{y_1^2}{4}\right), \left(0, \frac{y_2^2}{4}\right)\right) \\ &= \left|\frac{y_1^2}{4} - \frac{y_2^2}{4}\right| \\ &= \left(\frac{|y_1+y_2|}{4}\right)|y_1-y_2| \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{|y_1| + |y_2|}{4}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + 1}{4}\right)|y_1 - y_2| \\
&= \xi_S(0, y_1) d\big((0, y_1), (0, y_2)\big).
\end{aligned}$$

Hence S is a generalized proximal contraction of the first kind with respect to $g|_A$ with the function ξ_S.

Consider a function $\xi_T : B \to [0,1)$ defined by

$$\xi_T(1, y) = \frac{|y| + 1}{4}.$$

Then $\xi_T \in \Xi_S(g|_B)$. If $(1, y_1), (1, y_2) \in B$ such that

$$d\big(a, T(1, y_1)\big) = d(A, B) = 1 \quad \text{and} \quad d\big(b, T(1, y_2)\big) = d(A, B) = 1$$

for all $a, b \in B$, then we get

$$a = \left(1, \frac{y_1^2}{4}\right), \qquad b = \left(1, \frac{y_2^2}{4}\right).$$

Since

$$\begin{aligned}
d(a, b) &= d\left(\left(1, \frac{y_1^2}{4}\right), \left(1, \frac{y_2^2}{4}\right)\right) \\
&= \left|\frac{y_1^2}{4} - \frac{y_2^2}{4}\right| \\
&= \left(\frac{|y_1 + y_2|}{4}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + |y_2|}{4}\right)|y_1 - y_2| \\
&\leq \left(\frac{|y_1| + 1}{4}\right)|y_1 - y_2| \\
&= \xi_T(1, y_1) d\big((1, y_1), (1, y_2)\big),
\end{aligned}$$

we can conclude that T is a generalized proximal contraction of the first kind with respect to $g|_B$ with the function ξ_T.

Moreover, the pair (S, T) forms a proximal cyclic contraction and the other hypotheses of Theorem 1 are also satisfied. Further, it is easy to see that we have the unique elements $(0, 0) \in A$ and $(1, 0) \in B$ such that

$$d\big(g(0, 0), S(0, 0)\big) = d\big(g(1, 0), T(1, 0)\big) = d\big((0, 0), (1, 0)\big) = d(A, B).$$

Corollary 1 (Theorem 3.1 in [30]) *Let (X, d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 or B_0 is nonempty. Let $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ satisfy the following conditions*:

(a) *S and T are proximal contractions of the first kind.*
(b) *g is an isometry.*
(c) *The pair (S,T) is a proximal cyclic contraction.*
(d) $S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$.
(e) $A_0 \subseteq g(A_0)$ *and* $B_0 \subseteq g(B_0)$.

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(gx, Sx) = d(gy, Ty) = d(x,y) = d(A,B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A,B),$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A,B),$$

converges to the element y.

Furthermore, a sequence $\{u_n\}$ in A converges to x if there is a sequence of positive numbers $\{\epsilon_n\}$ such that

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{and} \quad d(u_{n+1}, z_{n+1}) \le \epsilon_n,$$

where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A,B)$.

Proof Since a proximal contractions of the first kind is a special case of a generalized proximal contraction of the first kind, we get this result from Theorem 1. □

If g is assumed to be the identity mapping, then Corollary 1 yields the following best proximity point theorem.

Corollary 2 (Corollary 3.3 in [30]) *Let (X,d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 or B_0 is nonempty. Let $S : A \to B$ and $T : B \to A$ satisfy the following conditions*:
(a) *S and T are proximal contractions of the first kind.*
(b) *The pair (S,T) is a proximal cyclic contraction.*
(c) $S(A_0) \subseteq B_0$, $T(B_0) \subseteq A_0$.

Then there exists a unique point $x \in A$ and there exists a unique point $y \in B$ such that

$$d(x, Sx) = d(y, Ty) = d(x,y) = d(A,B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Sx_n) = d(A,B),$$

converges to the element x. For any fixed $y_0 \in B_0$, the sequence $\{y_n\}$, defined by

$$d(y_{n+1}, Ty_n) = d(A,B),$$

converges to the element y.

Furthermore, a sequence $\{u_n\}$ *in* A *converges to* x *if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ *satisfies the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

Next, we establish a result for non-self-mappings which are generalized proximal contractions of the first kind and the second kind.

Theorem 2 *Let* (X, d) *be a complete metric space and* A *and* B *be nonempty, closed subsets of* X. *Further, suppose that* A_0 *or* B_0 *is nonempty. Let* $S : A \to B$ *and* $g : A \to A$ *satisfy the following conditions*:

(a) *S is generalized proximal contractions of first and second kinds with respect to g.*
(b) *g is an isometry.*
(c) *S preserves isometric distance with respect to g.*
(d) $S(A_0) \subseteq B_0$.
(e) $A_0 \subseteq g(A_0)$.

Then there exists a unique point $x \in A$ *such that*

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, *the sequence* $\{x_n\}$, *defined by*

$$d(gx_n, Sx_{n-1}) = d(A, B),$$

converges to the element x.

Furthermore, a sequence $\{u_n\}$ *in* A *converges to* x *if* $\{\xi_S(x_n) : n \in \mathbf{N}\}$ *bounded with constant* $M < 1$ *and there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ *satisfies the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof For fixed $x_0 \in A_0$, since $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, we can construct the sequence $\{x_n\}$ in A_0 similarly to Theorem 1 such that

$$d(gx_n, Sx_{n-1}) = d(A, B) \tag{25}$$

for all $n \in \mathbf{N}$. It follows from g being an isometry and by virtue of the fact that we have a generalized proximal contraction of the first kind with respect to g of S that

$$d(x_n, x_{n+1}) = d(gx_n, gx_{n+1}) \leq \xi_S(x_n) d(x_n, x_{n-1})$$

for all $n \in \mathbf{N}$. Similarly to the proof of Theorem 1, we can conclude that the sequence $\{x_n\}$ is a Cauchy sequence in A and so converges to some $x \in A$. As S is a generalized proximal contraction of the second kind with respect to g and preserves the isometric distance with respect to g,

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &= d(Sgx_n, Sgx_{n+1}) \\ &\leq \xi'_S(x_{n-1}) d(Sx_{n-1}, Sx_n) \\ &\leq \xi'_S(x_{n-2}) d(Sx_{n-1}, Sx_n) \\ &\quad \vdots \\ &\leq \xi'_S(x_0) d(Sx_{n-1}, Sx_n), \end{aligned} \tag{26}$$

which implies that $\{Sx_n\}$ is a Cauchy sequence in B and then it converges to some $y \in B$. Therefore, we can conclude that

$$d(gx, y) = \lim_{n\to\infty} d(gx_{n+1}, Sx_n) = d(A, B), \tag{27}$$

that is, $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, we have $gx = gz$ for some $z \in A_0$ and then $d(gx, gz) = 0$. By the fact that g is an isometry, we get $d(x, z) = d(gx, gz) = 0$. Hence x and z must be identical and so x becomes a point in A_0. As $S(A_0) \subseteq B_0$,

$$d(u, Sx) = d(A, B) \tag{28}$$

for some $u \in A$. It follows from (25), (28), and S being a generalized proximal contraction of the first kind with respect to g that

$$d(u, gx_{n+1}) \leq \xi_S(x) d(x, x_n) \tag{29}$$

for all $n \in \mathbf{N}$. Taking the limit as $n \to \infty$, we see that the sequence $\{gx_n\}$ converges to a point u. Owing to the fact that g is continuous, $\{gx_n\}$ converge to a point gx. By the uniqueness of the limit of the sequence, we conclude that $u = gx$. Therefore, we have the result that $d(gx, Sx) = d(u, Sx) = d(A, B)$. The uniqueness and the remaining part of the proof follow as in Theorem 1. This completes the proof of the theorem. □

Corollary 3 (Theorem 3.4 in [30]) *Let (X, d) be a complete metric space and A and B be nonempty, closed subsets of X. Further, suppose that A_0 or B_0 is nonempty. Let $S : A \to B$ and $g : A \to A$ satisfy the following conditions*:

(a) *S is proximal contractions of first and second kinds.*
(b) *g is an isometry.*
(c) *S preserves isometric distance with respect to g.*
(d) *$S(A_0) \subseteq B_0$.*
(e) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A$ such that

$$d(gx, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x.

Furthermore, a sequence $\{u_n\}$ *in* A *converges to* x *if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ *satisfies the condition that* $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof Since proximal contractions of the first kind and the second kind are special cases of generalized proximal contractions of the first and the second kinds, we get the result from Theorem 2. □

Corollary 4 (Corollary 3.5 in [30]) *Let* (X, d) *be a complete metric space and* A *and* B *be nonempty, closed subsets of* X. *Suppose that* A_0 *or* B_0 *is nonempty and* $S : A \to B$ *satisfy the following conditions*:

(a) *S is proximal contractions of first and second kinds*.

(b) $S(A_0) \subseteq B_0$.

Then there exists a unique point $x \in A$ *such that*

$$d(x, Sx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, *the sequence* $\{x_n\}$, *defined by*

$$d(x_{n+1}, Sx_n) = d(A, B),$$

converges to the element x.

Furthermore, a sequence $\{u_n\}$ *in* A *converges to* x *if there is a sequence of positive numbers* $\{\epsilon_n\}$ *such that*

$$\lim_{n\to\infty} \epsilon_n = 0 \quad \textit{and} \quad d(u_{n+1}, z_{n+1}) \leq \epsilon_n,$$

where $z_{n+1} \in A$ *satisfies the condition that* $d(z_{n+1}, Su_n) = d(A, B)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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