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Iterative methods for triple hierarchical variational inequalities and common fixed point problems

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Abstract
The purpose of this paper is to introduce a new iterative scheme for approximating the solution of a triple hierarchical variational inequality problem. Under some requirements on parameters, we study the convergence analysis of the proposed iterative scheme for the considered triple hierarchical variational inequality problem which is defined over the set of solutions of a variational inequality problem defined over the intersection of the set of common fixed points of a sequence of nearly nonexpansive mappings and the set of solutions of the classical variational inequality. Our strong convergence theorems extend and improve some known corresponding results in the contemporary literature for a wider class of nonexpansive type mappings in Hilbert spaces.
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1 Introduction

The classical variational inequality problem initially studied by Stampacchia [1] for a nonlinear operator $A : C \to H$ is a problem which provides us such $x^* \in D$ which satisfies

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in D, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and D is a nonempty closed convex subset of C. The variational inequality (1.1) is denoted by $\mathrm{VI}_D(C, A)$. The set of solutions of (1.1) is denoted by $\Omega_D(C, A)$, that is,

$$\Omega_D(C, A) = \{x^* \in D : \langle Ax^*, y - x^* \rangle \geq 0, \forall y \in D\}.$$

For $C = D$, we use $\mathrm{VI}(C, A) := \mathrm{VI}_D(C, A)$ and $\Omega(C, A) := \Omega_D(C, A)$.

In the framework of variational inequality problems, various problems arising in several branches of pure and applied sciences can be studied (see [2, 3]).

The equivalence relation between the variational inequality and fixed point problems can be seen by projection technique which plays an important role in developing an important role in developing some efficient methods for solving variational inequality problems and related optimization problems. The problem of finding the fixed points of a nonexpansive mapping is the subject of current interest related to variational inequality problems in functional analysis.

Over the set of fixed points of a nonexpansive mapping, several authors (see [4–7] *etc.*) have studied the variational inequality problem in a particular manner. This kind of variational inequality is named a hierarchical variational inequality; it is defined as follows:

$$\text{Find } x^* \in F(S) \text{ such that } \langle (I-T)x^*, y-x^* \rangle \geq 0, \quad \forall y \in F(S),$$

where T and S are two nonexpansive mappings from a nonempty closed convex subset C of a real Hilbert space H into itself, and $F(S)$ denotes the set of fixed points of the mapping S. One can easily observe that $\mathrm{VI}_{F(S)}(C, I-T)$ is equivalent to the fixed point problem $x^* = P_{F(S)}(Tx^*)$, that is, x^* is a fixed point of the nonexpansive mapping $P_{F(S)}(T)$, where $P_{F(S)}$ is the metric projection from H onto a nonempty closed convex subset $F(S)$ of H.

After all, in the scenario of variational inequality problem, we eagerly discuss such kind of variational inequality problem which is defined over the set of solutions of a variational inequality and the set of fixed points of a nonexpansive mapping, having a triple structure in contrast with bilevel programming problems or hierarchical constrained optimization problems or hierarchical fixed point problems. This kind of variational inequality is called the triple hierarchical variational inequality (see [8, 9]), which is also called the triple hierarchical constrained optimization problem (see [8]), and it is defined as follows:

$$\text{Find } x^* \in \Omega_{F(S)}(C,A) \text{ such that } \langle Fx^*, y-x^* \rangle \geq 0, \quad \forall y \in \Omega_{F(S)}(C,A), \tag{1.2}$$

where $\Omega_{F(S)}(C,A)$ is the set of solutions of $\mathrm{VI}_{F(S)}(C,A) \neq \emptyset$, and mappings A, F, and S are inverse strongly monotone, strongly monotone and Lipschitz continuous, and nonexpansive from a nonempty closed convex subset C of a real Hilbert space H into itself, respectively.

If $\Omega_{F(S)}(C, I-T)$ is nonempty, then the metric projection $P_{\Omega_{F(S)}(C,I-T)}$ is well defined. The minimum norm solution x^* of $\mathrm{VI}_{F(S)}(C, I-T)$ exists uniquely and is exactly the nearest point projection of the origin to $\Omega_{F(S)}(C, I-T)$, that is, $x^* = P_{\Omega_{F(S)}(C,I-T)}(0)$. Alternatively, x^* is the unique solution of the quadratic minimization problem:

$$\|x^*\|^2 = \min\{\|x\|^2 : x \in \Omega_{F(S)}(C, I-T)\}.$$

Finding of this minimum norm solution x^* is an interesting problem. In this context, Yao *et al.* [10] proposed two iterative schemes in an implicit and an explicit both ways to find the minimum norm solution x^* of $\mathrm{VI}_{F(S)}(C, I-T)$. They proved two strong convergence results by regularizing the nonexpansive mapping T using contractions.

Recently, Ceng *et al.* [11], motivated by the results of Yao *et al.* [10] introduced and studied two iterative schemes, one of which was an implicit while other was an explicit one. They proved two strong convergence results by the considered iterative schemes under suitable conditions on parameters for considered triple hierarchical variational inequalities for both cases. Some hybrid steepest-descent-like methods with variable parameters for triple hierarchical variational inequalities are also studied in Ceng *et al.* [12]. The importance of the triple hierarchical variational inequalities and a nice survey on this topic is given in [13].

In 2005, the first author introduced the class of nearly nonexpansive mappings [14, 15] which is an important generalization of the class of nonexpansive mappings. Let C be a nonempty subset of a Banach space X. Fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be nearly nonexpansive with respect to the sequence $\{a_n\}$ if for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n \quad \text{for all } x, y \in C.$$

We now discuss the notion of the sequence of nearly nonexpansive mappings.

Let C be a nonempty subset of a Banach space X. Let $\mathcal{T} := \{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into itself. We denote by $F(\mathcal{T})$ the set of common fixed points of the sequence $\mathcal{T}$, that is, $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$. Fix a sequence $\{a_n\}$ in $[0,\infty)$ with $a_n \to 0$, and let $\{T_n\}$ be a sequence of mappings from C into X. Then the sequence $\mathcal{T} := \{T_n\}$ is called a sequence of nearly nonexpansive mappings (see [16]) with respect to a sequence $\{a_n\}$ if

$$\|T_n x - T_n y\| \leq \|x - y\| + a_n \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

Clearly, the sequence of nearly nonexpansive mappings can easily be seen to be a wider class of sequence of nonexpansive mappings.

Motivated and inspired by the works mentioned above, we introduce an explicit iterative scheme that generates a sequence and prove that this sequence converges strongly to a unique solution of the considered triple hierarchical variational inequality problem defined over the set of solutions of a variational inequality problem which is defined over the intersection of the set of common fixed points of a sequence of nearly nonexpansive mappings and the set of solutions of the classical variational inequality problem. Our results generalize the result of Ceng *et al.* [11] in the context of the sequence of nearly nonexpansive mappings and in some other remarkable senses. Our results also extend the result of Yao *et al.* [10] and many other related works.

2 Preliminaries

Throughout this paper, we denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. The symbol $\mathbb{N}$ stands for the set of all natural numbers and $\omega_w(\{x_n\})$ denotes the set of all weak limits of the sequence $\{x_n\}$.

Let C be a nonempty subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. A mapping $T : C \to H$ is called

(1) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \text{for all } x, y \in C,$$

(2) η-strongly monotone if there exists a positive real number η such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2 \quad \text{for all } x, y \in C,$$

(3) α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \quad \text{for all } x, y \in C,$$

(4) k-Lipschitzian if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for all } x, y \in C,$$

(5) ρ-contraction if there exists a constant $\rho \in (0,1)$ such that

$$\|Tx - Ty\| \leq \rho\|x - y\| \quad \text{for all } x, y \in C,$$

(6) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$,

(7) λ-strictly pseudocontractive if there exists $\lambda \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in C,$$

where I is the identity mapping. Note that if $T : C \to H$ is λ-strictly pseudocontractive, then the mapping $A := I - T$ is $\frac{1-\lambda}{2}$-inverse strongly monotone.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| = \inf \|x - y\| =: d(x, C) \quad \text{for all } y \in C.$$

The mapping P_C is called the metric projection from H onto C (see Agarwal *et al.* [14] for some other information related to P_C).

Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $N_C(v)$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C(v) = \{w \in H : \langle v - y, w\rangle \geq 0 \text{ for all } y \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is a maximal monotone and $0 \in Tv$ if and only if $v \in \Omega(C, A)$.

Let C be a nonempty subset of a real Hilbert space H and let $T_1, T_2 : C \to H$ be two mappings. We denote $\mathcal{B}(C)$, the collection of all bounded subsets of C. The deviation between T_1 and T_2 on $B \in \mathcal{B}(C)$ [16], denoted by $\mathcal{D}_B(T_1, T_2)$, is defined by

$$\mathcal{D}_B(T_1, T_2) = \sup\{\|T_1x - T_2x\| : x \in B\}.$$

The following lemmas will be needed to prove our main results.

Lemma 2.1 ([17]) *The metric projection mapping P_C is characterized by the following properties*:

(i) $P_C(x) \in C$ *for all* $x \in H$;

(ii) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;

(iii) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for all* $x \in H$ *and* $y \in C$;

(iv) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2$ *for all* $x, y \in H$.

Lemma 2.2 ([18]) *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator on C. Define the mapping $W : C \to H$ by*

$$Wx = x - \lambda\mu F(x) \quad \text{for all } x \in C.$$

Then W is a contraction provided $\mu < \frac{2\eta}{k^2}$. More precisely, for $\mu \in (0, \frac{2\eta}{k^2})$,

$$\|Wx - Wy\| \le (1 - \lambda\tau)\|x - y\| \quad \text{for all } x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1]$.

Lemma 2.3 ([14]) *Let T be a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. Then $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to 0, then $x \in F(T)$.*

Lemma 2.4 ([19]) *Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of nonnegative real numbers which satisfy the conditions:

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \le 0$, *or*
(ii)′ $\sum_{n=1}^{\infty} \alpha_n\beta_n$ *is convergent.*

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([20]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\lambda_i > 0$ ($i = 1, 2, 3, \ldots, N$) such that $\sum_{i=1}^{N} \lambda_i = 1$. Let $T_1, T_2, T_3, \ldots, T_N : C \to C$ be nonexpansive mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{N} \lambda_i T_i$. Then T is nonexpansive from C into itself and $F(T) = \bigcap_{i=1}^{N} F(T_i)$.*

Proposition 2.1 ([21]) *Let C be a nonempty subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Then, the mapping $(I - tA)$ is nonexpansive from C into H, if $0 \le t \le 2\alpha$.*

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator, and $g : C \to H$ be a ρ-contraction mapping. Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $\mathcal{T} = \{T_n\}$ be a sequence of nearly nonexpansive mappings from C into itself with respect to a sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} \mathcal{D}_B(T_n, T_{n+1}) < \infty$ for all $B \in \mathcal{B}(C)$ and $F(\mathcal{T}) \cap \Omega(C, A) \neq \emptyset$ and let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Suppose that $F(T) = F(\mathcal{T})$, $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma \le \tau$, where*

$\tau = 1-\sqrt{1-\mu(2\eta-\mu k^2)}$. *Assume that* Ω, *the set of solutions of the hierarchical variational inequality of finding* $z^* \in F(\mathcal{T}) \cap \Omega(C,A)$ *such that*

$$\langle(\mu F-\gamma S)z^*, z-z^*\rangle \geq 0, \quad \forall z \in F(\mathcal{T}) \cap \Omega(C,A), \tag{3.1}$$

is nonempty. Consider the sequence $\{x_n\}$ *in* C *for arbitrary* $x_1 \in C$, *generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ y_n = T_n P_C[x_n - t_n A x_n], \\ x_{n+1} = P_C[\lambda_n \gamma(\alpha_n g(x_n) + (1-\alpha_n) S x_n) + (I-\lambda_n \mu F) y_n] \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}$, $\{\lambda_n\}$ *are sequences in* $(0,1)$, *and* $\{t_n\}$ *is a sequence in* $[a,b]$ (*for some* a, b *with* $0 < a < b < 2\alpha$) *satisfying the following conditions*:

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n \lambda_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}|}{\alpha_n \lambda_n^2} = 0$ *and* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n \lambda_n^2 \lambda_{n-1}} = 0$;

(iii) $\lim_{n\to\infty} \frac{\mathcal{D}_B(T_n, T_{n+1})}{\alpha_{n+1}\lambda_{n+1}^2} = 0$ *for each* $B \in \mathcal{B}(C)$ *and* $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$;

(iv) $\lim_{n\to\infty} \frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n} = 0$, $\lim_{n\to\infty} \frac{a_n}{\alpha_n \lambda_n^2} = 0$ *and* $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n \lambda_n^2} = 0$;

(v) *there are constants* $\bar{k} > 0$ *and* $\theta > 0$ *satisfying*

$$\|x - T_n x\| \geq \bar{k}\bigl[d\bigl(x, F(\mathcal{T}) \cap \Omega(C,A)\bigr)\bigr]^{\theta}, \quad \forall x \in C \text{ and } n \in \mathbb{N}.$$

If the generated sequence $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \frac{\|x_n - P_C[x_n - t_n A x_n]\|}{\lambda_n} = 0$, *then it converges strongly to the point* $x^* \in F(\mathcal{T}) \cap \Omega(C,A)$, *where* x^* *is the unique solution of the triple hierarchical variational inequality of finding* $x^* \in \Omega$ *such that*

$$\langle(\mu F-\gamma g)x^*, x-x^*\rangle \geq 0, \quad \forall x \in \Omega. \tag{3.3}$$

Proof First of all, we assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \frac{\|x_n - P_C[x_n - t_n A x_n]\|}{\lambda_n} = 0$. We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Set $u_n = \lambda_n \gamma(\alpha_n g(x_n) + (1-\alpha_n) S x_n) + (I - \lambda_n \mu F) y_n$ and $\gamma_n = (1-\rho)\gamma\lambda_n\alpha_n$. Then, we have

$$\begin{aligned} u_n - u_{n-1} = {} & \alpha_n \lambda_n \gamma\bigl[g(x_n) - g(x_{n-1})\bigr] + \lambda_n(1-\alpha_n)\gamma(S x_n - S x_{n-1}) \\ & + \bigl[(I - \lambda_n \mu F) y_n - (I - \lambda_n \mu F) y_{n-1}\bigr] \\ & + (\alpha_n \lambda_n - \alpha_{n-1}\lambda_{n-1})\gamma\bigl[g(x_{n-1}) - S x_{n-1}\bigr] \\ & + (\lambda_n - \lambda_{n-1})(\gamma S x_{n-1} - \mu F y_{n-1}). \end{aligned}$$

From (3.2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \bigl\|P_C(u_n) - P_C(u_{n-1})\bigr\| \\ &\leq \|u_n - u_{n-1}\| \\ &\leq \alpha_n \lambda_n \gamma \bigl\|g(x_n) - g(x_{n-1})\bigr\| + \lambda_n(1-\alpha_n)\gamma\|S x_n - S x_{n-1}\| \end{aligned}$$

$$\begin{aligned}
&+ \|(I-\lambda_n\mu F)y_n-(I-\lambda_n\mu F)y_{n-1}\| \\
&+ |\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}|\gamma\|g(x_{n-1})-Sx_{n-1}\| \\
&+ |\lambda_n-\lambda_{n-1}|\|\gamma Sx_{n-1}-\mu Fy_{n-1}\| \\
\le{}& \alpha_n\lambda_n\gamma\rho\|x_n-x_{n-1}\|+\lambda_n(1-\alpha_n)\gamma\|x_n-x_{n-1}\| \\
&+ (1-\lambda_n\tau)\|y_n-y_{n-1}\|+|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}|M+|\lambda_n-\lambda_{n-1}|M, \qquad (3.4)
\end{aligned}$$

where M is a constant such that

$$M=\sup_{n\in\mathbb{N}}\{\gamma\|g(x_n)-S(x_n)\|+\|\gamma Sx_n-\mu Fy_n\|\}.$$

Set $z_n := P_C(x_n - t_nAx_n)$ and $B = \{z_n\}$. Since $\{x_n\}$ is bounded, it follows that $B \in \mathcal{B}(C)$. Now, we have

$$\begin{aligned}
\|y_{n+1}-y_n\| ={}& \|T_{n+1}P_C(x_{n+1}-t_{n+1}Ax_{n+1})-T_nP_C(x_n-t_nAx_n)\| \\
\le{}& \|T_{n+1}P_C(x_{n+1}-t_{n+1}Ax_{n+1})-T_{n+1}P_C(x_n-t_nAx_n)\| \\
&+ \|T_{n+1}P_C(x_n-t_nAx_n)-T_nP_C(x_n-t_nAx_n)\| \\
\le{}& \|P_C(x_{n+1}-t_{n+1}Ax_{n+1})-P_C(x_n-t_nAx_n)\| \\
&+ \mathcal{D}_B(T_{n+1},T_n)+a_{n+1} \\
\le{}& \|(x_{n+1}-t_{n+1}Ax_{n+1})-(x_n-t_nAx_n)\|+\mathcal{D}_B(T_{n+1},T_n)+a_{n+1} \\
\le{}& \|x_{n+1}-x_n\|+|t_{n+1}-t_n|\|Ax_n\|+\mathcal{D}_B(T_{n+1},T_n)+a_{n+1}. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\| \le{}& \alpha_n\lambda_n\gamma\rho\|x_n-x_{n-1}\|+\lambda_n(1-\alpha_n)\gamma\|x_n-x_{n-1}\| \\
&+ |\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}|M+|\lambda_n-\lambda_{n-1}|M+(1-\lambda_n\tau)\big[\|x_n-x_{n-1}\| \\
&+ \mathcal{D}_B(T_n,T_{n-1})+|t_n-t_{n-1}|\|Ax_{n-1}\|+a_n\big] \\
={}& \big(1-(1-\rho)\gamma\lambda_n\alpha_n\big)\|x_n-x_{n-1}\|+M\big(|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}| \\
&+ |\lambda_n-\lambda_{n-1}|\big)+(1-\lambda_n\tau)\big[\mathcal{D}_B(T_n,T_{n-1})+|t_n-t_{n-1}|\|Ax_{n-1}\|+a_n\big] \\
\le{}& (1-\gamma_n)\|x_n-x_{n-1}\|+M\big(|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}|+|\lambda_n-\lambda_{n-1}|\big) \\
&+ \mathcal{D}_B(T_n,T_{n-1})+N|t_n-t_{n-1}|+a_n \\
\le{}& (1-\gamma_n)\|x_n-x_{n-1}\|+\gamma_n\left[M\left(\frac{|\alpha_n\lambda_n-\alpha_{n-1}\lambda_{n-1}|+|\lambda_n-\lambda_{n-1}|}{\gamma_n}\right)\right. \\
&\left.+ \frac{\mathcal{D}_B(T_n,T_{n-1})}{\gamma_n}+\frac{N|t_n-t_{n-1}|}{\gamma_n}+\frac{a_n}{\gamma_n}\right], \qquad (3.6)
\end{aligned}$$

where $N = \sup_{n\in\mathbb{N}}\{\|Ax_n\|\}$. Note that $\lim_{n\to\infty}\frac{a_n}{\alpha_n\lambda_n} = 0$ and $\sum_{n=1}^{\infty}\alpha_n\lambda_n = \infty$. Therefore, from conditions (ii), (iii), and Lemma 2.4, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 2. $\|Ax_n - Au\| \to 0$ for $u \in F(\mathcal{T}) \cap \Omega(C,A)$ and $\frac{\|x_{n+1}-x_n\|}{\lambda_n} \to 0$ as $n \to \infty$.

Set $d_n = 2a_n\|z_n - u\| + a_n^2$ and $\varepsilon_n = 2\lambda_n\|\gamma(\alpha_n g(x_n) + (1-\alpha_n)Sx_n) - \mu Fu\|\|y_n - u\| + d_n$. One can observe that

$$\begin{aligned}\|z_n - u\|^2 &= \big\|P_C(x_n - t_nAx_n) - P_C(u - t_nAu)\big\|^2\\ &\le \big\|(x_n - t_nAx_n) - (u - t_nAu)\big\|^2\\ &= \big\|(x_n - u) - t_n(Ax_n - Au)\big\|^2\\ &\le \|x_n - u\|^2 - 2t_n\langle x_n - u, Ax_n - Au\rangle + t_n^2\|Ax_n - Au\|^2\\ &\le \|x_n - u\|^2 - t_n(2\alpha - t_n)\|Ax_n - Au\|^2\\ &\le \|x_n - u\|^2 - a(2\alpha - b)\|Ax_n - Au\|^2.\end{aligned}$$

We also have

$$\begin{aligned}\|y_n - u\|^2 &= \|T_nz_n - T_nu\|^2\\ &\le \big(\|z_n - u\| + a_n\big)^2\\ &\le \|z_n - u\|^2 + 2a_n\|z_n - u\| + a_n^2\\ &= \|z_n - u\|^2 + d_n.\end{aligned}$$

From (3.2), we have

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \big\|P_C\big[\lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n\big] - P_C(u)\big\|^2\\ &\le \big\|\lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n - u\big\|^2\\ &= \big\|\lambda_n\big(\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big) + (I - \lambda_n\mu F)(y_n)\\ &\quad - (I - \lambda_n\mu F)(u)\big\|^2\\ &\le \big[\lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\| + (1 - \lambda_n\tau)\|y_n - u\|\big]^2\\ &\le \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2\\ &\quad + (1 - \lambda_n\tau)\big(\|z_n - u\|^2 + d_n\big)\\ &\quad + 2\lambda_n(1 - \lambda_n\tau)\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|\|y_n - u\|\\ &\le \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2 + \|z_n - u\|^2 + \varepsilon_n\\ &\le \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2 + \|x_n - u\|^2\\ &\quad - a(2\alpha - b)\|Ax_n - Au\|^2 + \varepsilon_n.\end{aligned}\tag{3.7}$$

Thus, we get

$$\begin{aligned}a(2\alpha - b)\|Ax_n - Au\|^2 &\le \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2\\ &\quad + \big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\big) + \varepsilon_n\\ &\le \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2\\ &\quad + \|x_n - x_{n+1}\|\big(\|x_n - u\| + \|x_{n+1} - u\|\big) + \varepsilon_n.\end{aligned}$$

Since $\lambda_n \to 0$, $\varepsilon_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Au\| \to 0$ as $n \to \infty$. From (3.6), we have

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\lambda_n} &\leq (1-\gamma_n)\frac{\|x_n - x_{n-1}\|}{\lambda_n} + \frac{M(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|)}{\lambda_n} \\
&\quad + \frac{\mathcal{D}_B(T_n, T_{n-1})}{\lambda_n} + \frac{N|t_n - t_{n-1}|}{\lambda_n} + \frac{a_n}{\lambda_n} \\
&= (1-\gamma_n)\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} + (1-\gamma_n)\left(\frac{\|x_n - x_{n-1}\|}{\lambda_n} - \frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}}\right) \\
&\quad + \frac{M(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|)}{\lambda_n} \\
&\quad + \frac{\mathcal{D}_B(T_n, T_{n-1})}{\lambda_n} + \frac{N|t_n - t_{n-1}|}{\lambda_n} + \frac{a_n}{\lambda_n} \\
&\leq (1-\gamma_n)\frac{\|x_n - x_{n-1}\|}{\lambda_{n-1}} + \alpha_n\lambda_n\|x_n - x_{n-1}\|\frac{1}{\alpha_n\lambda_n}\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\
&\quad + \frac{M\alpha_n\lambda_n(|\alpha_n\lambda_n - \alpha_{n-1}\lambda_{n-1}| + |\lambda_n - \lambda_{n-1}|)}{\alpha_n\lambda_n^2} \\
&\quad + \alpha_n\lambda_n\left(\frac{\mathcal{D}_B(T_n, T_{n-1})}{\alpha_n\lambda_n^2} + \frac{N|t_n - t_{n-1}|}{\alpha_n\lambda_n^2} + \frac{a_n}{\alpha_n\lambda_n^2}\right).
\end{aligned}$$

Noticing that $\lim_{n\to\infty} \frac{a_n}{\alpha_n\lambda_n^2} = 0$, $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n\lambda_n^2} = 0$, and $\sum_{n=1}^{\infty} \alpha_n\lambda_n = \infty$. Thus, using conditions (ii) and (iii), and applying Lemma 2.4, we have

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\lambda_n} = 0. \tag{3.8}$$

Step 3. $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Let $u \in F(\mathcal{T}) \cap \Omega(C, A)$. Then using Lemma 2.1(iv), we have

$$\begin{aligned}
\|z_n - u\|^2 &= \left\|P_C(x_n - t_nAx_n) - P_C(u - t_nAu)\right\|^2 \\
&\leq \left\langle (x_n - t_nAx_n) - (u - t_nAu), z_n - u\right\rangle \\
&= \frac{1}{2}\Big[\left\|(x_n - t_nAx_n) - (u - t_nAu)\right\|^2 + \|z_n - u\|^2 \\
&\quad - \left\|(x_n - t_nAx_n) - (u - t_nAu) - (z_n - u)\right\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|x_n - u\|^2 + \|z_n - u\|^2 - \left\|(x_n - z_n) - t_n(Ax_n - Au)\right\|^2\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_n - u\|^2 &\leq \|x_n - u\|^2 - \|x_n - z_n\|^2 \\
&\quad + 2t_n\langle x_n - z_n, Ax_n - Au\rangle - t_n^2\|Ax_n - Au\|^2.
\end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we have

$$\begin{aligned}
\|x_{n+1} - u\|^2 &\leq \lambda_n\left\|\gamma\left(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\right) - \mu Fu\right\|^2 + \|x_n - u\|^2 \\
&\quad - \|x_n - z_n\|^2 + 2t_n\langle x_n - z_n, Ax_n - Au\rangle - t_n^2\|Ax_n - Au\|^2 + \varepsilon_n,
\end{aligned}$$

which gives

$$\begin{aligned}\|x_n - z_n\|^2 &\le \lambda_n \big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2 \\ &\quad + \big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\big) \\ &\quad + 2t_n\langle x_n - z_n, Ax_n - Au\rangle - t_n^2\|Ax_n - Au\|^2 + \varepsilon_n \\ &\le \lambda_n \big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fu\big\|^2 \\ &\quad + \big(\|x_n - u\| + \|x_{n+1} - u\|\big)\|x_n - x_{n+1}\| \\ &\quad + 2t_n\|x_n - z_n\|\|Ax_n - Au\| - t_n^2\|Ax_n - Au\|^2 + \varepsilon_n.\end{aligned}$$

We have $\|x_{n+1} - x_n\| \to 0$, $\lambda_n \to 0$, $\varepsilon_n \to 0$, and $\|Ax_n - Au\| \to 0$ as $n \to \infty$. Therefore, we have $\|x_n - z_n\| \to 0$ as $n \to \infty$.

Step 4. $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Since $y_n = T_n z_n$, we get

$$\begin{aligned}\|x_{n+1} - T_n z_n\| &= \big\|P_C\big[\lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n\big] - P_C(T_n z_n)\big\| \\ &\le \big\|\lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n - T_n z_n\big\| \\ &= \big\|\lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + y_n - \lambda_n\mu F y_n - T_n z_n\big\| \\ &= \lambda_n\big\|\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) - \mu F y_n\big\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - T_n z_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n z_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.10}$$

Also, we get

$$\|z_n - T_n z_n\| \le \|z_n - x_n\| + \|x_n - T_n z_n\| \to 0 \quad \text{as } n \to \infty.$$

Note that

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - T_n z_n\| + \|T_n z_n - T_n x_n\| \\ &\le \|x_n - T_n z_n\| + \|z_n - x_n\| + a_n \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus,

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - Tz_n\| + \|Tz_n - T_n z_n\| + \|T_n z_n - x_n\| \\ &\le \|x_n - z_n\| + \mathcal{D}_B(T_n, T) + \|T_n z_n - x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Step 5. $\omega_w(\{x_n\}) \subset F(\mathcal{T}) \cap \Omega(C, A)$.

Note that A is an α-inverse strongly monotone mapping so that it is $\frac{1}{\alpha}$-Lipschitz continuous. Therefore, we have

$$\lim_{n\to\infty} \|Az_n - Ax_n\| = 0.$$

Since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $\hat{x}$ in C. Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, it follows, from the demiclosedness principle of nonexpansive mappings, that $\hat{x} \in F(\mathcal{T})$. Now, let us show that $\hat{x} \in \Omega(C,A)$. Let

$$\mathbb{A}v = \begin{cases} Av + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Note that $\mathbb{A}$ is maximal monotone and $0 \in \mathbb{A}v$ if and only if $v \in \Omega(C,A)$. Let $(v,w) \in G(\mathbb{A})$, the graph of $\mathbb{A}$. Then, we have $w \in \mathbb{A}v = Av + N_C(v)$ and hence $w - Av \in N_C(v)$. Thus, we have

$$\langle w - Av, v - u\rangle \geq 0 \quad \text{for all } u \in C.$$

On the other hand, from $z_n = P_C(x_n - t_n Ax_n)$ and $v \in C$, we have

$$\langle x_n - t_n Ax_n - z_n, z_n - v\rangle \geq 0,$$

and hence

$$\left\langle v - z_n, \frac{z_n - x_n}{t_n} + Ax_n \right\rangle \geq 0.$$

Therefore, from $w - Av \in N_C(v)$ and $z_{n_i} \in C$, we have

$$\begin{aligned} \langle v - z_{n_i}, w\rangle &\geq \langle v - z_{n_i}, Av\rangle \\ &\geq \langle v - z_{n_i}, Av\rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{t_{n_i}} + Ax_{n_i} \right\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle \\ &\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{t_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{t_{n_i}} \right\rangle. \end{aligned}$$

Letting limit $n_i \to \infty$ we obtain $\langle v - \hat{x}, w\rangle \geq 0$. Thus, $\hat{x} \in \mathbb{A}^{-1}0$ together with the maximal monotonicity of $\mathbb{A}$ imply $\hat{x} \in \Omega(C,A)$.

Step 6. $\limsup_{n\to\infty}\langle(\mu F - \gamma g)x^*, x_n - x^*\rangle \geq 0$.

From (3.2), we have

$$x_{n+1} = P_C(u_n) - u_n + \lambda_n\gamma\big(\alpha_n g(x_n) + (1 - \alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n.$$

Therefore, we have

$$\begin{aligned} x_n - x_{n+1} &= u_n - P_C(u_n) + \alpha_n\lambda_n(\mu F - \gamma g)x_n + \lambda_n(1 - \alpha_n)(\mu F - \gamma S)x_n \\ &\quad + (1 - \lambda_n)(x_n - y_n) + \lambda_n\big[(I - \mu F)x_n - (I - \mu F)y_n\big]. \end{aligned}$$

Set $v_n := \frac{x_n - x_{n+1}}{\lambda_n(1-\alpha_n)}$, $\forall n \in \mathbb{N}$. Note $x_n = P_C(u_{n-1})$. Then, we have

$$\begin{aligned} v_n = {} & \frac{u_n - P_C(u_n)}{\lambda_n(1-\alpha_n)} + \frac{\alpha_n}{1-\alpha_n}(\mu F - \gamma g)x_n + (\mu F - \gamma S)x_n \\ & + \frac{(1-\lambda_n)}{\lambda_n(1-\alpha_n)}(x_n - y_n) + \frac{1}{1-\alpha_n}\big[(I-\mu F)x_n - (I-\mu F)y_n\big]. \end{aligned}$$

Let $w \in F(\mathcal{T}) \cap \Omega(C,A)$. Observe that

$$\begin{aligned} \langle v_n, x_n - w\rangle = {} & \frac{1}{\lambda_n(1-\alpha_n)}\big\langle u_n - P_C(u_n), P_C(u_{n-1}) - w\big\rangle \\ & + \frac{\alpha_n}{1-\alpha_n}\big\langle(\mu F - \gamma g)x_n, x_n - w\big\rangle \\ & + \big\langle(\mu F - \gamma S)x_n, x_n - w\big\rangle + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle x_n - y_n, x_n - w\rangle \\ & + \frac{1}{1-\alpha_n}\big\langle(I-\mu F)x_n - (I-\mu F)y_n, x_n - w\big\rangle \\ = {} & \frac{1}{\lambda_n(1-\alpha_n)}\big\langle u_n - P_C(u_n), P_C(u_n) - w\big\rangle \\ & + \frac{1}{\lambda_n(1-\alpha_n)}\big\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\big\rangle \\ & + \big\langle(\mu F - \gamma S)w, x_n - w\big\rangle + \big\langle(\mu F - \gamma S)x_n - (\mu F - \gamma S)w, x_n - w\big\rangle \\ & + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle x_n - y_n, x_n - w\rangle + \frac{\alpha_n}{1-\alpha_n}\big\langle(\mu F - \gamma g)x_n, x_n - w\big\rangle \\ & + \frac{1}{1-\alpha_n}\big\langle(I-\mu F)x_n - (I-\mu F)y_n, x_n - w\big\rangle. \end{aligned} \tag{3.11}$$

The first and fourth terms in (3.11) are nonnegative due to the property of the projection operator given in Lemma 2.1(ii), and the monotonicity of $(\mu F - \gamma S)$, respectively. Note $x_{n+1} = P_C(u_n)$. Thus, from (3.11), we have

$$\begin{aligned} \langle v_n, x_n - w\rangle \geq {} & \frac{1}{\lambda_n(1-\alpha_n)}\big\langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n)\big\rangle \\ & + \big\langle(\mu F - \gamma S)w, x_n - w\big\rangle + \frac{\alpha_n}{1-\alpha_n}\big\langle(\mu F - \gamma g)x_n, x_n - w\big\rangle \\ & + \frac{1}{1-\alpha_n}\big\langle(I-\mu F)x_n - (I-\mu F)y_n, x_n - w\big\rangle \\ & + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle x_n - y_n, x_n - w\rangle \\ = {} & \big\langle u_n - P_C(u_n), v_n\big\rangle + \big\langle(\mu F - \gamma S)w, x_n - w\big\rangle \\ & + \frac{\alpha_n}{1-\alpha_n}\big\langle(\mu F - \gamma g)x_n, x_n - w\big\rangle \\ & + \frac{1}{1-\alpha_n}\big\langle(I-\mu F)x_n - (I-\mu F)y_n, x_n - w\big\rangle \\ & + \frac{1-\lambda_n}{\lambda_n(1-\alpha_n)}\langle x_n - y_n, x_n - w\rangle. \end{aligned} \tag{3.12}$$

Noticing, from (3.10), that $\|x_n - y_n\| \to 0$, we have $\|(I-\mu F)x_n - (I-\mu F)y_n\| \to 0$. It is clear from (3.8) that $v_n \to 0$. By assumption $\alpha_n \to 0$ and the sequence $\{x_n\}$ is bounded; we see that $\{u_n\}$ is bounded. Thus, from (3.12), we have

$$\limsup_{n\to\infty}\langle(\mu F - \gamma S)w, x_n - w\rangle \le 0, \quad \forall w \in F(\mathcal{T}) \cap \Omega(C,A). \tag{3.13}$$

This is sufficient to guarantee that $\omega_w(\{x_n\}) \subseteq \Omega$, *i.e.*, every weak limit point of the sequence $\{x_n\}$ solves the hierarchical variational inequality (3.1). In fact, if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{x} \in \omega_w(\{x_n\})$, then, from (3.13), we have

$$\langle(\mu F - \gamma S)w, \tilde{x} - w\rangle = \limsup_{n\to\infty}\langle(\mu F - \gamma S)w, x_n - w\rangle \le 0, \quad \forall w \in F(\mathcal{T}) \cap \Omega(C,A),$$

that is,

$$\langle(\mu F - \gamma S)w, w - \tilde{x}\rangle \ge 0, \quad \forall w \in F(\mathcal{T}) \cap \Omega(C,A). \tag{3.14}$$

Note that $\omega_w(\{x_n\}) \subseteq F(\mathcal{T}) \cap \Omega(C,A)$. Moreover, $(\mu F - \gamma S)$ is monotone and Lipschitz continuous, and $F(\mathcal{T}) \cap \Omega(C,A) \neq \emptyset$ is closed and convex. Therefore, the inequality (3.14) is equivalent to the inequality (3.1) by the Minty lemma (see [22]). Thus, we have $\tilde{x} \in \Omega$.

Now, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle(\mu F - \gamma g)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle(\mu F - \gamma g)x^*, x_{n_k} - x^*\rangle.$$

Without loss of generality, we may further assume that $x_{n_k} \rightharpoonup \tilde{x}$. Note that $\tilde{x} \in \Omega$. As x^* is a solution of the triple hierarchical variational inequality (3.3), we obtain

$$\limsup_{n\to\infty}\langle(\mu F - \gamma g)x^*, x_n - x^*\rangle = \langle(\mu F - \gamma g)x^*, \tilde{x} - x^*\rangle \ge 0.$$

Step 7. $x_n \to x^*$ as $n \to \infty$.

Noticing that $y_n = T_n z_n$, $\gamma_n = (1-\rho)\gamma\lambda_n\alpha_n$, and $x_{n+1} = P_C(u_n)$. Set $\chi_n = \alpha_n\lambda_n\chi_n' + \chi_n''$, where $\chi_n' := \langle(\gamma g - \mu F)x^*, x_{n+1} - x^*\rangle$ and $\chi_n'' = \lambda_n(1-\alpha_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle$. From (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle u_n - x^*, x_{n+1} - x^*\rangle + \langle P_C(u_n) - u_n, P_C(u_n) - x^*\rangle \\
&\le \langle u_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle \lambda_n\gamma\big(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n - x^*, x_{n+1} - x^*\rangle \\
&= \langle (I - \lambda_n\mu F)y_n - (I - \lambda_n\mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n\lambda_n\gamma\langle g(x_n) - g(x^*), x_{n+1} - x^*\rangle + \lambda_n(1-\alpha_n)\gamma\langle Sx_n - Sx^*, x_{n+1} - x^*\rangle \\
&\quad + \alpha_n\lambda_n\langle(\gamma g - \mu F)x^*, x_{n+1} - x^*\rangle + \lambda_n(1-\alpha_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\le (1 - \lambda_n\tau)\big(\|z_n - x^*\| + a_n\big)\|x_{n+1} - x^*\| + \alpha_n\lambda_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \lambda_n(1-\alpha_n)\gamma\|x_n - x^*\|\|x_{n+1} - x^*\| + \chi_n \\
&\le (1 - \lambda_n\tau)\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\lambda_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_n(1-\alpha_n)\gamma\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\lambda_n\tau)a_n\|x_{n+1} - x^*\| + \chi_n \\
&\leq \big[1 - \lambda_n\tau + \alpha_n\lambda_n\gamma\rho + \lambda_n(1-\alpha_n)\gamma\big]\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + a_n\|x_{n+1} - x^*\| + \chi_n \\
&\leq \big[1 - \alpha_n\lambda_n\gamma(1-\rho)\big]\|x_n - x^*\|\|x_{n+1} - x^*\| + a_n\|x_{n+1} - x^*\| + \chi_n \\
&\leq \big[1 - \alpha_n\lambda_n\gamma(1-\rho)\big]\frac{1}{2}\big(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\big) \\
&\quad + a_n\|x_{n+1} - x^*\| + \chi_n.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \frac{1 - \alpha_n\lambda_n\gamma(1-\rho)}{1 + \alpha_n\lambda_n\gamma(1-\rho)}\|x_n - x^*\|^2 + \frac{2}{1+\gamma_n}\chi_n + \frac{2a_n}{1+\gamma_n}R \\
&\leq \big[1 - \alpha_n\lambda_n\gamma(1-\rho)\big]\|x_n - x^*\|^2 + \frac{2\chi_n}{1+\gamma_n} + \frac{2a_nR}{1+\gamma_n}
\end{aligned}
\tag{3.15}
$$

for some $R > 0$. Since $x^* \in \Omega$, by using condition (v) we have

$$
\begin{aligned}
\big\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\big\rangle &= \big\langle(\gamma S - \mu F)x^*, x_{n+1} - P_{F(\mathcal{T})\cap\Omega(C,A)}(x_{n+1})\big\rangle \\
&\quad + \big\langle(\gamma S - \mu F)x^*, P_{F(\mathcal{T})\cap\Omega(C,A)}(x_{n+1}) - x^*\big\rangle \\
&\leq \big\langle(\gamma S - \mu F)x^*, x_{n+1} - P_{F(\mathcal{T})\cap\Omega(C,A)}(x_{n+1})\big\rangle \\
&\leq \big\|(\gamma S - \mu F)x^*\big\| d\big(x_{n+1}, F(\mathcal{T}) \cap \Omega(C,A)\big) \\
&\leq \big\|(\gamma S - \mu F)x^*\big\|\left(\frac{1}{k}\|x_{n+1} - T_nx_{n+1}\|\right)^{\frac{1}{\theta}}.
\end{aligned}
\tag{3.16}
$$

Note that

$$
\begin{aligned}
\|x_{n+1} - T_nx_n\| &= \big\|P_C(u_n) - P_C(T_nx_n)\big\| \\
&\leq \|u_n - T_nx_n\| \\
&= \big\|\lambda_n\gamma\big(\alpha_ng(x_n) + (1-\alpha_n)Sx_n\big) + (I - \lambda_n\mu F)y_n - T_nx_n\big\| \\
&\leq \lambda_n\big\|\gamma\big(\alpha_ng(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fy_n\big\| + \|T_nz_n - T_nx_n\| \\
&\leq \lambda_n\big\|\gamma\big(\alpha_ng(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fy_n\big\| + \|z_n - x_n\| + a_n.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
\|x_{n+1} - T_nx_{n+1}\| &\leq \|x_{n+1} - T_nx_n\| + \|T_nx_n - T_nx_{n+1}\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| + a_n \\
&\leq \|x_n - x_{n+1}\| + \lambda_n\big\|\gamma\big(\alpha_ng(x_n) + (1-\alpha_n)Sx_n\big) - \mu Fy_n\big\| \\
&\quad + \|z_n - x_n\| + 2a_n. \\
&\leq \|x_n - x_{n+1}\| + \lambda_nM + \|z_n - x_n\| + 2a_n.
\end{aligned}
$$

Hence from (3.16), we get

$$\begin{aligned}
&\bigl\langle (\gamma S - \mu F)x^*, x_{n+1} - x^* \bigr\rangle \\
&\quad \le \left(\frac{1}{\bar{k}}\right)^{\frac{1}{\theta}} \bigl\|(\gamma S - \mu F)x^*\bigr\| \bigl(\|x_n - x_{n+1}\| + M\lambda_n + \|z_n - x_n\| + 2a_n\bigr)^{\frac{1}{\theta}} \\
&\quad \le \lambda_n^{\frac{1}{\theta}} M'' \left(1 + \frac{\|x_n - x_{n+1}\|}{\lambda_n} + \frac{\|z_n - x_n\|}{\lambda_n} + \frac{a_n}{\lambda_n}\right)^{\frac{1}{\theta}} \qquad (3.17)
\end{aligned}$$

for some constant M''. Therefore from (3.15) and (3.17), we have

$$\begin{aligned}
\bigl\|x_{n+1} - x^*\bigr\|^2 &\le \bigl[1 - \gamma(1-\rho)\alpha_n\lambda_n\bigr]\bigl\|x_n - x^*\bigr\|^2 \\
&\quad + \frac{2\alpha_n\lambda_n}{1+\gamma_n}\left[\chi_n' + M''\frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n}\left(1 + \frac{\|x_n - x_{n+1}\|}{\lambda_n} + \frac{\|z_n - x_n\|}{\lambda_n} + \frac{a_n}{\lambda_n}\right)^{\frac{1}{\theta}}\right] + \frac{2a_nR}{1+\gamma_n} \\
&= (1-\gamma_n)\bigl\|x_n - x^*\bigr\|^2 + \sigma_n + \frac{2a_nR}{1+\gamma_n},
\end{aligned}$$

where

$$\sigma_n = \frac{2\alpha_n\lambda_n}{1+\gamma_n}\left[\chi_n' + M''\frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n}\left(1 + \frac{\|x_n - x_{n+1}\|}{\lambda_n} + \frac{\|z_n - x_n\|}{\lambda_n} + \frac{a_n}{\lambda_n}\right)^{\frac{1}{\theta}}\right].$$

Note that $\lim_{n\to\infty}\frac{a_n}{\lambda_n} = 0$ and $\sum_{n=1}^{\infty}\alpha_n\lambda_n = \infty$. Using Lemma 2.4, we obtain $x_n \to x^*$. This completes the proof. □

If we put $g = 0$ in (3.3), then this triple hierarchical variational inequality reduces to the variational inequality (3.18). Thus, the following is the direct consequence of Theorem 3.1.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator, and $S : C \to C$ be a nonexpansive mapping. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $\mathcal{T} = \{T_n\}$ be a sequence of nearly nonexpansive mappings from C into itself with respect to a sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty}\mathcal{D}_B(T_n, T_{n+1}) < \infty$ for all $B \in \mathcal{B}(C)$ and $F(\mathcal{T}) \cap \Omega(C, A) \neq \emptyset$ and let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Suppose that $F(T) = F(\mathcal{T})$, $0 < \mu < \frac{2\eta}{k^2}$, and $0 < \gamma \le \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Assume that Ω, the set of solutions of the hierarchical variational inequality* (3.1), *is nonempty. Consider the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process*:

$$\begin{cases}
x_1 \in C, \\
y_n = T_n P_C[x_n - t_n A x_n], \\
x_{n+1} = P_C[\lambda_n(1-\alpha_n)\gamma S x_n + (I - \lambda_n\mu F)y_n]
\end{cases}$$

for all $n \in \mathbb{N}$, to be bounded and $\lim_{n\to\infty}\frac{\|x_n - P_C[x_n - t_n A x_n]\|}{\lambda_n} = 0$, where $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{t_n\}$ are sequences mentioned in Theorem 3.1 *satisfying all the conditions of Theorem* 3.1. *Then the sequence $\{x_n\}$ converges strongly to a unique solution x^* of the variational inequality of* *finding* $x^* \in \Omega$ *such that*

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{3.18}$$

Take $T_n = T$ and $A = 0$ in Theorem 3.1, we have the following.

Corollary 3.1 (Ceng *et al.* [11, Theorem 4.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator, and $g : C \to H$ be a ρ-contraction mapping. Let S and T be nonexpansive mappings from C into itself such that $F(T) \neq \emptyset$. Suppose that $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Assume that Ω, the set of solutions of the hierarchical variational inequality of finding $z^* \in F(T)$ such that*

$$\langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0, \quad \forall z \in F(T),$$

is nonempty. Consider the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[\lambda_n \gamma(\alpha_n g(x_n) + (1 - \alpha_n)Sx_n) + (I - \lambda_n \mu F)Tx_n] \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$ *satisfying the conditions* (i)-(ii) *of Theorem* 3.1. *Suppose that $\lim_{n\to\infty} \frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n} = 0$, and $\|x - Tx\| \geq \bar{k}[d(x, F(T))]^\theta$, $\forall x \in C$, where $\bar{k} > 0$ and $\theta > 0$ are constants. Then the following hold*:

(a) *If the generated sequence $\{x_n\}$ is bounded, then the sequence $\{x_n\}$ converges strongly to the point $x^* \in F(T)$, where x^* is the unique solution of the triple hierarchical variational inequality of finding $x^* \in \Omega$ such that*

$$\langle (\mu F - \gamma g)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

(b) *If the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[\lambda_n(1 - \alpha_n)\gamma Sx_n + (I - \lambda_n \mu F)Tx_n] \end{cases}$$

for all $n \in \mathbb{N}$, is bounded, then the sequence $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality of finding $x^* \in \Omega$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

We now derive the result of Yao *et al.* [10, Theorem 4.1] as a corollary.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a ρ-contraction mapping. Let S and T be nonexpansive mappings from C into itself such that $F(T) \neq \emptyset$. Assume that Ω, the set of solutions of the hierarchical variational inequality of finding $x^* \in F(T)$ such that*

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{3.19}$$

is nonempty. Consider the sequence $\{x_n\}$ *in* C *for arbitrary* $x_1 \in C$*, generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[\lambda_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\lambda_n)Tx_n] \end{cases}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0,1)$ *satisfying the conditions* (i)-(ii) *of Theorem* 3.1. *Suppose that* $\lim_{n\to\infty} \frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n} = 0$ *and* $\|x - Tx\| \geq \bar{k}[d(x, F(T))]^\theta$, $\forall x \in C$*, where* $\bar{k} > 0$ *and* $\theta > 0$ *are constants. Then*:

(a) *If the generated sequence* $\{x_n\}$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to the point* $x^* \in F(T)$*, where* x^* *is the unique solution of the variational inequality of finding* $x^* \in \Omega$ *such that*

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

(b) *If the sequence* $\{x_n\}$ *in* C *for arbitrary* $x_1 \in C$*, generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C[\lambda_n(1-\alpha_n)Sx_n + (1-\lambda_n)Tx_n] \end{cases}$$

for all $n \in \mathbb{N}$*, is bounded, then the sequence* $\{x_n\}$ *converges strongly to a minimum norm solution of the hierarchical variational inequality* (3.19).

Again, we derive the following result as a corollary for S and T being two nonexpansive mappings.

Corollary 3.3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $F : C \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator, and* $g : C \to H$ *be a* ρ*-contraction mapping. Let* S *and* T *be nonexpansive mappings from* C *into itself and* $A : C \to H$ *be an* α*-inverse strongly monotone mapping such that* $F(T) \cap \Omega(C, A) \neq \emptyset$*. Suppose that* $0 < \mu < \frac{2\eta}{k^2}$ *and* $0 < \gamma \leq \tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$*. Assume that* Ω*, the set of solutions of the hierarchical variational inequality of finding* $z^* \in F(T) \cap \Omega(C, A)$ *such that*

$$\langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0, \quad \forall z \in F(T) \cap \Omega(C, A),$$

is nonempty. Consider the sequence $\{x_n\}$ *in* C *for arbitrary* $x_1 \in C$*, generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ y_n = TP_C[x_n - t_n A x_n], \\ x_{n+1} = P_C[\lambda_n \gamma(\alpha_n g(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n \mu F)y_n] \end{cases}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}$, $\{\lambda_n\}$ *are sequences in* $(0,1)$ *and* $\{t_n\}$ *is a sequence in* $[a,b]$ (*for some* a, b *with* $0 < a < b < 2\alpha$) *satisfying the conditions* (i)-(ii) *of Theorem* 3.1. *Suppose that*

$\lim_{n\to\infty}\frac{\lambda_n^{\frac{1}{\theta}}}{\alpha_n}=0$, $\sum_{n=1}^{\infty}|t_{n+1}-t_n|<\infty$, $\lim_{n\to\infty}\frac{|t_n-t_{n-1}|}{\alpha_n\lambda_n^2}=0$ *and* $\|x-Tx\|\geq\bar{k}[d(x,F(T)\cap\Omega(C,A))]^{\theta}$, $\forall x\in C$, *where* $\bar{k}>0$ *and* $\theta>0$ *are constants. Then the following hold*:

(a) *If the generated sequence* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty}\frac{\|x_n-P_C[x_n-t_nAx_n]\|}{\lambda_n}=0$, *then the sequence* $\{x_n\}$ *converges strongly to the point* $x^*\in F(T)\cap\Omega(C,A)$, *where* x^* *is the unique solution of the triple hierarchical variational inequality of finding* $x^*\in\Omega$ *such that*

$$\langle(\mu F-\gamma g)x^*,x-x^*\rangle\geq 0,\quad\forall x\in\Omega.$$

(b) *If the sequence* $\{x_n\}$ *in* C *for arbitrary* $x_1\in C$, *generated by the following iterative process*:

$$\begin{cases}x_1\in C,\\ y_n=TP_C[x_n-t_nAx_n],\\ x_{n+1}=P_C[\lambda_n(1-\alpha_n)\gamma Sx_n+(I-\lambda_n\mu F)y_n]\end{cases}$$

for all $n\in\mathbb{N}$, *is bounded and* $\lim_{n\to\infty}\frac{\|x_n-P_C[x_n-t_nAx_n]\|}{\lambda_n}=0$, *then the sequence* $\{x_n\}$ *converges strongly to the unique solution* x^* *of the variational inequality of finding* $x^*\in\Omega$ *such that*

$$\langle Fx^*,x-x^*\rangle\geq 0,\quad\forall x\in\Omega.$$

4 Applications

In this section, we present two applications of Theorem 3.1. The first application is concerned with the image recovery problem which is equivalent to finding a common fixed point of finitely many nonexpansive self mappings. The first application improves a number of results related to this context. The second application deals with a strictly pseudocontractive mapping.

Theorem 4.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $F:C\to H$ *be a* k*-Lipschitzian and* η*-strongly monotone operator, and* $g:C\to H$ *be a* ρ*-contraction mapping. Let* $S:C\to C$ *be a nonexpansive mapping and* $A:C\to H$ *be an* α*-inverse strongly monotone mapping. Let* $t_1,t_2,t_3,\ldots,t_N>0$ *such that* $\sum_{i=1}^N t_i=1$. *Let* $T_1,T_2,T_3,\ldots,T_N:C\to C$ *be nonexpansive mappings such that* $\bigcap_{i=1}^N F(T_i)\cap\Omega(C,A)\neq\emptyset$ *and assume that* $T=\sum_{i=1}^N t_iT_i$. *Suppose that* $0<\mu<\frac{2\eta}{k^2}$ *and* $0<\gamma\leq\tau$, *where* $\tau=1-\sqrt{1-\mu(2\eta-\mu k^2)}$. *Assume that* Ω, *the set of solutions of the hierarchical variational inequality of finding* $z^*\in\bigcap_{i=1}^N F(T_i)\cap\Omega(C,A)$ *such that*

$$\langle(\mu F-\gamma S)z^*,z-z^*\rangle\geq 0,\quad\forall z\in\bigcap_{i=1}^N F(T_i)\cap\Omega(C,A),$$

is nonempty. Consider the sequence $\{x_n\}$ *in* C *for arbitrary* $x_1\in C$, *generated by the following iterative process*:

$$\begin{cases}x_1\in C,\\ y_n=\sum_{i=1}^N t_iT_iP_C[x_n-t_nAx_n],\\ x_{n+1}=P_C[\lambda_n\gamma(\alpha_ng(x_n)+(1-\alpha_n)Sx_n)+(I-\lambda_n\mu F)y_n]\end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$ and $\{t_n\}$ is a sequence in $[a,b]$ (for some a, b with $0 < a < b < 2\alpha$) satisfying all the conditions of Corollary 3.3. Then the following hold:

(a) *If the generated sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \frac{\|x_n - P_C[x_n - t_n Ax_n]\|}{\lambda_n} = 0$, then the sequence $\{x_n\}$ converges strongly to the point $x^* \in \bigcap_{i=1}^N F(T_i) \cap \Omega(C,A)$, where x^* is the unique solution of the triple hierarchical variational inequality of finding $x^* \in \Omega$ such that*

$$\langle (\mu F - \gamma g)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

(b) *If the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ y_n = \sum_{i=1}^N t_i T_i P_C[x_n - t_n Ax_n], \\ x_{n+1} = P_C[\lambda_n(1-\alpha_n)\gamma Sx_n + (I - \lambda_n \mu F)y_n] \end{cases}$$

for all $n \in \mathbb{N}$, is bounded and $\lim_{n\to\infty} \frac{\|x_n - P_C[x_n - t_n Ax_n]\|}{\lambda_n} = 0$, then the sequence $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality of finding $x^* \in \Omega$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

Proof Lemma 2.5 implies that T is nonexpansive from C into itself and $F(T) = \bigcap_{i=1}^N F(T_i)$. Hence, the result follows from Corollary 3.3. □

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator, and $g : C \to H$ be a ρ-contraction mapping. Let $S : C \to C$ be a nonexpansive mapping. Let $U : C \to C$ be a λ-strictly pseudocontractive mapping and $\mathcal{T} = \{T_n\}$ be a sequence of nearly nonexpansive mappings from C into itself with respect to a sequence $\{a_n\}$ such that $\sum_{n=1}^\infty \mathcal{D}_B(T_n, T_{n+1}) < \infty$ for all $B \in \mathcal{B}(C)$ and $F(\mathcal{T}) \cap F(U) \neq \emptyset$ and let T be a mapping from C into itself defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$. Suppose that $F(T) = F(\mathcal{T})$, $0 < \mu < \frac{2\eta}{k^2}$, and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Assume that Ω, the set of solutions of the hierarchical variational inequality of finding $z^* \in F(\mathcal{T}) \cap F(U)$ such that*

$$\langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0, \quad \forall z \in F(\mathcal{T}) \cap F(U),$$

is nonempty. Consider the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process:

$$\begin{cases} x_1 \in C, \\ y_n = T_n[(1-t_n)x_n + t_n Ux_n], \\ x_{n+1} = P_C[\lambda_n \gamma(\alpha_n g(x_n) + (1-\alpha_n)Sx_n) + (I - \lambda_n \mu F)y_n] \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$ and $\{t_n\}$ is a sequence in $[a,b]$ (for some a, b with $0 < a < b < 1 - \lambda$) satisfying the conditions (i)-(iv) of Theorem 3.1. Suppose that $\|x - T_n x\| \geq \bar{k}[d(x, F(\mathcal{T}) \cap F(U))]^\theta$, $\forall x \in C$ and $n \in \mathbb{N}$, where $\bar{k} > 0$ and $\theta > 0$ are constants. Then the following hold:

(a) *If the generated sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \frac{\|x_n - Ux_n\|}{\lambda_n} = 0$, then the sequence $\{x_n\}$ converges strongly to the point $x^* \in F(\mathcal{T}) \cap F(U)$, where x^* is the unique solution of the triple hierarchical variational inequality of finding $x^* \in \Omega$ such that*

$$\langle (\mu F - \gamma g)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

(b) *If the sequence $\{x_n\}$ in C for arbitrary $x_1 \in C$, generated by the following iterative process:*

$$\begin{cases} x_1 \in C, \\ y_n = T_n[(1 - t_n)x_n + t_n U x_n], \\ x_{n+1} = P_C[\lambda_n(1 - \alpha_n)\gamma S x_n + (I - \lambda_n \mu F) y_n] \end{cases}$$

for all $n \in \mathbb{N}$, is bounded and $\lim_{n\to\infty} \frac{\|x_n - Ux_n\|}{\lambda_n} = 0$, then the sequence $\{x_n\}$ converges strongly to the unique solution x^* of the variational inequality of finding $x^* \in \Omega$ such that

$$\langle F x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

Proof Put $A = I - U$ in Theorem 3.1, then A is $\frac{(1-\lambda)}{2}$-inverse strongly monotone. We also have $F(U) = \Omega(C, A)$ and $P_C(x_n - t_n A x_n) = (1 - t_n)x_n + t_n U x_n$. Therefore, the conclusion follows from Theorem 3.1 and Theorem 3.2. □

5 Numerical example

In this section, we discuss the following example which shows the effectiveness and convergence of iteratively generated sequence $\{x_n\}$ by the considered scheme (3.2) of Theorem 3.1.

Example 5.1 Let $H = \mathbb{R}$ and $C = [0,1]$. Let A, S, and T be mappings defined by $A(x) = 2x - 1$, $S(x) = x$, and $T(x) = 1 - x$ for all $x \in C$.

Let $F, g : C \to H$ be mappings defined by $F(x) = 2x$ and $g(x) = \frac{x}{2} + 1$ for all $x \in C$. Define $\{t_n\}$, $\{\alpha_n\}$, and $\{\lambda_n\}$ in $(0,1)$ by $t_n = \frac{1}{2}$, $\alpha_n = \frac{1}{(n+2)^p}$, and $\lambda_n = \frac{1}{(n+2)^q}$, where $0 < p + 2q < 1$. It is clear that S and T are nonexpansive self mappings, and A is 2-inverse strongly monotone. Note F is a 2-Lipschitzian and 2-strongly monotone, and g is a $\frac{1}{2}$-contraction mapping. Here, $k = 2$, $\eta = 2$, and $\rho = \frac{1}{2}$. We take $\mu = \frac{1}{4}$, $\gamma = \tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} = \frac{1}{2}$, $p = \frac{1}{2}$, $q = \frac{1}{6}$, and $\theta = \frac{1}{4}$. Note that $0 < \mu < \frac{2\eta}{k^2}$. Observe that $T_n = T$ with $a_n = 0$ for all $n \in \mathbb{N}$ and $F(\mathcal{T}) \cap \Omega(C, A) = \{\frac{1}{2}\}$. The iterative algorithm (3.2) can be written as

$$x_{n+1} = P_C(u_n) \quad \text{for all } n \in \mathbb{N},$$

where

$$u_n = \lambda_n \gamma \big(\alpha_n g(x_n) + (1 - \alpha_n) S x_n\big) + (I - \lambda_n \mu F) y_n,$$

$$y_n = T_n z_n, \qquad z_n = P_C(x_n - t_n A x_n).$$

We observe that

$$z_n = P_C(x_n - t_n A x_n) = P_C\left(x_n - \frac{1}{2}(2x_n - 1)\right) = P_C\left(\frac{1}{2}\right) = \frac{1}{2},$$
$$y_n = T_n z_n = T(1/2) = 1/2$$

and

$$\begin{aligned} u_n &= \lambda_n \gamma\left(\alpha_n g(x_n) + (1-\alpha_n)Sx_n\right) + (I - \lambda_n \mu F)y_n \\ &= \frac{\lambda_n}{2}\left(\alpha_n\left(\frac{x_n}{2} + 1\right) + (1-\alpha_n)x_n\right) + \frac{1}{2} - \frac{\lambda_n}{4}F\left(\frac{1}{2}\right) \\ &= \frac{\lambda_n}{2}\left(\alpha_n + \left(1 - \frac{\alpha_n}{2}\right)x_n\right) + \frac{1}{2} - \frac{\lambda_n}{4}. \end{aligned}$$

Let $x_1 \in C$. For $n = 1$, we have

$$\begin{aligned} u_1 &= \frac{\lambda_1}{2}\left(\alpha_1 + \left(1 - \frac{\alpha_1}{2}\right)x_1\right) + \frac{1}{2} - \frac{\lambda_1}{4} \\ &\le \frac{\lambda_1}{2}\left(\alpha_1 + \left(1 - \frac{\alpha_1}{2}\right)\right) + \frac{1}{2} - \frac{\lambda_1}{4} \\ &= \frac{1}{2(3)^{\frac{1}{6}}}\left(\frac{1}{2(3)^{\frac{1}{2}}} + 1\right) + \frac{1}{2} - \frac{1}{4(3)^{\frac{1}{6}}} \\ &= \frac{1}{4(3)^{\frac{1}{6}}} + \frac{1}{4(3)^{\frac{2}{3}}} + \frac{1}{2} = 0.8284 < 1. \end{aligned}$$

Thus, $u_1 \in C$.

Next we show that $x_n \in C$ for all $n \in \mathbb{N}$. Note $u_1 \in C$. Suppose that $u_k \in C$ for some $k \in \mathbb{N}$. Now, for $n = k + 1$, we have

$$\begin{aligned} u_{k+1} &= \frac{\lambda_{k+1}}{2}\left(\alpha_{k+1} + \left(1 - \frac{\alpha_{k+1}}{2}\right)x_{k+1}\right) + \frac{1}{2} - \frac{\lambda_{k+1}}{4} \\ &= \frac{\lambda_{k+1}}{2}\left(\alpha_{k+1} + \left(1 - \frac{\alpha_{k+1}}{2}\right)P_C u_k\right) + \frac{1}{2} - \frac{\lambda_{k+1}}{4} \\ &= \frac{\lambda_{k+1}}{2}\left(\alpha_{k+1} + \left(1 - \frac{\alpha_{k+1}}{2}\right)u_k\right) + \frac{1}{2} - \frac{\lambda_{k+1}}{4} \\ &\le \frac{\lambda_{k+1}}{2}\left(\alpha_{k+1} + \left(1 - \frac{\alpha_{k+1}}{2}\right)\right) + \frac{1}{2} - \frac{\lambda_{k+1}}{4} \\ &= \frac{\lambda_{k+1}}{2}\left(\frac{\alpha_{k+1}}{2} + 1\right) + \frac{1}{2} - \frac{\lambda_{k+1}}{4} \\ &= \frac{\lambda_{k+1}}{4} + \frac{1}{2} + \frac{\lambda_{k+1}\alpha_{k+1}}{4} \\ &< \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1. \end{aligned}$$

Thus, by mathematical induction, we get $u_n \in C$ for all $n \in \mathbb{N}$. Therefore, $x_n \in C$ for all $n \in \mathbb{N}$. It can be seen from Table 1 and Figure 1 that $\{\frac{|x_n - z_n|}{\lambda_n}\}$ converges to 0 (see also (5.3)).

Table 1 The numerical values of $\frac{|x_n - z_n|}{\lambda_n}$ up to n = 10,000

n	$\frac{\|x_n-z_n\|}{\lambda_n}$	n	$\frac{\|x_n-z_n\|}{\lambda_n}$	n	$\frac{\|x_n-z_n\|}{\lambda_n}$
1	0.600468477588001	3,400	0.007373276312992	6,800	0.005133489777493
200	0.033090574752038	3,600	0.007155873637358	7,000	0.005056543877199
400	0.022842404623134	3,800	0.006956215006767	7,200	0.004982885604091
600	0.018408162394866	4,000	0.006772024658859	7,400	0.004912288613870
800	0.015801751117225	4,200	0.006601414499865	7,600	0.004844547738281
1,000	0.014040689141910	4,400	0.006442803256853	7,800	0.004779476507962
1,200	0.012750658965951	4,600	0.006294855306963	8,000	0.004716905020154
1,400	0.011754246690473	4,800	0.006156433766950	8,200	0.004656678095976
1,600	0.010955215843824	5,000	0.006026564076254	8,400	0.004598653681922
1,800	0.010296300001474	5,200	0.005904405409570	8,600	0.004542701458279
2,000	0.009741032549748	5,400	0.005789228005585	8,800	0.004488701623540
2,200	0.009264955188978	5,600	0.005680395018282	9,000	0.004436543829128
2,400	0.008850976030794	5,800	0.005577347862578	9,200	0.004386126242951
2,600	0.008486746121476	6,000	0.005479594286560	9,400	0.004337354723797
2,800	0.008163093477629	6,200	0.005386698590768	9,600	0.004290142091386
3,000	0.007873045437275	6,400	0.005298273552540	9,800	0.004244407479276
3,200	0.007611194925153	6,600	0.005213973715158	10,000	0.004200075759726

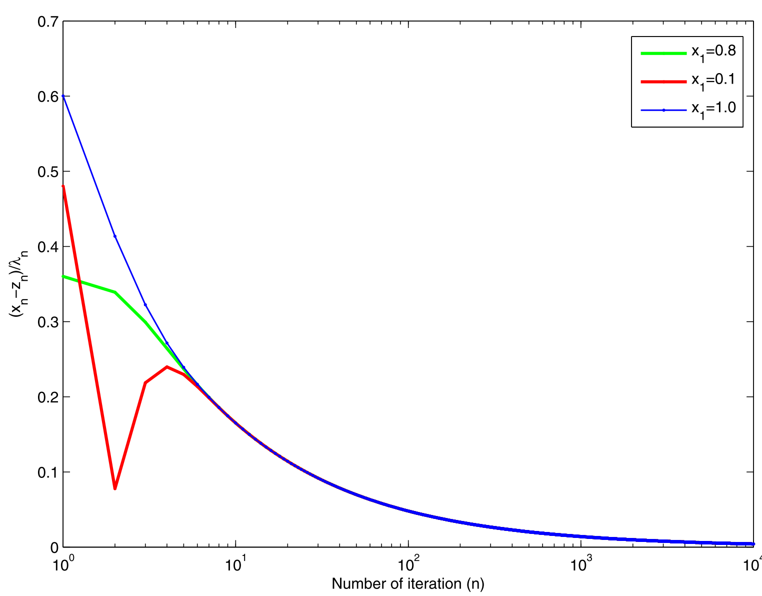

Figure 1 Convergence of sequence $\{\frac{|x_n - z_n|}{\lambda_n}\}$.

Thus, all the assumptions of Theorem 3.1 are satisfied. Therefore, the iteratively generated sequence $\{x_n\}$ defined by (3.2) converges strongly to $\{\frac{1}{2}\}$, which is also the unique solution of the triple hierarchical variational inequality (3.3).

The numerical values of x_n up to $n = 10{,}000$ have been calculated in Table 2 and convergence of sequence $\{x_n\}$ is given in Figure 2. Finally, mathematically, we show that $x_n \to \frac{1}{2}$ and $\frac{|x_n - z_n|}{\lambda_n} \to 0$ as $n \to \infty$. Note

$$\begin{aligned} x_{n+1} &= P_C(u_n) = u_n = \frac{\lambda_n}{2}\left(\alpha_n + \left(1 - \frac{\alpha_n}{2}\right)x_n\right) + \frac{1}{2} - \frac{\lambda_n}{4} \\ &\le \frac{\lambda_n}{2}\left(\alpha_n + \left(1 - \frac{\alpha_n}{2}\right)\right) + \frac{1}{2} - \frac{\lambda_n}{4} = \frac{\alpha_n \lambda_n}{4} + \frac{\lambda_n}{4} + \frac{1}{2}, \end{aligned} \tag{5.1}$$

Table 2 The numerical values of x_n up to $n = 10{,}000$

n	x_n	n	x_n	n	x_n
1	0.000000000000000	3,400	0.501901245173150	6,800	0.501179342447758
200	0.513660892766142	3,600	0.501827701962605	7,000	0.501156068130334
400	0.508408221515529	3,800	0.501760776817138	7,200	0.501133892973359
600	0.506334966973673	4,000	0.501699569844454	7,400	0.501112736570979
800	0.505184133249873	4,200	0.501643340889305	7,600	0.501092526371813
1,000	0.504438577466294	4,400	0.501591475155260	7,800	0.501073196728994
1,200	0.503910346287644	4,600	0.501543457407883	8,000	0.501054688086088
1,400	0.503513474443515	4,800	0.501498852346845	8,200	0.501036946276588
1,600	0.503202657672226	5,000	0.501457289482619	8,400	0.501019921918822
1,800	0.502951585650275	5,200	0.501418451349507	8,600	0.501003569891334
2,000	0.502743853895386	5,400	0.501382064222004	8,800	0.500987848876446
2,200	0.502568662846337	5,600	0.501347890731946	9,000	0.500972720961835
2,400	0.502418590433880	5,800	0.501315723944806	9,200	0.500958151291612
2,600	0.502288354629419	6,000	0.501285382567478	9,400	0.500944107759859
2,800	0.502174086142696	6,200	0.501256707041723	9,600	0.500930560740660
3,000	0.502072880709147	6,400	0.501229556336889	9,800	0.500917482849616
3,200	0.501982512601766	6,600	0.501203805299252	10,000	0.500904848732640

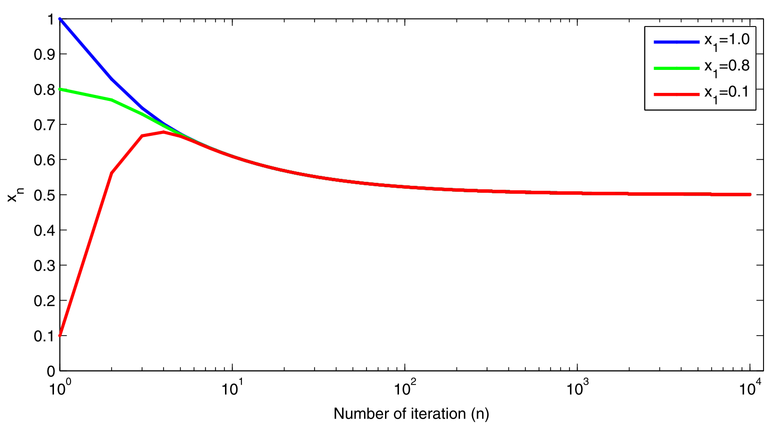

Figure 2 Convergence of the sequence $\{x_n\}$.

which implies that

$$\left|x_{n+1} - \frac{1}{2}\right| \leq \frac{\alpha_n \lambda_n}{4} + \frac{\lambda_n}{4} \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

Therefore, from (5.2) we get $x_n \to \frac{1}{2}$ as $n \to \infty$.

From (5.1), we have

$$x_{n+1} - \frac{1}{2} = \frac{\lambda_n}{2}\left[\alpha_n + x_n - \frac{1}{2} - \frac{\alpha_n x_n}{2}\right],$$

which implies that

$$\frac{|x_{n+1} - \frac{1}{2}|}{\lambda_{n+1}} = \frac{\lambda_n}{2\lambda_{n+1}}\left|\alpha_n + x_n - \frac{1}{2} - \frac{\alpha_n x_n}{2}\right|.$$

We have $\alpha_n \to 0$ and $\lambda_n \to 0$ as $n \to \infty$. Thus, we obtain

$$\frac{|x_{n+1} - z_n|}{\lambda_{n+1}} = \frac{\lambda_n}{2\lambda_{n+1}} \left| \alpha_n + x_n - \frac{1}{2} - \frac{\alpha_n x_n}{2} \right| \to 0 \quad \text{as } n \to \infty. \tag{5.3}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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