Karapınar and Samet *Fixed Point Theory and Applications* 2014, 2014:26
http://www.fixedpointtheoryandapplications.com/content/2014/1/26

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A note on 'ψ-Geraghty type contractions'

Erdal Karapınar[1*] and Bessem Samet[2]

*Correspondence: erdalkarapinar@yahoo.com; ekarapinar@atilim.edu.tr
[1]Department of Mathematics, Atilim University, Incek, Ankara 06836, Turkey
Full list of author information is available at the end of the article

Abstract
Very recently, the notion of a ψ-Geraghty type contraction was defined by Gordji *et al.* (Fixed Point Theory and Applications 2012:74, 2012). In this short note, we realize that the main result via ψ-Geraghty type contraction is equivalent to an existing related result in the literature. Consequently, all results inspired by the paper of Gordji *et al.* in (Fixed Point Theory and Applications 2012:74, 2012) can be derived in the same way.

MSC: 47H10; 54H25; 46J10; 46J15

Keywords: Geraghty type contraction; auxiliary function

1 Introduction and preliminaries

One of the celebrated generalizations of the Banach contraction (mapping) principle was given by Geraghty [1].

Theorem 1.1 (Geraghty [1]) *Let (X,d) be a complete metric space and $T: X \to X$ be an operator. Suppose that there exists $\beta: (0,\infty) \to [0,1)$ satisfying the condition*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0. \tag{1}$$

If T satisfies the following inequality:

$$d(Tx,Ty) \le \beta\big(d(x,y)\big)d(x,y), \quad \textit{for any } x,y \in X, \tag{2}$$

then T has a unique fixed point.

Let $\mathcal{S}$ denote the set of all functions $\beta: (0,\infty) \to [0,1)$ satisfying (1). This nice result of Geraghty [1] has been studied by a number of authors, see *e.g.* [2–10] and references therein.

In the following Harandi and Emami [2] reconsidered Theorem 1.1 in the framework of partially ordered metric spaces (see also [11]).

Theorem 1.2 *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $f: X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. If there exists $\alpha \in \mathcal{S}$ such that*

$$d(fx,fy) \le \alpha\big(d(x,y)\big)d(x,y), \tag{3}$$

for each $x,y \in X$ with $x \succeq y$, then f has a fixed point provided that either f is continuous or X is such that if an increasing sequence $\{x_n\} \to x$ in X; then $x_n \preceq x$, for all n. Besides, if for *each $x,y \in X$ there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point.*

Very recently, Gordji *et al.* [12] supposedly improved and extended Theorem 1.2 in the following way via the auxiliary function defined below. Let Ψ denote the class of the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(ψ_1) ψ is nondecreasing;
(ψ_2) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$;
(ψ_3) ψ is continuous;
(ψ_4) $\psi(t) = 0 \Leftrightarrow t = 0$.

The following is the main theorem of Gordji *et al.* [12].

Theorem 1.3 *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exist $\alpha \in \mathcal{S}$ and $\psi \in \Psi$ such that*

$$\psi\bigl(d(fx,fy)\bigr) \leq \alpha\bigl(\psi\bigl(d(x,y)\bigr)\bigr)\psi\bigl(d(x,y)\bigr), \tag{4}$$

for all $x,y \in X$ with $x \preceq y$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ converges to x, then $x_n \preceq x$ for each $n \geq 1$. Then f has a fixed point.

2 Main results

We start this section with the following lemma, which is the skeleton of this note.

Lemma 2.1 *Let (X,d) be a metric space and $\psi \in \Psi$. Then, a function $d_\psi : X \times X \to [0,\infty)$ defined by $d_\psi(x,y) = \psi(d(x,y))$ forms a metric on X. Moreover, (X,d) is complete if and only if (X,d_ψ) is complete.*

Proof

(1) If $x = y$, then $d(x,y) = 0$. Due to (ψ_4), we have $\psi(d(x,y)) = 0$. The converse is obtained analogously.
(2) $d_\psi(x,y) = \psi(d(x,y)) = \psi(d(y,x)) = d_\psi(y,x)$.
(3) Since ψ is nondecreasing, we have $\psi(d(x,y)) \leq \psi(d(x,z) + d(z,y))$. Regarding the subadditivity of ψ, we derived

$$\begin{aligned} d_\psi(x,y) = \psi\bigl(d(x,y)\bigr) &\leq \psi\bigl(d(x,z) + d(z,y)\bigr) \\ &\leq \psi\bigl(d(x,z)\bigr) + \psi\bigl(d(z,y)\bigr) \\ &= d_\psi(x,z) + d_\psi(z,y). \end{aligned}$$

Notice that the completeness of (X,d_ψ) follows from (ψ_3) and (ψ_4). □

The following is the main result of this note.

Theorem 2.2 *Theorem* 1.3 *is a consequence of Theorem* 1.2.

Proof Due to Lemma 2.1, we derived the result that (X, d_ψ) is a complete metric space. Furthermore, the condition (4) turns into

$$d_\psi(fx, fy) \leq \alpha\big(d_\psi(x,y)\big)d_\psi(x,y). \tag{5}$$

Hence all conditions of Theorem 1.2 are satisfied. □

3 The best proximity case

Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$\begin{aligned} A_0 &= \big\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\big\}, \\ B_0 &= \big\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\big\}, \end{aligned} \tag{6}$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

In [13, 14], the author introduces the following definition.

Definition 3.1 Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A,B) \quad \text{and} \quad d(x_2, y_2) = d(A,B) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2). \tag{7}$$

Caballero *et al.* proved the following result.

Theorem 3.2 (See [8]) *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty contraction, i.e. there exists $\beta \in \mathcal{S}$ such that*

$$d(Tx, Ty) \leq \beta\big(d(x,y)\big)d(x,y), \quad \textit{for any } x, y \in A. \tag{8}$$

Suppose that T is continuous and satisfies $T(A_0) \subseteq B_0$. Suppose also that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.

Inspired by Gordji *et al.* [12] and Caballero *et al.* [8], Karapinar [7] reported the following result.

Theorem 3.3 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be ψ-Geraghty contraction, i.e. there exists $\beta \in \mathcal{S}$ such that*

$$\psi\big(d(Tx, Ty)\big) \leq \alpha\big(\psi\big(d(x,y)\big)\big)\psi\big(d(x,y)\big), \quad \textit{for any } x, y \in A. \tag{9}$$

Suppose that T is continuous and satisfies $T(A_0) \subseteq B_0$. Suppose also that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.

The following lemmas belong to Akbar and Gabeleh [15].

Lemma 3.4 [15] *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the P-property. Then (A_0, B_0) is a closed pair of subsets of X.*

Lemma 3.5 [15] *Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d) such that A_0 is nonempty. Assume that the pair (A, B) has the P-property. Then there exists a bijective isometry $g : A_0 \to B_0$ such that $d(x, gx) = dist(A, B)$.*

Very recently, by using Lemma 3.4 and Lemma 3.5, Akbar and Gabeleh [15] proved that the best proximity point results via P-property can be obtained from the associate results in fixed point theory. In particular they proved the following theorem.

Theorem 3.6 *Theorem* 3.2 *is a consequence of Theorem* 1.1.

As a consequence of Theorem 2.2 we can observe the following result.

Corollary 3.7 *Theorem* 3.3 *is a consequence of Theorem* 3.2.

Regarding the analogy, we omit the proof.

Theorem 3.8 *Theorem* 3.3 *is a consequence of Theorem* 1.1.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Atilim University, Incek, Ankara 06836, Turkey. [2]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia.

Acknowledgements
The authors thank the Visiting Professor Programming at King Saud University for funding this work. The authors thank the anonymous referees for their remarkable comments, suggestion, and ideas, which helped to improve this paper.

Received: 10 December 2013 Accepted: 23 January 2014 Published: 03 Feb 2014

References
1. Geraghty, M: On contractive mappings. Proc. Am. Math. Soc. **40**, 604-608 (1973)
2. Amini-Harandi, A, Emami, H: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Nonlinear Anal., Theory Methods Appl. **72**(5), 2238-2242 (2010)
3. Amini-Harandi, A, Fakhar, M, Hajisharifi, H, Hussain, N: Some new results on fixed and best proximity points in preordered metric spaces. Fixed Point Theory Appl. **2013**, 263 (2013)
4. Bilgili, N, Karapinar, E, Sadarangani, K: A generalization for the best proximity point of Geraghty-contractions. J. Inequal. Appl. **2013**, 286 (2013)
5. Mongkolkeha, C, Cho, Y, Kumam, P: Best proximity points for Geraghty's proximal contraction mappings. Fixed Point Theory Appl. **2013**, 180 (2013)
6. Kim, J, Chandok, S: Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces. Fixed Point Theory Appl. **2013**, 307 (2013)
7. Karapinar, E: On best proximity point of ψ-Geraghty contractions. Fixed Point Theory Appl. **2013**, 200 (2013)
8. Caballero, J, Harjani, J, Sadarangani, K: A best proximity point theorem for Geraghty-contractions. Fixed Point Theory Appl. **2012**, 231 (2012)
9. Abbas, M, Sintunavarat, W, Kumam, P: Coupled fixed point of generalized contractive mappings on partially ordered *G*-metric spaces. Fixed Point Theory Appl. **2012**, 31 (2012)
10. Sintunavarat, W: Generalized Ulam-Hyers stability, well-posedness and limit shadowing of fixed point problems for $\alpha - \beta$-contraction mapping in metric spaces. Sci. World J. (in press)
11. Cho, S-H, Bae, J-S, Karapinar, E: Fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. **2013**, 329 (2013)

12. Gordji, ME, Ramezani, M, Cho, YJ, Pirbavafa, S: A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations. Fixed Point Theory Appl. **2012**, 74 (2012)
13. Raj, VS: A best proximity theorem for weakly contractive non-self mappings. Nonlinear Anal. **74**, 4804-4808 (2011)
14. Raj, VS: Banach's contraction principle for non-self mappings (preprint)
15. Abkar, A, Gabeleh, M: A note on some best proximity point theorems proved under *P*-property. Abstr. Appl. Anal. **2013**, Article ID 189567 (2013). doi:10.1155/2013/189567

10.1186/1687-1812-2014-26
Cite this article as: Karapınar and Samet: **A note on 'ψ-Geraghty type contractions'.** *Fixed Point Theory and Applications* 2014, 2014:26