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Strong convergence theorem for quasi-Bregman strictly pseudocontractive mappings and equilibrium problems in Banach spaces

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Abstract

In this paper, we introduce a new iterative scheme by a hybrid method and prove a strong convergence theorem of a common element in the set of fixed points of a finite family of closed quasi-Bregman strictly pseudocontractive mappings and common solutions to a system of equilibrium problems in reflexive Banach space. Our results extend important recent results announced by many authors.

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1 Introduction

Let E be a real Banach space and C a nonempty closed convex subset of E. The normalized duality map from E to 2^{E^*} (E^* is the dual space of E) denoted by J is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

Let $T : C \to C$ be a map, a point $x \in C$ is called a fixed point of T if $Tx = x$, and the set of all fixed points of T is denoted by $F(T)$. The mapping T is called L-Lipschitzian or simply Lipschitz if there exists $L > 0$, such that $\|Tx - Ty\| \leq L\|x - y\|$, $\forall x, y \in C$ and if $L = 1$, then the map T is called nonexpansive.

Let $g : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem with respect to g is to find

$$z \in C \text{ such that } g(z, y) \geq 0, \quad \forall y \in C.$$

The set of solution of equilibrium problem is denoted by $\mathrm{EP}(g)$. Thus

$$\mathrm{EP}(g) := \{z \in C : g(z, y) \geq 0, \forall y \in C\}.$$

Numerous problems in physics, optimization and economics reduce to finding a solution of equilibrium problem. Some methods have been proposed to solve the equilibrium problem in Hilbert spaces; see for example Blum and Oettli [1], Combettes and Hirstoaga [2]. Recently, Tada and Takahashi [3, 4] and Takahashi and Takahashi [5] obtain weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and set of fixed points of a nonexpansive mapping in Hilbert space. In particular, Takahashi and Zembayashi [4] established a strong convergence theorem for finding a common element of the two sets by using the hybrid method introduced in Nakajo and Takahashi [6]. They also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

Reich and Sabach [7] and Kassay *et al.* [8] proved some convergence theorems for the solution of some equilibrium and variational inequality problems in the setting of reflexive Banach spaces.

Let $\phi : E \times E \to [0,\infty)$ denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E.$$

A mapping $T : C \to C$ is said to be quasi-ϕ strictly pseudocontractive, see [9], if $F(T) \neq \emptyset$ and there exists a constant $k \in (0,1]$ such that

$$\phi(p,Tx) \leq \phi(p,x) + k\phi(x,Tx), \quad \forall x \in C \text{ and } p \in F(T).$$

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Throughout this paper, we shall assume $f : E \to (-\infty,+\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$ the domain of f.

Let $x \in \operatorname{int}\operatorname{dom} f$; the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty,+\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

A function f on E is coercive [10] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

A function f on E is said be strongly coercive [11] if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x,y) := \lim_{t\to0^+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^\circ(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\operatorname{int}\operatorname{dom} f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\operatorname{int}\operatorname{dom} f$ (see also [12, 13]). We will need the following results.

Lemma 1.1 [14] *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 1.2 [15] The function f is said to be:

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom}\partial f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.3 Let E be a reflexive Banach space. Then we have:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [15], Theorem 5.4);
(ii) $(\partial f)^{-1} = \partial f^*$ (see [13]);
(iii) f is Legendre if and only if f^* is Legendre (see [15], Corollary 5.5);
(iv) if f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$ and $\operatorname{ran}\nabla f^* = \operatorname{dom} f = \operatorname{int}\operatorname{dom} f$ (see [15], Theorem 5.10).

Examples of Legendre functions were given in [15, 16]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E, *i.e.*, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$, defined as follows:

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle, \tag{1.1}$$

is called the Bregman distance with respect to f (see [17]). It is obvious from the definition of D_f that

$$D_f(z,x) = D_f(z,y) + D_f(y,x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle. \tag{1.2}$$

Recall that the Bregman projection [18] of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty, closed, and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f\big(P_C^f(x), x\big) = \inf\big\{D_f(y,x) : y \in C\big\}.$$

Concerning the Bregman projection, the following are well known.

Lemma 1.4 [19] *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*:

(a) $z = P^f_C(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$, $\forall y \in C$;

(b) $D_f(y, P^f_C(x)) + D_f(P^f_C(x), x) \leq D_f(y, x)$, $\forall x \in E$, $y \in C$.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of the total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\bigl\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\bigr\}.$$

The function f is called totally convex at x if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$ and is said to be totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\bigl\{\nu_f(x, t) : x \in B \cap \operatorname{dom} f\bigr\}.$$

Lemma 1.5 [20] *If $x \in \operatorname{dom} f$, then the following statements are equivalent*:

(i) *the function f is totally convex at x*;

(ii) *for any sequence $\{y_n\} \subset \operatorname{dom} f$*,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \quad \Rightarrow \quad \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Recall that the function f called sequentially consistent [19] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

Lemma 1.6 [21] *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 1.7 [22] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 1.8 [22] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let C be a nonempty, closed, and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(P^f_C(x_0), x_0)$ for any $n \in \mathbb{R}$, then $\{x_n\}$ converges strongly to $P^f_C(x_0)$.*

A mapping T is said to be Bregman firmly nonexpansive [23], if for all $x, y \in C$,

$$\bigl\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\bigr\rangle \leq \bigl\langle \nabla f(x) - \nabla f(y), Tx - Ty\bigr\rangle$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

A point $p \in C$ is said to be asymptotic fixed point of a map T, if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A point $p \in C$ is said to be strong asymptotic fixed point of a map T, if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\tilde{F}(T)$ the set of strong asymptotic fixed points of T. Let $f : E \to \mathbb{R}$, a mapping $T : C \to C$ is said to be Bregman relatively nonexpansive [24] if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and $D_f(p, T(x)) \leq D_f(p,x)$ for all $x \in C$ and $p \in F(T)$. The map $T : C \to C$ is said to be Bregman weak relatively nonexpansive if $F(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and $D_f(p, T(x)) \leq D_f(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$, and $D_f(p, T(x)) \leq D_f(p,x)$ for all $x \in C$ and $p \in F(T)$. In [22] quasi-Bregman relatively nonexpansive is called left quasi-Bregman relatively nonexpansive. A map $T : C \to C$ is called right quasi-Bregman relatively nonexpansive [25] if $F(T) \neq \emptyset$, and $D_f(T(x),p) \leq D_f(x,p)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-Bregman strictly pseudocontractive if there exist a constant $k \in [0,1)$ and $F(T) \neq \emptyset$ such that $D_f(p, Tx) \leq D_f(p,x) + kD_f(x, Tx)$ for all $x \in C$ and $p \in F(T)$. In particular, T is said to be quasi-Bregman relatively nonexpansive if $k = 0$ and T is said to be quasi-Bregman pseudocontractive if $k = 1$.

Very recently, Zhou and Gao [9] introduced this definition of a quasi-strict pseudocontraction related to the function ϕ and proved the convergence of a hybrid projection algorithm to a fixed point of a closed and quasi-strict pseudocontraction in a smooth and uniformly convex Banach space. They studied the strong convergence of the following scheme:

$$\begin{cases} x_0 \in E, \\ C_1 = C, \\ x_1 = \prod_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, Tx_n) \leq \frac{2}{1-k}\langle x_n - z, Jx_n - JTx_n\rangle\}, \\ x_{n+1} = \prod_{C_{n+1}}(x_0), \end{cases}$$

where $\prod_{C_{n+1}}$ is the generalized projection from E onto C_{n+1}. They proved that the sequence $\{x_n\}$ converges strongly to $\prod_{F(T)}(x_0)$.

Recently, Zegeye and Shahzad [26] proved a strong convergence theorem for the common fixed point of a finite family of right Bregman strongly nonexpansive mappings in a reflexive Banach space. Alghamdi *et al.* [27] proved a strong convergence theorem for the common fixed point of a finite family of quasi-Bregman nonexpansive mappings. Pang *et al.* [28] proved weak convergence theorems for Bregman relatively nonexpansive mappings. Shahzad and Zegeye [29] proved a strong convergence theorem for multivalued Bregman relatively nonexpansive mappings, while Zegeye and Shahzad [30] proved a strong convergence theorem for a finite family of Bregman weak relatively nonexpansive mappings.

Motivated and inspired by the above works, in this paper, we prove a new strong convergence theorem for a finite family of closed quasi-Bregman strictly pseudocontractive mapping and a system of equilibrium problems in a real reflexive Banach space. These results generalize and improve several recent results. We showed by an example that the class of quasi-Bregman strictly pseudocontractive mappings is a proper generalization of the class of quasi-ϕ-Bregman strictly pseudocontractive mappings.

2 Preliminaries

The next lemma will be useful in the proof of our main results.

Lemma 2.1 *Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subsets of E, let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a quasi-Bregman strictly pseudocontractive mapping with respect to f. Then, for any $x \in C$, $p \in F(T)$ and $k \in [0,1)$ the following hold*:

$$D_f(x, Tx) \le \frac{1}{1-k}\langle \nabla f(x) - \nabla f(Tx), x - p\rangle. \tag{2.1}$$

Proof Let $x \in C$, $p \in F(T)$ and $k \in [0,1)$, by definition of T, we have

$$D_f(p, Tx) \le D_f(p,x) + kD_f(x, Tx)$$

and, from (1.2), we obtain

$$D_f(p,x) + D_f(x, Tx) + \langle \nabla f(x) - \nabla f(Tx), p - x\rangle \le D_f(p,x) + kD_f(x, Tx),$$

which implies

$$D_f(x, Tx) \le \frac{1}{1-k}\langle \nabla f(x) - \nabla f(Tx), x - p\rangle.$$

This completes the proof. □

Lemma 2.2 [31] *Let E be a real reflexive Banach space, $f : E \to (-\infty, +\infty]$ be a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^N t_i D_f(z, x_i). \tag{2.2}$$

In order to solve the equilibrium problem, let us assume that a bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions [1]:

(A1) $g(x,x) = 0$, $\forall x \in C$;
(A2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \le 0$, $\forall x,y \in C$;
(A3) $\limsup_{t\downarrow 0} g(x + t(z-x), y) \le g(x,y)$, $\forall x,z,y \in C$;
(A4) the function $y \mapsto g(x,y)$ is convex and lower semi-continuous.

The resolvent of a bifunction g [2] is the operator $\mathrm{Res}_g^f : E \to 2^C$ defined by

$$\mathrm{Res}_g^f(x) = \{z \in C : g(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \ge 0, \forall y \in C\}. \tag{2.3}$$

From Lemma 1, in [32], if $f : (-\infty, +\infty] \to \mathbb{R}$ is a strongly coercive and Gâteaux differentiable function, and g satisfies conditions (A1)-(A4), then $\mathrm{dom}(\mathrm{Res}_g^f) = E$. The following lemma gives some characterization of the resolvent Res_g^f.

Lemma 2.3 [32] *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the conditions* (A1)-(A4), *then the following hold*:

(i) Res_g^f *is single-valued*;
(ii) Res_g^f *is a Bregman firmly nonexpansive operator*;
(iii) $F(\mathrm{Res}_g^f) = \mathrm{EP}(g)$;
(iv) $\mathrm{EP}(g)$ *is closed and convex subset of* C;
(v) *for all* $x \in E$ *and for all* $q \in F(\mathrm{Res}_g^f)$, *we have*

$$D_f\big(q, \mathrm{Res}_g^f(x)\big) + D_f\big(\mathrm{Res}_g^f(x), x\big) \le D_f(q, x). \tag{2.4}$$

3 Main result

Lemma 3.1 *Let* $f : E \to \mathbb{R}$ *be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of* E, *let* C *be a nonempty, closed, and convex subset of* E *and let* $T : C \to C$ *be a quasi-Bregman strictly pseudocontractive mapping with respect to* f. *Then* $F(T)$ *is closed and convex.*

Proof Let $F(T)$ be nonempty set. First we show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to z$ as $n \to \infty$, we need to show that $z \in F(T)$. From Lemma 2.1, we obtain

$$D_f(z, Tz) \le \frac{1}{1-k}\big\langle \nabla f(z) - \nabla f(Tz), z - x_n\big\rangle. \tag{3.1}$$

From (3.1), we have $D_f(z, Tz) \le 0$, and from [15], Lemma 7.3, it follows that $Tz = z$. Therefore $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $z_1, z_2 \in F(T)$, for any $t \in (0,1)$; putting $z = tz_1 + (1-t)z_2$, we need to show that $z \in F(T)$. From Lemma 2.1, we obtain, respectively,

$$D_f(z, Tz) \le \frac{1}{1-k}\big\langle \nabla f(z) - \nabla f(Tz), z - z_1\big\rangle \tag{3.2}$$

and

$$D_f(z, Tz) \le \frac{1}{1-k}\big\langle \nabla f(z) - \nabla f(Tz), z - z_2\big\rangle. \tag{3.3}$$

Multiplying (3.2) by t and (3.3) by $(1-t)$ and adding the results, we obtain

$$D_f(z, Tz) \le \frac{1}{1-k}\big\langle \nabla f(z) - \nabla f(Tz), z - z\big\rangle, \tag{3.4}$$

which implies $D_f(z, Tz) \le 0$, and from [15], Lemma 7.3, it follows that $Tz = z$. Therefore $F(T)$ is also convex. This completes the proof. □

We now prove the following theorem.

Theorem 3.2 *Let* C *be a nonempty, closed, and convex subset of a real reflexive Banach space* E *and* $f : E \to \mathbb{R}$ *a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of* E. *For each* $k = 1, 2, \ldots, m$, *let* g_k *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4) *and let* $\{T_{i=1}^N\}$ *be a finite family of* L_i*-Lipschitzian,* $i = 1, 2, 3, \ldots, N$, *closed and quasi-Bregman strictly pseudocontractive self* *mappings of C such that $F := (\bigcap_{k=1}^{m} \mathrm{EP}(g_k)) \cap (\bigcap_{i=1}^{N} F(T_i)) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 = x \in C$, $C_1 = C$ and*

$$\begin{cases} x_1 \in C, \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_n x_n)), \\ u_{j,n} = \mathrm{Res}_{g_j}^f y_n, \quad j = 1,2,3,\ldots,m, \\ w_n = \nabla f^*(\sum_{j=1}^{m} \beta_{j,n} \nabla f(u_{j,n})), \\ C_{n+1} = \{w \in C_n : D_f(x_n, w_n) \leq \frac{1}{1-k}\langle \nabla f(x_n) \\ \qquad - \nabla f(T_n x_n), x_n - w\rangle + \langle \nabla f(T_n x_n) - \nabla f(w_n), x_n - w\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^f(x), \quad n \in \mathbb{N}, \end{cases} \tag{3.5}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$, and $k \in [0,1)$, for each $i = 1,2,\ldots,N$, T_i is uniformly continuous; suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_{j,n}\}_{n=1}^{\infty}$, $j = 1,2,\ldots,m$ are sequences in $(0,1)$ such that (i) $\liminf_{n\to\infty}(1-\alpha_n) > 0$, (ii) $\sum_{j=1}^{m} \beta_{j,n} = 1$, $n \geq 1$. *Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_F^f(x)$, where P_F^f is the Bregman projection of E onto F.*

Proof The proof is divided into six steps.

Step I. Show that $F = (\bigcap_{j=1}^{m} \mathrm{EP}(g_j)) \cap (\bigcap_{i=1}^{N} F(T_i))$ is closed and convex. From Lemma 3.1, $\bigcap_{i=1}^{N} F(T_i)$ is closed and convex and from (iv) of Lemma 2.3, $\bigcap_{j=1}^{m} \mathrm{EP}(g_j)$ is closed and convex. So, $F = (\bigcap_{j=1}^{m} \mathrm{EP}(g_j)) \cap (\bigcap_{i=1}^{N} F(T_i))$ is closed and convex.

Step II. Show that C_n is closed and convex for all $n \geq 1$. For $n = 1$, $C_1 = C$ is closed and convex. Assume that C_h is closed and convex for some $h > 1$. For $w \in C_{h+1}$, one obtains

$$\begin{aligned} D_f(x_h, w_h) \leq{}& \frac{1}{1-k}\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\rangle \\ &+ \langle \nabla f(T_h x_h) - \nabla f(w_h), x_h - w\rangle; \end{aligned}$$

using the fact that $\langle \nabla f(x_h) - \nabla f(T_h x_h), \cdot\rangle$ and $\langle \nabla f(T_h x_h) - \nabla f(w_h), \cdot\rangle$ are continuous and linear in E, for $h \geq 1$, C_{h+1} is closed and convex.

Step III. Show that $F \subset C_n$ for every $n \geq 1$. Note that $F \subset C_1 = C$. Suppose $F \subset C_h$, for $h \geq 1$, then for all $w \in F \subset C_h$, since $u_{j,h} = \mathrm{Res}_{g_j}^f(y_h)$ for each $j = 1,2,\ldots,m$, from (2.2) and Lemma 2.3, we have

$$\begin{aligned} D_f(w, w_h) &= D_f\left(w, \nabla f^*\left(\sum_{j=1}^{m} \beta_{j,n} \nabla f(u_{j,n})\right)\right) \\ &\leq \sum_{j=1}^{m} \beta_{jh} D_f(w, u_{jh}) \\ &\leq \sum_{j=1}^{m} \beta_{jh} D_f(w, y_h) \\ &= D_f(w, y_h); \end{aligned} \tag{3.6}$$

also from (2.2) and (2.1), we obtain

$$\begin{aligned} D_f(w, y_h) &= D_f\big(w, \nabla f^*\big(\alpha_h \nabla f(x_h) + (1-\alpha_h)\nabla f(T_h x_h)\big)\big) \\ &\leq \alpha_h D_f(w, x_h) + (1-\alpha_h) D_f(w, T_h x_h) \end{aligned}$$

$$\begin{aligned}&\le \alpha_h D_f(w,x_h) + (1-\alpha_h)\big[D_f(w,x_h) + kD_f(x_h, T_h x_h)\big]\\&\le D_f(w,x_h) + kD_f(x_h, T_h x_h)\\&\le D_f(w,x_h) + \frac{k}{1-k}\big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle.\end{aligned}\tag{3.7}$$

But, from (1.2),

$$D_f(w,w_h) = D_f(w,x_h) + D_f(x_h,w_h) + \big\langle \nabla f(x_h) - \nabla f(w_h), w - x_h\big\rangle.\tag{3.8}$$

From (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned}D_f(x_h,w_h) &\le \frac{k}{1-k}\big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle\\&\quad + \big\langle \nabla f(x_h) - \nabla f(w_h), x_h - w\big\rangle\\&= \frac{k}{1-k}\big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle\\&\quad + \big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle\\&\quad + \big\langle \nabla f(T_h x_h) - \nabla f(w_h), x_h - w\big\rangle\\&= \left(\frac{k}{1-k} + 1\right)\big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle\\&\quad + \big\langle \nabla f(T_h x_h) - \nabla f(w_h), x_h - w\big\rangle\\&= \frac{1}{1-k}\big\langle \nabla f(x_h) - \nabla f(T_h x_h), x_h - w\big\rangle\\&\quad + \big\langle \nabla f(T_h x_h) - \nabla f(w_h), x_h - w\big\rangle.\end{aligned}\tag{3.9}$$

This shows that $w \in C_{h+1}$, which implies $F \subset C_n$ for every $n \ge 1$.

Step IV. Show that $\lim_{n\to\infty} D_f(x_n,x)$ exists. From (3.5), $x_n = P^f_{C_n}x$, which from (a) of Lemma 1.4 implies

$$\big\langle \nabla f(x) - \nabla f(x_n), y - x_n\big\rangle \le 0, \quad \forall y \in C_n.$$

Since $F \subset C_n$, we have

$$\big\langle \nabla f(x) - \nabla f(x_n), w - x_n\big\rangle \le 0, \quad \forall w \in F.\tag{3.10}$$

From (b) of Lemma 1.4 we have

$$\begin{aligned}D_f(x_n,x) &= D_f\big(P^f_{C_n}x, x\big) \le D_f(w,x) - D_f\big(w, P^f_{C_n}x\big)\\&\le D_f(w,x), \quad \forall n \ge 1, w \in F.\end{aligned}\tag{3.11}$$

This implies that $\{D_f(x_n,x)\}$ is bounded, from Lemma 1.7, $\{x_n\}$ is bounded. By the construction of C_n, we have $x_m \in C_m \subset C_n$, and $x_n = P^f_{C_n}x$, for any positive integer $m \ge n$. Then we obtain

$$\begin{aligned}D_f(x_m,x_n) &= D_f\big(x_m, P^f_{C_n}x\big) \le D_f(x_m,x) - D_f\big(P^f_{C_n}x, x\big)\\&= D_f(x_m,x) - D_f(x_n,x).\end{aligned}\tag{3.12}$$

In particular,

$$D_f(x_{n+1},x_n) \le D_f(x_{n+1},x) - D_f(x_n,x).$$

Since $x_n = P^f_{C_n}x$ and $x_{n+1} = P^f_{C_{n+1}}x \in C_{n+1} \subset C_n$, we obtain $D_f(x_n,x) \le D_f(x_{n+1},x)$, $\forall n \ge 1$. This shows that $\{D_f(x_n,x)\}$ is nondecreasing and hence the limit $\lim_{n\to\infty} D_f(x_n,x)$ exists. Thus from (3.12), taking the limit as $m,n \to \infty$, we obtain $\lim_{n\to\infty} D_f(x_m,x_n) = 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent (see [17]). It follows that $\|x_m - x_n\| \to 0$ as $m,n \to \infty$. Hence $\{x_n\}$ is Cauchy sequence in C. As $\{x_n\}$ is Cauchy in a complete space E, there exists $p \in E$ such that $x_n \to p$ as $n \to \infty$. Clearly $p \in C$.

Since $D_f(x_m,x_n) \to 0$, as $m,n \to \infty$, we have in particular

$$\lim_{n\to\infty} D_f(x_{n+1},x_n) = 0, \tag{3.13}$$

and this further implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Step V. Next we show that $x_n \to p \in F$.

Since $x_{n+1} = P^f_{C_{n+1}}x \in C_{n+1}$, we have from (3.5)

$$D_f(x_n,w_n) \le \frac{1}{1-k}\bigl\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - x_{n+1}\bigr\rangle \tag{3.15}$$

$$+ \bigl\langle \nabla f(T_n x_n) - \nabla f(w_n), x_n - x_{n+1}\bigr\rangle, \tag{3.16}$$

which implies that $\lim_{n\to\infty} D_f(x_n,w_n) = 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent (see [17]). It follows that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.17}$$

From (3.14) and (3.17), we have

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0. \tag{3.18}$$

Since f is uniformly Fréchet differentiable, it follows from Lemma 1.1 that ∇f is uniformly continuous and f is uniformly continuous on bounded subsets of E (see [33], Theorem 1.8). Hence

$$\lim_{n\to\infty} \bigl\| \nabla f(x_{n+1}) - \nabla f(w_n)\bigr\| = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty} \bigl| f(x_{n+1}) - f(w_n)\bigr| = 0. \tag{3.20}$$

Since $x_{n+1} \in C_{n+1}$, it follows from (3.6), (3.7) that

$$f(x_{n+1}) - f(w_n) - \bigl\langle \nabla f(w_n), x_{n+1} - w_n\bigr\rangle$$
$$= D_f(x_{n+1},w_n) \le D_f(x_{n+1},y_n) \le D_f(x_{n+1},x_n) + \frac{k}{1-k}\bigl\langle \nabla f(x_n) - \nabla f(T_n x_n), x_n - x_{n+1}\bigr\rangle,$$

which implies from (3.20), (3.18), (3.13), and (3.14) that

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0.$$

From the sequential consistency of f, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0; \tag{3.21}$$

from (3.14) and (3.21), we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0, \tag{3.22}$$

which implies that $y_n \to p \in C$, since $x_n \to p \in C$. From the uniform continuity of ∇f, we have

$$\lim_{n\to\infty} \big\|\nabla f(x_n) - \nabla f(y_n)\big\| = 0. \tag{3.23}$$

From (3.5), we have

$$\big\|\nabla f(T_n x_n) - \nabla f(x_n)\big\| = \frac{1}{1-\alpha_n}\big\|\nabla f(x_n) - \nabla f(y_n)\big\|,$$

which implies from (3.23) that

$$\lim_{n\to\infty} \big\|\nabla f(T_n x_n) - \nabla f(x_n)\big\| = 0. \tag{3.24}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets, and from (3.24) we obtain

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \tag{3.25}$$

On the other hand, we see that

$$\begin{aligned}\|x_n - T_{n+l}x_n\| &\le \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| \\ &\quad + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ &\le (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\|\end{aligned}$$

for all $l \in \{1, 2, \dots, N\}$, where $L := \sup_{1\le i\le N} L_i$. It follows from (3.14) and (3.25) that

$$\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0$$

for all $l \in \{1, 2, \dots, N\}$. Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \tag{3.26}$$

for all $l \in \{1,2,\dots,N\}$. Since $x_n \to p$ as $n \to \infty$, by the closedness of T_l for each $l \in \{1,2,\dots,N\}$, we obtain $p \in \bigcap_{l=1}^N F(T_l)$.

Also, since $y_n \to p$ as $n \to \infty$, we have from Lemma 2.3, for each $j = 1,2,\dots,m$,

$$0 \leq D_f(p,u_{jn}) = D_f\bigl(p, \operatorname{Res}_{g_j}^f y_n\bigr) \leq D_f(p,y_n) \to 0 \quad \text{as } n \to \infty.$$

Then we have from Lemma 1.5 that $\lim_{n\to\infty} \|p - u_{jn}\| = 0$, for each $j = 1,2,\dots,m$. Consequently, we have

$$\|u_{jn} - y_n\| \leq \|u_{jn} - p\| + \|p - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.27}$$

From the uniform continuity of ∇f, for each $j = 1,2,\dots,m$ we have

$$\lim_{n\to\infty} \bigl\|\nabla f(u_{jn}) - \nabla f(y_n)\bigr\| = 0. \tag{3.28}$$

From (2.3), we have, for $j = 1,2,\dots,m$,

$$g_j(u_{jn},y) + \bigl\langle \nabla f(u_{jn}) - \nabla f(y_n), y - u_{jn}\bigr\rangle \geq 0, \quad \forall y \in C.$$

Furthermore, using (A2) in the last inequality, we obtain

$$\bigl\langle \nabla f(u_{jn}) - \nabla f(y_n), y - u_{jn}\bigr\rangle \geq g_j(y,u_{jn}), \quad \forall y \in C.$$

By (A4), (3.28), and $u_{jn} \to p$ as $n \to \infty$, we have

$$g_j(y,p) \leq 0, \quad \forall y \in C. \tag{3.29}$$

Let $z_t := ty + (1-t)p$ for $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. This yields $g_j(z_t,p) \leq 0$. It follows from (A1) and (A4) that

$$\begin{aligned} 0 &= g_j(z_t,z_t) \leq t g_j(z_t,y) + (1-t) g_j(z_t,p) \\ &\leq t g_j(z_t,y), \end{aligned}$$

and hence

$$0 \leq g_j(z_t,y).$$

From condition (A3), we obtain

$$g_j(p,y) \geq 0, \quad \forall y \in C \text{ and } \forall j \in \{1,2,3,\dots,m\}.$$

This implies that $p \in \operatorname{EP}(g_j)$, for each $j = 1,2,\dots,m$. Thus, $p \in \bigcap_{j=1}^m \operatorname{EP}(g_j)$. Hence we have $p \in F = (\bigcap_{i=1}^N F(T_i)) \cap (\bigcap_{j=1}^m \operatorname{EP}(g_j))$.

Step VI. Finally, we show that $p = P_F^f x$. Setting $n \to \infty$ in (3.10), we obtain

$$\bigl\langle \nabla f(x) - \nabla f(p), w - p\bigr\rangle \leq 0, \quad \forall w \in F.$$

By (a) of Lemma 1.4, we have $p = P_F^f x$. □

Here we give an example of a quasi-Bregman strictly peudocontractive mapping which is not quasi-ϕ strictly pseudocontractive mapping; this shows that the former class is a generalization of the latter.

Example 3.3 Let $E = \mathbb{R}$, $C = [-1,0]$ and define $T, f : [-1,0] \to \mathbb{R}$ by $f(x) = x$ and $Tx = 2x$, for all $x \in [-1,0]$. We want to show that T is a quasi-Bregman strictly pseudocontractive but not quasi-ϕ strictly pseudocontractive.

Proof From the definition it is clear that f is proper, lower semi-continuous, and convex, and also $F(T) = \{0\}$. By the definition of quasi-Bregman strict pseudocontractivity, we find $k \in [0,1)$ such that $D_f(p,Tx) \leq D_f(p,x) + kD_f(x,Tx)$ for all $x \in C$ and $p \in F(T)$. Now,

$$\begin{aligned} D(0,Tx) &= f(0) - f(Tx) - \langle \nabla f(Tx), 0 - Tx \rangle \\ &= 0 - 2x - \langle \nabla f(2x), 0 - 2x \rangle \\ &= -2x - \langle 2, -2x \rangle \\ &= -2x + 4x = 2x, \end{aligned} \tag{3.30}$$

$$\begin{aligned} D(0,x) &= f(0) - f(x) - \langle \nabla f(x), 0 - x \rangle \\ &= 0 - x - \langle 1, -x \rangle \\ &= -x + x = 0 \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} D(x,Tx) &= f(x) - f(Tx) - \langle \nabla f(Tx), x - Tx \rangle \\ &= x - 2x - \langle 2, x - 2x \rangle \\ &= -x + 2x = x. \end{aligned} \tag{3.32}$$

From (3.30), (3.31), and (3.32), we obtain

$$\begin{aligned} D(0,Tx) &= 2x \leq x \\ &\leq 0 + kx, \quad \forall x \in [-1,0], k \in [0,1) \\ &\leq D(0,x) + kD(x,Tx), \quad \forall x \in [-1,0], k \in [0,1). \end{aligned}$$

Therefore

$$D(0,Tx) \leq D(0,x) + kD(x,Tx), \quad \forall x \in [-1,0], k \in [0,1).$$

Hence, T is a quasi-Bregman strictly pseudocontractive map.

Further,

$$\begin{aligned} \phi(0,Tx) &= |0|^2 - 2\langle 0, J(Tx) \rangle + |Tx|^2 \\ &= 0 - 2\langle 0, J(2x) \rangle + |2x|^2 \\ &= 4|x|^2, \end{aligned} \tag{3.33}$$

$$\begin{aligned}\phi(0,x) &= |0|^2 - 2\langle 0, J(x)\rangle + |x|^2 \\ &= 0 - 2\langle 0, J(x)\rangle + |x|^2 \\ &= |x|^2 \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\phi(x,Tx) &= |x|^2 - 2\langle x, J(Tx)\rangle + |Tx|^2 \\ &= |x|^2 - 2\langle x, J(2x)\rangle + 4|x|^2 \\ &= |x|^2 - 4\langle x, J(x)\rangle + 4|x|^2 \\ &= |x|^2 - 4|x|^2 + 4|x|^2 \\ &= |x|^2. \end{aligned} \tag{3.35}$$

Since $4|x|^2 > |x|^2 + k|x|^2$, for all $k \in [0,1)$ and for all $x \in [-1,0]$,

$$\phi(0,Tx) \le \phi(0,x) + k\phi(x,Tx), \quad \forall x \in [-1,0]$$

cannot hold for any $k \in [0,1)$. Hence, T is not a quasi-ϕ strictly pseudocontractive map. □

4 Numerical example

In this section we discuss the direct application of Theorem 3.2 on a typical example on a real line. Consider the following:

$$\begin{aligned}
&\mathbb{E} = \mathbb{R}, \qquad C = [-1,1], \qquad g(z,y) = y^2 + yz - 2z^2, \\
&f(x) = \frac{2}{3}x^2, \qquad \nabla f(x) = \frac{4}{3}x, \qquad Tx = -2x, \\
&f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{E}\}, \\
&f^*(z) = \frac{3}{8}z^2, \qquad \nabla f^*(z) = \frac{3}{4}z, \qquad \alpha_n = \frac{n+1}{4n}, \\
&\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n) = \frac{-(5n-3)}{3n}x_n, \\
&k = 1/2, \qquad x_1 = 1/2 \in C,
\end{aligned}$$

then the scheme can be simplified as

$$\begin{aligned}
&y_n = \nabla f^*\bigl(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\bigr), \\
&\therefore y_n = \frac{-(5n-3)}{4n}x_n, \\
&u_n = \operatorname{Res}_g^f y_n = \frac{4}{13}y_n, \\
&w_n = \nabla f^*\bigl(\nabla f(u_n)\bigr) = u_n, \\
&C_{n+1} = \left\{ w \in C_n : w \le x_n - \frac{(1-k)(x_n - w_n)^2}{2[(1+2k)x_n - (1-k)w_n]} \right\}, \\
&x_{n+1} = P^f_{C_{n+1}}(x_1) = x_n - \frac{(1-k)(x_n - w_n)^2}{2[(1+2k)x_n - (1-k)w_n]}.
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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