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Fixed points of set-valued Caristi-type mappings on semi-metric spaces via partial order relations

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Abstract

In this paper, we present some fixed-point theorems that are related to a set-valued Caristi-type mapping. The main results extend the recent work which was presented by Jiang and Li (Fixed Point Theory Appl. 2013:74, 2013) from a single-valued setting to a set-valued case. Further, the presented results also improve essentially many results that have appeared, because we have removed some conditions from the auxiliary function. Meanwhile, we give some partial answers to an important problem which was raised by Kirk (Colloq. Math. 36:81-86, 1976).

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1 Introduction

Let (X, d) be a complete metric space and a self-map T on X (not necessarily continuous). In 1976, Caristi [1] asserted that T must have a fixed point provided that there exists a nonnegative real-valued function φ, which is lower semicontinuous, such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \tag{1.1}$$

for all $x \in X$. A number of extensions of the Banach contraction principle have appeared in the literature and Caristi's fixed-point theorem is one of the most important extensions of the Banach contraction principle. It is well known that Caristi's fixed-point theorem is a variation of the ε-variational principle of Ekeland [2], which is an important tool in nonlinear analysis, such as optimization, variational inequalities, differential equations, and control theory. Of course, many authors have studied and generalized Caristi's fixed-point theorem in various directions. Notice that Caristi's original proof involved an intricate transfinite induction argument. Subsequently, Kirk [3] gave an elegant proof of Caristi's fixed-point theorem by considering a partial order on X, which has been defined by Brøndsted [4], as follows:

$$x \preceq y \quad \Leftrightarrow \quad d(x, y) \leq \varphi(x) - \varphi(y) \tag{1.2}$$

for all $x, y \in X$. Kirk also considered a class of mappings, so-called *Caristi-type mapping*, that is, a map $T : X \to X$ such that for all $x \in X$

$$\eta\bigl(d(x, Tx)\bigr) \le \varphi(x) - \varphi(Tx) \tag{1.3}$$

for some a function $\eta : [0, +\infty) \to [0, +\infty)$ and a function $\varphi : X \to (-\infty, +\infty)$. He raised a problem relating to generalize Caristi's fixed-point theorem stating whether a Caristi-type mapping T has a fixed point for a complete metric space (X, d).

Recall that a function η on a subset A of real numbers is called *subadditive* if

$$\eta(t + s) \le \eta(t) + \eta(s)$$

for all $t, s \in A$. In 1998, Jachymski [5] showed that each Caristi-type mapping T has a fixed point whenever we assume φ is a nonnegative lower semicontinuous on X and η is a nondecreasing and subadditive function, continuous at 0 and such that $\eta^{-1}(\{0\}) = \{0\}$. Later, in 2006, Feng and Liu [6] considered a relation $\preceq$ which is defined on X by

$$x \preceq y \quad \Leftrightarrow \quad \eta\bigl(d(x,y)\bigr) \le \varphi(x) - \varphi(y) \tag{1.4}$$

for all $x, y \in X$. By assuming that η is nondecreasing, continuous, subadditive, and $\eta^{-1}(\{0\}) = \{0\}$, they showed that the relation defined by (1.4) is a partial order on X and proved that each Caristi-type mapping has a fixed point, by inquiring the existence of maximal element of a partially ordered complete metric space $(X, \preceq)$ provided that φ is bounded from below and lower semicontinuous. It was noticed that the subadditivity of η is necessarily assumed for insisting that the relation in (1.4) is a partial order. Thus if η is not a subadditive function, then the relationship $\preceq$ defined by (1.4) may not be a partial order in X, and consequently the technique used in [5, 6] becomes invalid. Observing this one, by introducing a partial order on a complete subset of X, Khamsi [7] removed the subadditivity of η and showed the existence theorem of fixed point for a Caristi type mapping. Recently, Li [8] noticed that $\eta(0) = 0$ is an essential condition for the existence of a fixed point for Caristi-type mapping.

Let X be a nonempty set, and $\preceq$ be a partial ordering on X. For each $x \in X$, we set

$$[x, +\infty)_{\preceq} := \{z \in X : x \preceq z\}$$

and

$$(-\infty, x]_{\preceq} := \{z \in X : z \preceq x\}.$$

Recall that a single-valued mapping T is said to be an *isotone mapping* if

$$Tx \preceq Ty \quad \text{for all } x, y \in X, x \preceq y. \tag{1.5}$$

Very recently, in 2013, Jiang and Li [9] considered the following condition: for each $x, y \in X$,

$$x \preceq y \quad \Rightarrow \quad \eta\bigl(d(x,y)\bigr) \le \varphi(x) - \varphi(y). \tag{1.6}$$

Without the subadditivity of η and the continuity of both η and φ, they successfully obtained the following fixed-point theorem.

Theorem 1.1 *Let (X,d) be a complete metric space, let $\varphi : X \to (-\infty,+\infty)$ be a bounded below functional, let $\eta : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function with $\eta^{-1}(\{0\}) = \{0\}$, and let $\preceq$ be a partial order on X such that (1.6) and $[x,+\infty)_{\preceq}$ and $(-\infty,x]_{\preceq}$ are closed for each $x,y \in X$. Let $T : X \to X$ be an isotone mapping. Assume that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then*

(i) *T has a maximal fixed point $x^* \in [x_0,+\infty)_{\preceq}$, i.e., let $x \in [x_0,+\infty)_{\preceq}$ be a fixed point of T, then $x^* \preceq x$ implies $x = x^*$;*

(ii) *T has a least fixed point $x_* \in [x_0,+\infty)_{\preceq}$, i.e., let $x \in [x_0,+\infty)_{\preceq}$ be a fixed point of T, then $x_* \preceq x$.*

Inspired by the above literature, in this work, we will present some extensions of results of Jiang and Li [9], say from a single-valued setting to a set-valued case. Further, meanwhile, our results also improve essentially many results. This is because we will remove some conditions, such as the assumption of an auxiliary function, as η, being nondecreasing. Also, some related results will be discussed. To complete our plan, the following useful basic concepts are needed.

A *semi-metric* for a nonempty set X is a nonnegative real-valued function d from $X \times X$ into $\mathbb{R}$ such that

(S1) $d(x,y) = 0 \Leftrightarrow x = y$;

(S2) $d(x,y) = d(y,x)$,

for each $x,y \in X$. A pair (X,d) is called a *semi-metric space*.

A subset A of a semi-metric space (X,d) is said to be *closed* iff $\overline{A}^d = A$, where $\overline{A}^d = \{x \in X : d(x,A) = 0\}$ and $d(x,A) = \inf\{d(x,y) : y \in A\}$. Let d be a semi-metric on X and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *convergent* to $x \in X$ if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$, for all $n \geq n_0$. $(x_n)_{n\in\mathbb{N}}$ is called *Cauchy* if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m,x_n) < \varepsilon$, for all $m,n \geq n_0$. A semi-metric space (X,d) is said to be *complete* if every Cauchy sequence is a convergent sequence.

For more information on semi-metric space, readers may consult [10–15].

Let $(X,\preceq)$ be a partial order set and 2^X be denoted for a class of all nonempty subset of X. As the core of this work, we are interested in the following class of set-valued mappings: a set-valued mapping $T : X \to 2^X$ is called *isotone* if, whenever $x \preceq y$ in X,

(I1) for each $u \in T(x)$ there is $v \in T(y)$ such that $u \preceq v$, and

(I2) dually, for each $v \in T(y)$ there is $u \in T(x)$ such that $u \preceq v$.

Isotone mappings were first studied by Smithson [16], and it is well known that one of these conditions may be satisfied while the other is not. In particular, a single-valued case of (I1) and (I2) must be similar, *i.e.*, for all $x,y \in X$ with $x \preceq y$ we have $T(x) \preceq T(y)$.

2 Main results

In this section, we let (X,d) be a semi-metric space, $\eta : [0,+\infty) \to (-\infty,+\infty)$ and $\varphi : X \to (-\infty,+\infty)$ be functions such that

(H1) η is nonnegative on $\{d(x,y) : x,y \in X\}$ with $\eta(0) = 0$, and there exist $c > 0$ and $\varepsilon > 0$ such that

$$\eta(t) \geq ct \quad \text{for each } t \in W_{\eta,\varepsilon},$$

where $W_{\eta,\varepsilon} = \{t \geq 0 : \eta(t) \leq \varepsilon\} \cap \{d(x,y) : x,y \in X\}$.

Further if x_0 is an element in X, we assume that there is a partially ordered set $(X, \preceq)$ such that the following conditions hold:

(H2) φ is bounded below function on $[x_0, +\infty)_{\preceq}$.

(H3) We have

$$\eta\big(d(x,y)\big) \leq \varphi(x) - \varphi(y) \quad \text{for each } x, y \in [x_0, +\infty)_{\preceq}, \text{with } x \preceq y.$$

(H4) $[x, +\infty)_{\preceq}$ is closed for each $x \in X$ such that $x_0 \preceq x$.

(H5) The set $P := \{x \in [x_0, +\infty)_{\preceq} : x \preceq u, \exists u \in T(x)\}$ contains every limit points of increasing convergent sequence in P.

Remark 2.1 If η is a nonnegative and subadditive function on $[0, +\infty)$ with $\eta(0) = 0$, we know that

$$\lim_{t \to 0} \frac{\eta(t)}{t} = \sup\left\{\frac{\eta(x)}{x} : x > 0\right\},$$

see [17]. By using this, we know that (H1) is always satisfied, see [7].

The following two examples show that (H4) and (H5) are different.

Example 2.2 Let $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual metric d. Define a partially ordered $\preceq$ on X by $x \preceq y \Leftrightarrow y \leq x$ for all $x, y \in X$. Pick $x_0 = 1$. One can observe that, for each $1 \preceq x$, we have $[x, +\infty)_{\preceq}$ is a closed set. This means that (H4) is satisfied, whereas (H5) is not satisfied. Indeed, if we consider a mapping

$$T(x) = \begin{cases} X \setminus \{0\}; & x = 0, \\ \{0, 1\}; & \text{otherwise.} \end{cases}$$

We see that $P = \{\frac{1}{n} : n \in \mathbb{N}\}$. One can notice that there is an increasing convergent sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ in P which converges to 0, but $0 \notin P$.

Example 2.3 Let X and d be the same as in Example 2.2. Define a partial ordering $\preceq$ on X by $x \preceq y$ if and only if (i) $y \leq x$, and (ii) $\frac{x}{y} \in \mathbb{N}$. Define a mapping T by

$$T\left(\frac{1}{n}\right) = \begin{cases} 1; & x = 0, \\ \{\frac{1}{n+1}, \frac{1}{n+2}\}; & \text{otherwise.} \end{cases}$$

For $x_0 = 1$, we notice that the set P is nothing else but $\{1, \frac{1}{2}\}$. It follows that (H5) must be satisfied. This notwithstanding, since $[1, +\infty)_{\preceq} = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not a closed set, (H4) is not satisfied.

Now we show our main results.

Theorem 2.4 *Let (X, d) be a complete semi-metric space and $T : X \to 2^X$ be a set-valued mapping. Assume that there exist a partially ordered $\preceq$ on X and a point $x_0 \in X$ such that*

$$x_0 \preceq u \quad \textit{for some } u \in T(x_0). \tag{2.1}$$

If T is an isotone mapping (I1) *and conditions* (H1)-(H5) *are satisfied, then T has a fixed point x^*. Moreover, x^* is a maximal fixed point in $[x_0, +\infty)_{\preceq}$.*

Proof Firstly, by (2.1), we notice that the set $P := \{x \in [x_0, +\infty)_{\preceq} : x \preceq u, \exists u \in T(x)\}$ is a nonempty set. Now, let $(x_\alpha)_{\alpha\in\Gamma}$ be an increasing chain in P, where Γ is a directed set. By condition (H3), we know that $(\varphi(x_\alpha))_{\alpha\in\Gamma}$ is a decreasing net of real numbers. Further, by condition (H2), we know that $s := \inf\{\varphi(x_\alpha) : \alpha \in \Gamma\}$ does exist. Related to this number s, let us define now a nonempty subset Δ of Γ by

$$\Delta = \{\alpha \in \Gamma : \varphi(x_\alpha) \leq s + \varepsilon\},$$

where ε is a positive real number as appeared in condition (H1).

On the other hand, we know that there is an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of elements from Γ such that

$$\lim_{n\to\infty} \varphi(x_{\alpha_n}) = s.$$

We claim that $(x_{\alpha_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Otherwise, there exist a subsequence $(x_{\alpha_{n_i}})_{i\in\mathbb{N}}$ of $(x_{\alpha_n})_{n\in\mathbb{N}}$ and $\delta > 0$ such that

$$d(x_{\alpha_{n_i}}, x_{\alpha_{n_{i+1}}}) \geq \delta, \quad \forall i \in \mathbb{N}. \tag{2.2}$$

It follows by the increasing of $(x_{\alpha_{n_i}})_{i\in\mathbb{N}}$ and (H3) that

$$0 \leq \eta\bigl(d(x_{\alpha_{n_i}}, x_{\alpha_{n_{i+1}}})\bigr) \leq \varphi(x_{\alpha_{n_i}}) - \varphi(x_{\alpha_{n_{i+1}}}), \quad \forall i \in \mathbb{N}. \tag{2.3}$$

Let us pick $i_0 \in \mathbb{N}$, which is the first natural number such that $\alpha_{n_{i_0}} \in \Delta$. Then, by the decreasing of $(\varphi(x_{\alpha_n}))_{n\in\mathbb{N}}$, we have

$$s \leq \varphi(x_{\alpha_{n_{i+1}}}) \leq \varphi(x_{\alpha_{n_i}}) \leq s + \varepsilon, \quad \forall i \geq i_0. \tag{2.4}$$

Subsequently, by (2.3) and (2.4), we obtain

$$0 \leq \eta\bigl(d(x_{\alpha_{n_i}}, x_{\alpha_{n_{i+1}}})\bigr) \leq \varepsilon, \quad \forall i \geq i_0.$$

Using this one, in view of (2.2) together with (H1) and (H3), we have

$$c\delta \leq cd(x_{\alpha_{n_i}}, x_{\alpha_{n_{i+1}}}) \leq \eta\bigl(d(x_{\alpha_{n_i}}, x_{\alpha_{n_{i+1}}})\bigr) \leq \varphi(x_{\alpha_{n_i}}) - \varphi(x_{\alpha_{n_{i+1}}}), \quad \forall i \geq i_0.$$

This implies

$$ic\delta \leq \varphi(x_{\alpha_{n_{i_0}}}) - \varphi(x_{\alpha_{n_{i_0+i}}}), \quad \forall i \geq i_0.$$

Since $\lim_{n\to\infty} \varphi(x_{\alpha_n}) = \inf_{\alpha\in\Gamma} \varphi(x_\alpha)$, we obtain from i approaches infinity that

$$\inf_{\alpha\in\Gamma} \varphi(x_\alpha) = \lim_{i\to\infty} \varphi(x_{\alpha_{n_{i_0+i}}}) \leq \lim_{i\to\infty} \bigl(\varphi(x_{\alpha_{n_{i_0}}}) - ic\delta\bigr) = -\infty.$$

This is a contradiction. Hence $(x_{\alpha_n})_{n\in\mathbb{N}}$ must be a Cauchy sequence in P. Thus, since X is complete, there exists $\bar{x} \in X$ such that

$$\lim_{n\to\infty} x_{\alpha_n} = \bar{x}.$$

Further, by (H5), we have $\bar{x} \in P$.

Next, we will show that $(x_\alpha)_{\alpha\in\Gamma}$ has an upper bound in P. We consider the following two cases.

Case I. *For each* $\alpha \in \Gamma$, *there exists* $n_\Gamma \in \mathbb{N}$ *such that* $x_\alpha \preceq x_{\alpha_{n_\Gamma}}$.

Let $\alpha \in \Gamma$ be fixed. By the closedness of $[x_\alpha, +\infty)_\preceq$ and passing to a subsequence $(x_{\alpha_j})_{j=n_\Gamma+1}^{\infty}$ of an increasing sequence $(x_{\alpha_n})_{n\in\mathbb{N}}$, we know that $x_\alpha \preceq \bar{x}$. Since $\alpha \in \Gamma$ is arbitrary and $\bar{x} \in P$, we reach the required result.

Case II. *There exists* $\beta \in \Gamma$ *such that* $x_{\alpha_n} \preceq x_\beta$ *for all* $n \in \mathbb{N}$.

Obviously, in this case $x_\beta \in P$. Further, by condition (H3), we also have

$$\varphi(x_\beta) \le \varphi(x_{\alpha_n}) \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$

This also implies that $\beta \in \Delta$. Subsequently, by the decreasing of $\varphi((x_{\alpha_n}))_{n\in\mathbb{N}}$, we have

$$\varphi(x_\beta) = \inf_{\alpha\in\Gamma} \varphi(x_\alpha). \tag{2.6}$$

We claim that x_β is an upper bound of $(x_\alpha)_{\alpha\in\Gamma}$. Assume that there is $\alpha_0 \in \Gamma$ such that $x_\beta \preceq x_{\alpha_0}$ and $x_\beta \neq x_{\alpha_0}$. Thus, by (H3), we would have

$$0 \le \eta\bigl(d(x_\beta, x_{\alpha_0})\bigr) \le \varphi(x_\beta) - \varphi(x_{\alpha_0})$$

and

$$s \le \varphi(x_{\alpha_0}) \le \varphi(x_\beta) \le s + \varepsilon.$$

Using these inequalities, we see that $0 \le \eta(d(x_\beta, x_{\alpha_0})) \le \varepsilon$. So, by (H2), we get

$$0 < cd(x_\beta, x_{\alpha_0}) \le \eta\bigl(d(x_\beta, x_{\alpha_0})\bigr) \le \varphi(x_\beta) - \varphi(x_{\alpha_0}).$$

This implies that $\varphi(x_{\alpha_0}) < \varphi(x_\beta)$, and this contradicts to (2.6). Hence, our claim is asserted.

By Cases I and II, we can conclude that $(x_\alpha)_{\alpha\in\Gamma}$ has an upper bounded in P. Consequently, since $(x_\alpha)_{\alpha\in\Gamma}$ is an arbitrary chain in P, Zorn's lemma will therefore imply that $(P, \preceq)$ has a maximal element, say x^*.

Next, since $x^* \in P$, we know that

$$x^* \preceq u \quad \text{for some } u \in T\bigl(x^*\bigr). \tag{2.7}$$

Consequently, in view of the isotonicity (I1) of T, we can find $v \in T(u)$ such that $u \preceq v$. This, alternatively, implies that $u \in P$. Thus, by (2.7) together with the maximality of x^*, we obtain $x^* = u \in T(x^*)$. This means x^* is a maximal fixed point of T, and the proof is completed. □

Remark 2.5 Assume that conditions (H1)-(H5) are satisfied and T is linked to X by the following stronger relationships: there exists a point $x_0 \in X$ such that $x_0 \preceq u$ for all $u \in T(x_0)$, and

(SI) if $x, y \in X$ with $x \preceq y$, $x \neq y$ then $u \preceq v$ for each $u \in T(x)$, $v \in T(y)$.

Then T has an end point $\tilde{x}$. Moreover, $\tilde{x}$ is the maximal endpoint in $[x_0, +\infty)_{\preceq}$. Indeed, let $\tilde{P} = \{x \in [x_0, +\infty)_{\preceq} : x \preceq u, \forall u \in T(x)\}$ and also $(x_\alpha)_{\alpha \in \Gamma}$ be an increasing chain in $\tilde{P}$, where Γ is a directed set. It follows from the proof of Theorem 2.4 that there is $\tilde{x} \in \tilde{P}$ such that it is a maximal element of $(\tilde{P}, \preceq)$. Let $y \in T(\tilde{x})$ be arbitrary. It follows by $\tilde{x} \in \tilde{P}$ that $x_0 \preceq \tilde{x} \preceq y$. Subsequently, by (SI), we obtain

$$y \preceq v \quad \text{for all } v \in T(y).$$

This implies that $y \in \tilde{P}$. Now, by the maximality of $\tilde{x}$, we see that $\tilde{x} = y$. Since y is arbitrary, we conclude that $\{\tilde{x}\} = T(\tilde{x})$.

By using Theorem 2.4, we also have the following results.

Theorem 2.6 *Let (X, d) be a complete semi-metric space and $T : X \to 2^X$ be a set-valued mapping. Assume that there exist a partially ordered $\preceq$ on X and a point $x_0 \in X$ such that*

$$v \preceq x_0 \quad \textit{for some } v \in T(x_0). \tag{2.8}$$

Assume that T is an isotone mapping (I2) *on $(-\infty, x_0]_{\preceq}$ and the conditions* (H2), (H3) *and the following are satisfied*:

(H1)′ *φ is a bounded above functional on $(-\infty, x_0]_{\preceq}$*;
(H4)′ *$(-\infty, x]_{\preceq}$ is closed for each $x \in X$ such that $x \preceq x_0$*;
(H5)′ *the set $\hat{P} := \{x \in (-\infty, x_0]_{\preceq} : u \preceq x, \exists u \in T(x)\}$ contains every limit points of decreasing convergent sequence in $\hat{P}$.*

Then T has a fixed point x^*. Moreover, x^* is a minimal fixed point in $[-\infty, x_0)_{\preceq}$.

Proof Let $\preceq_1$ be the inverse partial order of $\preceq$ and $\varphi_1 := -\varphi$. Clearly, $x_0 \preceq_1 v$ for some $v \in T(x_0)$ and φ_1 is a bounded below functional on $(-\infty, x_0]_{\preceq}$. Set $[x, +\infty)_{\preceq_1} = \{z \in X : x \preceq_1 z\}$. Then, by (H4)′, we find that $[x, +\infty)_{\preceq_1}$ is a closed set for each $x \in X$ such that $x_0 \preceq_1 x$. It is easy to check that (H3) is satisfied for $\preceq_1$ and T is an isotone mapping (I1) on $[x_0, +\infty)_1$. Moreover, any decreasing sequence in $\hat{P}$ is an increasing sequence in P. Applying Theorem 2.4 with respect to $\preceq_1$, we have T has a maximal fixed point $x^* \in [x_0, +\infty)_{\preceq_1}$. This gives the result that T has a minimal fixed point in $(-\infty, x_0]_{\preceq}$. □

Remark 2.7 Similarly, as we have mentioned in Remark 2.5, if all assumptions of Theorem 2.6 are satisfied and T is linked to X by the stronger relationship (SI), and there exists a point $x_0 \in X$ such that such that $v \preceq x_0$ for all $v \in T(x_0)$, then one can show that there is a minimal end point $\tilde{x} \in (-\infty, x_0]_{\preceq}$ of T.

Now we give an example to demonstrate Theorem 2.4.

Example 2.8 Let $X = \{0\} \cup \{\frac{1}{n} : n = 1,2,3,\ldots\}$ with a semi-metric $d : X \times X \to [0,+\infty)$ defined by

$$\begin{cases} d(0,1) = 1 = d(1,0), \\ d(1,1) = 0, \\ d(1,\frac{1}{n}) = \frac{1}{3} = d(\frac{1}{n},1) \quad \text{for } n \geq 2, \\ d(x,y) = |x-y| \quad \text{for } x,y \in X - \{1\}. \end{cases}$$

We know that (X,d) is a complete semi-metric space, see [18].

Let us consider a set-valued mapping $T : X \to 2^X$ which is defined by

$$T(0) = T(1) = \{0,1\}$$

and

$$T\left(\frac{1}{n}\right) = \left\{\frac{1}{m} : \frac{m}{n} \in \mathbb{N}\right\}, \quad \forall n \geq 2,$$

where functions $\eta : [0,+\infty) \to [0,+\infty)$ and $\varphi : X \to (-\infty,+\infty)$ are defined by

$$\eta(t) = \begin{cases} \frac{t}{2}; & t \text{ is a rational number}, \\ \frac{1}{4}; & \text{otherwise} \end{cases}$$

for all $t \in [0,+\infty)$, and $\varphi(x) = \frac{3x}{4}$ for all $x \in X$, respectively.

Consider a partial ordering relation $\preceq$ on X which is defined by $x \preceq y$ if and only if (i) $y \leq x$, and (ii) $\frac{x}{y} \in \mathbb{N}$ or $xy = 0$, where $\leq$ is the usual order of real numbers. Related to this partial ordering $\preceq$, one can check that the mapping T is an isotone (I1) and $[x,+\infty)_{\preceq} \cap T(x) \neq \emptyset$, for each $x \in X$. Moreover, we note that η satisfies condition (H1) in Theorem 2.4 and φ is a bounded below functional on $[1,+\infty)_{\preceq}$. Further, we have

$$\eta\bigl(d(x,y)\bigr) = \begin{cases} \frac{1}{2} \leq \frac{3}{4} = \varphi(x) - \varphi(y); & x = 1, y = 0, \\ \frac{1}{2n} \leq \frac{3}{4n} = \varphi(x) - \varphi(y); & x = \frac{1}{n}, \forall n \geq 2, y = 0, \\ \frac{1}{6} \leq \frac{4n-3}{4n} = \varphi(x) - \varphi(y); & x = 1, y = \frac{1}{n}, \forall n \geq 2, \\ \frac{m-n}{2mn} \leq \frac{3(m-n)}{4mn} = \varphi(x) - \varphi(y); & x = \frac{1}{n}, \forall n \geq 2, y = \frac{1}{m}, \forall m \geq n, \end{cases}$$

which implies that (H3) is satisfied. Also, for each $x \in X$, we have

$$[x,+\infty)_{\preceq} = \{z \in X : x \preceq z\} = \begin{cases} \{0\}; & x = 0, \\ \{0\} \cup \{\frac{1}{m} : \frac{m}{n} \in \mathbb{N}\}; & x = \frac{1}{n}, n \geq 1. \end{cases}$$

Note that $\{0\}$, $\{0\} \cup \{\frac{1}{m} : \frac{m}{n} \in \mathbb{N}\}$ $(n \geq 1)$ are closed sets. Then for each $x \in X$, $[x,+\infty)_{\preceq}$ is a closed set. This means (H4) is satisfied. Therefore, all assumptions of Theorem 2.4 are satisfied. In fact, we see that 0 is a maximal fixed point in $[1,+\infty)_{\preceq}$.

3 Further results

By using Theorems 2.4 and 2.6, in this section, we will provide some largest and least fixed-point theorems for a single-valued mapping on semi-metric space.

Theorem 3.1 *Let (X,d) be a complete semi-metric space and $T : X \to X$ be an isotone mapping. Assume that there exist a partial ordering $\preceq$ on X and a point $x_0 \in X$ such that $x_0 \preceq T(x_0)$. If conditions* (H1)-(H4) *hold and the following condition is satisfied*:

(H6) $(-\infty, x]_{\preceq}$ *is a closed set for each $x \in X$ such that $x_0 \preceq x$,*

then

(i) *T has a maximal fixed point $x^* \in [x_0, +\infty)_{\preceq}$;*

(ii) *T has a least fixed point $x_* \in [x_0, +\infty)_{\preceq}$, that is, if $x \in [x_0, +\infty)_{\preceq}$ is a fixed point of T, then $x_* \preceq x$.*

Proof (i) Set $P = \{x \in [x_0, +\infty)_{\preceq} : x \preceq T(x)\}$; in this case, it suffices to show that the condition (H5) in Theorem 2.4 can be removed. Indeed, let an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in P be such that $x_n \to \bar{x}$, for some $\bar{x} \in X$. Since $(x_n)_{n\in\mathbb{N}} \subset P$, we have $x_n \preceq T(x_n)$, for all $n \in \mathbb{N}$. Furthermore, for each $n_0 \in \mathbb{N}$, we have $x_{n_0} \preceq x_n$, for each $n \geq n_0$. It follows from $[x_{n_0}, +\infty)_{\preceq}$ is a closed set that $\bar{x} \in [x_{n_0}, +\infty)_{\preceq}$. So, we have $x_{n_0} \preceq \bar{x}$. Since n_0 is arbitrary, we obtain $x_n \preceq \bar{x}$, for each $n \in \mathbb{N}$. This implies that $x_n \preceq T(x_n) \preceq T(\bar{x})$, for each $n \in \mathbb{N}$. By taking n approaching infinity and the closedness of $(-\infty, T(\bar{x})]_{\preceq}$, we have $\bar{x} \preceq T(\bar{x})$ and hence $\bar{x} \in P$. Therefore, the result can be obtained immediately from Theorem 2.4.

(ii) The proof is akin to Theorem 1 of [9]. For the sake of completeness, we present here its proof. Set $F(T) = \{x \in [x_0, +\infty)_{\preceq} : x = T(x)\}$. Clearly, by (i), we see that $F(T) \neq \emptyset$. Set

$$S := \bigl\{I = [x, +\infty)_{\preceq} : x \in [x_0, +\infty)_{\preceq}, x \preceq T(x), F(T) \subset I\bigr\}. \tag{3.1}$$

Clearly, $S \neq \emptyset$ since $[x_0, +\infty)_{\preceq} \in S$. Define a relation on S by

$$I_1 \preceq_s I_2 \quad \Leftrightarrow \quad I_1 \subset I_2 \quad \text{for all } I_1, I_2 \in S. \tag{3.2}$$

It is easy to check that the relation $\preceq_s$ is a partial order on S.

Let $(I_\alpha)_{\alpha\in\Gamma}$ be a decreasing chain in S, where $I_\alpha = [x_\alpha, +\infty)_{\preceq}$. Then we see that $(x_\alpha)_{\alpha\in\Gamma}$ is an increasing chain of M, where $M = \{x \in [x_0, +\infty)_{\preceq} : x \preceq T(x), F(T) \subset [x, +\infty)_{\preceq}\}$. Clearly $M \subset P$. Following the proof of Theorem 2.4, we know that there exists an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in M with $\lim_{n\to\infty} \varphi(x_{\alpha_n}) = \inf_{\alpha\in\Gamma} \varphi(x_\alpha)$ and $\bar{x} \in P$ such that $\lim_{n\to\infty} x_{\alpha_n} = \bar{x}$ and $x_{\alpha_n} \preceq \bar{x}$, for all $n \in \mathbb{N}$. Since $(x_{\alpha_n})_{n\in\mathbb{N}} \subset M$, we have $x_{\alpha_n} \preceq x$ for each $x \in F(T)$ and $n \in \mathbb{N}$. By the fact that T is an isotone mapping, we have

$$x_{\alpha_n} \preceq T(x_{\alpha_n}) \preceq T(x) = x, \quad \forall x \in F(T), n \in \mathbb{N}.$$

By letting $n \to \infty$ and the closedness of $(-\infty, x]_{\preceq}$, we obtain

$$\bar{x} \preceq x, \quad \forall x \in F(T).$$

It follows that $\bar{x} \in M$. In analogy to the proof of Theorem 2.4, we can show that $(x_\alpha)_{\alpha\in\Gamma}$ has an upper bound in M, denote it by $\hat{x}$. Set $\hat{I} = [\hat{x}, +\infty)_{\preceq}$, then, by $\hat{x} \in M$ and (3.1), we have $\hat{I} \in S$. Since $\hat{x}$ is an upper bound of $(x_\alpha)_{\alpha\in\Gamma}$ in M, we have

$$\hat{I} \subset I_\alpha, \quad \forall \alpha \in \Gamma,$$

which together with (3.2) implies that

$$\hat{I} \preceq_s I_\alpha, \quad \forall \alpha \in \Gamma.$$

This means that $\hat{I}$ is a lower bound of $(I_\alpha)_{\alpha\in\Gamma}$ in S. By Zorn's lemma, $(S, \preceq_s)$ has a minimal element, denote it by $I^* = [x_*, +\infty)_\preceq$. By (3.1) we have $x_0 \preceq x_* \preceq T(x_*)$ and

$$x_* \preceq x, \quad \forall x \in F(T). \tag{3.3}$$

By the fact that T is an isotone mapping, we have $x_0 \preceq x_* \preceq T(x_*) \preceq T(T(x_*))$ and $T(x_*) \preceq T(x) = x$ for each $x \in F(T)$. Set $\tilde{I} = [T(x_*), +\infty)_\preceq$. Clearly, $\tilde{I} \in S$ and $\tilde{I} \subset I^*$ by (3.1). Thus, we have $\tilde{I} \preceq_s I^*$ by (3.2). By the minimality of I^* in S, we can conclude that $\tilde{I} = I^*$. This implies that $x_* = T(x_*)$. Therefore x_* is a least fixed point of T in $[x_0, +\infty)_\preceq$. □

By Theorem 3.1 and using a technique as in Theorem 2.6, we can obtain the following result.

Theorem 3.2 *Let (X, d) be a complete semi-metric space and $T : X \to X$ be an isotone mapping. Assume that there exist a partial ordering $\preceq$ on X and a point $x_0 \in X$ such that $T(x_0) \preceq x_0$. If conditions* (H1)-(H4) *and* (H6) *are satisfied, then*

(i) *T has a minimal fixed point $x^* \in [x_0, +\infty)_\preceq$;*

(ii) *T has a largest fixed point $x_* \in (-\infty, x_0]_\preceq$, that is, if $x \in (-\infty, x_0]_\preceq$ be a fixed point of T, then $x \preceq x_*$.*

Remark 3.3 In Theorems 3.1 and 3.2, we replace the conditions that η is a nondecreasing function on $[0, +\infty)$ and $\eta^{-1}(\{0\}) = \{0\}$, which have proposed in Theorem 1.1, by the condition (H1).

The following example is inspired by Example 2 in [9].

Example 3.4 Let X, d, φ, $\preceq$ be the same as appearing in Example 2.8. Let $T : X \to X$ and $\eta : [0, +\infty) \to [0, +\infty)$ be defined by

$$T(x) = \begin{cases} 0; & x = 0, \\ \frac{1}{2}; & x = \frac{1}{2}, \\ \frac{1}{n+1}; & x = \frac{1}{n}, n = 1, 2, 3, \ldots \end{cases} \tag{3.4}$$

and

$$\eta(t) = \begin{cases} \frac{t}{2}; & t \text{ is rational number}, \\ \frac{1-\sqrt{3}t}{2}; & \text{otherwise}, \end{cases} \tag{3.5}$$

respectively. We know that T is an isotone mapping. Moreover, for each $x \in X$, $(x, \infty]_\preceq$ and $(-\infty, x]_\preceq$ are closed sets, see also [9]. Further, with respect to the given function η and as showed in Example 2.8, we know that the condition (H3) is also satisfied. In fact, 0 is the largest fixed point and $\frac{1}{2}$ is the least fixed point in $[\frac{1}{2}, +\infty)_\preceq$.

Remark 3.5 In Example 3.4, the considered function η is not a decreasing function and $\eta^{-1}\{0\} = \{0, \frac{1}{\sqrt{3}}\}$. Thus, Theorem 1.1 cannot be applied in this situation.

4 Conclusion

In this work, the set-valued Caristi-type mapping in the setting of generalized metric space, as a semimetric space, is considered. Evidently, the presented results improve essentially many results because we also removed some conditions, such as a nondecreasingness assumption from an auxiliary function, which have been imposed in the literature. In fact, we would like to point out that this paper gives some partial answers to an important problem which was raised by Kirk [3].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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