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Krasnosel'skiĭ-Mann-Opial type iterative solution of m-accretive operator equation and its stability in arbitrary Banach spaces

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Abstract

Let X be a Banach space. Suppose that $A : X \rightarrow X$ is a Lipschitz accretive operator. The objective of this note is to discuss simultaneously the existence and uniqueness of solution of the equation $x + Ax = f$ for any given $f \in X$, and its convergence, estimate of convergent rate, and stability of Krasnosel'skiĭ-Mann-Opial type iterative solution $\{x_n\} \subseteq X$. If an iterative parameter is selected suitably then the iterative procedure converges strongly to a unique solution of the equation and the iterative process is stable in arbitrary Banach space without any convexity or reflexivity. In particular, if A is nonexpansive then an estimate of the convergence rate can be written as $\|x_{n+1} - q\| \leq (\frac{17}{18})^{n+1} \|x_0 - q\|$ where $q \in X$ is a solution of $x + AX = f$.

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1 Introduction and preliminaries

Throughout this paper, X is assumed to be a real Banach space.

An operator A with domain $D(A) \subset X$ and range $R(A) \subset X$ is said to be accretive if and only if we have the inequality

$$\|x - y\| \leq \|x - y + r(Ax - Ay)\| \tag{1.1}$$

for all $x, y \in D(A)$ and $r > 0$. An accretive operator A is said to be m-accretive if $R(I + \lambda A) = X$ for all (or, equivalently, for some) $\lambda > 0$, where I stands for the identity operator on X.

The notion of accretive operators was introduced and studied independently by Browder [1] and Kato [2]. The interest and importance of accretive operators stems mainly from the fact that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{dx(t)}{dt} + Ax(t) = 0, \quad x(0) = x_0. \tag{1.2}$$

Here A is an accretive operator in a Hilbert space or an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equation. One of the earliest problems in the theory of accretive operators

was to solve (1.2). A fundamental result in the theory, due to Browder [1], states that the initial value problem (1.2) is solvable if A is locally Lipschitzian and accretive on H (a Hilbert space). In particular, if A is m-accretive then the equation

$$x + Ax = f \quad x \in X \tag{1.3}$$

has a solution for any given $f \in X$. Martin [3] extended the results, that is, he proved that (1.2) is solvable if A is continuous and accretive on X. Furthermore, if A is continuous and accretive then A must be m-accretive. It implies that (1.3) has a solution $q \in X$ for any given $f \in X$. If q^* also is a solution of (1.3), then we have the inequality

$$\left\| q - q^* \right\| \leq \left\| q - q^* + r\left(Aq - Aq^*\right) \right\| = |1 - r| \left\| q - q^* \right\| \tag{1.4}$$

for all $r > 0$. To pick $r = 1$, we have $q = q^*$. It follows that for the equation $x + Ax = f$ there exists a unique solution $q \in X$.

It is well known that the approximative solution of operator equation is closely related with the iterative sequence convergence of a fixed point of the mapping. Therefore a brief history on iterative methods is reviewed in the following.

In 1922, using the Picard iterative method, Banach's fixed point theorem was obtained where $S : D(S) \subseteq X \to X$ is a contractive type operator. For a given nonexpansive mapping S of a closed convex set C of a Banach space X into itself and $\lambda \in (0, 1)$, the operator $S_\lambda = \lambda I + (1 - \lambda)S$ also is nonexpansive. Moreover, as has been pointed out by Krasnosel'skiĭ [4] for $\lambda = 1/2$ and by Schaefer [5] for an arbitrary λ, if X is uniformly convex and S has at least one fixed point in C, then the operator S_λ is asymptotically regular. For any $x_0 \in C$ and any $\lambda \in (0, 1)$, Opial proved that the iterative sequence $\{S_\lambda^{n+1} x_0\}$ (or $\{\lambda x_n + (1 - \lambda)Sx_n\}$) is weakly convergent to a fixed point of S under some assumptions (see, *e.g.*, Theorem 3 of [6]).

On the other hand, the so-called general Mann [7] iterates $\{x_n\}_{n \geq 0}$ of the mapping S are defined by

$$x_0 \in X, \qquad x_{n+1} = S(v_n), \quad v_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n \geq 0. \tag{1.5}$$

Here $(a_{nj}) = A$ is an infinite real matrix under some conditions. Most of the literature deals with the specialized Mann iteration method defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where C is a closed convex subset of X and $\{\alpha_n\}$ is a real sequence satisfying the following conditions:

$$\text{(i) } \alpha_0 = 1, \qquad \text{(ii) } 0 < \alpha_n < 1, \quad n \geq 1, \qquad \text{(iii) } \sum_{n=0}^{\infty} \alpha_n = \infty. \tag{1.7}$$

In this note, we need an iterative method, the so-called Krasnosel'skiĭ-Mann-Opial type iterative procedure which is defined as follows.

Definition 1.1 For any given $x_0 \in X$, the iterative sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha)x_n + \alpha Sx_n, \quad n \geq 0, \tag{1.8}$$

where the iterative parameter $\alpha \in (0, 1)$.

In 1986, Chidume [8, 9] proved that the Mann iterative sequence converges strongly to a solution of the equation $x + Ax = f$ where A is a Lipschitz accretive operator defined on the Hilbert space H or the space L_p. The result was generalized by Ding and Deng [10], that is, they proved that the Ishikawa iterative sequence converges strongly to a solution of the equation $x + Ax = f$ where A is a Lipschitz accretive operator defined on p-uniformly smooth Banach space X. Zhu [11] proved that the Mann iterative sequence converges strongly to the unique solution of the equation $x + Ax = f$ under slightly different conditions where $A : D(A) \subseteq X \to X$ is a Lipschitz m-accretive operator and $D(A)$ is an open subset of a uniformly smooth Banach space X. Recently, Chidume and Osilike further extended the above results to the Mann iterative sequence (see, *e.g.*, Theorem 5 of [12]), where A is Lipschitz m-accretive and $D(A)$ is a closed subset of a real Banach space X which is both uniformly convex and p-uniformly smooth.

The objective of this note is to discuss simultaneously the existence and uniqueness of a solution of the equation $x + Ax = f$ for any given $f \in X$, and its convergence, estimate of convergent rate, and stability of a Krasnosel'skiĭ-Mann-Opial type iterative solution. If the iterative parameter is selected suitably then the iterative procedure converges strongly to a unique solution of the equation and the iterative process is stable in arbitrary Banach space without any convexity or reflexivity.

Remark 1.2 A class of operators closely related to the class of accretive operators is the class of dissipative operators. An operator $S : D(A) \subseteq X \to X$ is said to be dissipative if and only if $(-S)$ is accretive and S is called m-dissipative if $(-S)$ is m-accretive (see, *e.g.*, Barbu [13]). Some related results of the equation $x - \lambda Tx = f$ are also proved in [9, 10, 14] where $\lambda > 0$ and T is a Lipschitzian dissipative operator on X.

To set the framework we recall some basic notations as follows.

Definition 1.3 [15] Let $S : X \to X$ be an operator. Suppose that the iterative sequence $\{x_n\}_{n=0}^{\infty} \subset X$ defined by $x_{n+1} = f(S, x_n)$, and $\{x_n\}$ converges strongly to $q \in F(S) = \{x \in X : Sx = x\} \neq \emptyset$. Furthermore, suppose that $\{\epsilon_n\}$ is a sequence in $(0, \infty)$ given by $\epsilon_n = \|z_{n+1} - f(S, z_n)\|$ where $\{z_n\}_{n=0}^{\infty}$ is arbitrary sequence in X. Then the iteration procedure $\{x_n\}$ is said to be stable with respect to S (or, simply, S-stable) if $\lim_{n \to \infty} \epsilon_n = 0$ implies that $\lim_{n \to \infty} z_n = q$.

Lemma 1.4 [16] *Let $\{a_n\}$ be a nonnegative real sequence and $\{\delta_n\}$ be a real sequence in $[0, 1]$ such $\sum_{n=0}^{\infty} \delta_n = \infty$. If there exists a positive integer n_1 such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n \epsilon_n \quad \forall n \geq n_1, \tag{1.9}$$

where $\epsilon_n \geq 0$ for all $n \geq 0$ and $\epsilon_n \to 0$ (as $n \to \infty$), then we have $\lim_{n \to \infty} a_n = 0$.

2 Iterative solution of *m*-accretive operator equation and its stability

We now prove the following propositions.

Proposition 2.1 *If $A : X \to X$ is a Lipschitz accretive operator with Lipschitz constant L ($1 \le L$), then the sequence $\{x_n\}$ is defined by (1.8) is an iterative solution of (1.3), where the iterative parameter $\alpha_0 \in (0, 1/(L+1)^2)$.*

Proof Since A is a continuous accretive operator, the equation $x + Ax = f$ has a solution $q \in X$. Setting $S : X \to X$ by $Sx = f - Ax$, we have two facts: first, q is a solution of $x + Ax = f$ if and only if q is a fixed point of S. It implies that S is a dissipative operator; next, it is easy to see that L also is a Lipschitz constant of S. In fact, we have

$$\|Sx - Sy\| = \|Ay - Ax\| \le L\|x - y\| \tag{2.1}$$

for all $x, y \in X$. Also, (1.8) can be written as

$$\begin{cases} x_0 \in X, \\ x_{n+1} = (1 - \alpha)x_n + \alpha Sx_n \end{cases} \quad (n \ge 0). \tag{2.2}$$

From (1.1), we obtain

$$\|x - y\| \le \|x - y - r(Sx - Sy)\| \tag{2.3}$$

for all $x, y \in X$ and $r > 0$. For any given $x_0 \in X$, utilizing (2.2) and (2.3), we have

$$\begin{aligned}
(1 + \alpha) & \left[(x_{n+1} - q) - \alpha(1 + \alpha)^{-1}(Sx_{n+1} - q) \right] \\
&= \left(1 - \alpha^2 \right)(x_n - q) + \alpha(1 + \alpha)(Sx_n - q) - \alpha(Sx_{n+1} - q) \\
&= (x_n - q) + \alpha^2(Sx_n - x_n) + \alpha(Sx_n - Sx_{n+1}),
\end{aligned} \tag{2.4}$$

$$\|Sx_n - x_n\| \le \|x_n - q\| + \|Sx_n - q\| \le (L + 1)\|x_n - q\| \tag{2.5}$$

and

$$\|Sx_n - Sx_{n+1}\| \le \alpha L\|Sx_n - x_n\| \le \alpha L(L + 1)\|x_n - q\| \tag{2.6}$$

for any $n \ge 0$. Putting $r = \alpha/(1 + \alpha)$ in (2.3), it follows from (2.2), (2.4), (2.5), and (2.6) that

$$\begin{aligned}
(1 + \alpha)\|x_{n+1} - q\| &\le (1 + \alpha)\left\| x_{n+1} - q - \alpha(1 + \alpha)^{-1}(Sx_{n+1} - q) \right\| \\
&= \left\| x_n - q + \alpha^2(Sx_n - x_n) + \alpha(Sx_n - Sx_{n+1}) \right\| \\
&\le \|x_n - q\| + \alpha^2\|Sx_n - x_n\| + \alpha\|Sx_n - Sx_{n+1}\| \\
&\le \left(1 + \alpha^2 \right)\|x_n - q\| + \alpha^2 L(L + 2)\|x_n - q\| \\
&= \left[1 + (L + 1)^2\alpha^2 \right]\|x_n - q\|
\end{aligned} \tag{2.7}$$

for all $n \ge 0$. Thus (2.7) can be written as

$$\|x_{n+1} - q\| \le \frac{1 + (L + 1)^2\alpha^2}{1 + \alpha}\|x_n - q\|. \tag{2.8}$$

Putting

$$Q(\alpha) = \frac{1 + (L+1)^2 \alpha^2}{1 + \alpha} < 1, \tag{2.9}$$

we obtain $\alpha < \frac{1}{(L+1)^2}$. Thus, (2.9) can be written as

$$\|x_{n+1} - q\| \leq Q(\alpha)\|x_n - q\| \leq \left[Q(\alpha)\right]^{n+1}\|x_0 - q\|. \tag{2.10}$$

It implies that $\{x_n\}$ converges strong to q if $\alpha < \frac{1}{(L+1)^2}$, *i.e.* $\{x_n\}$ is an iterative solution of (1.3) and

$$\lim_{n \to +\infty} \|x_n - q\| = 0. \tag{2.11}$$

This completes the proof. □

Remark 2.2 In Proposition 2.1, there exists an $\alpha^* \in (0, 1/(L+1)^2)$ such that the convergent rate is fastest in (2.10). In fact, it follows from

$$Q(\alpha) = \frac{1 + (L+1)^2 \alpha^2}{1 + \alpha} \tag{2.12}$$

that

$$\alpha^* = \frac{\sqrt{L^2 + 2L + 2} - L - 1}{L + 1}, \tag{2.13}$$

and

$$Q(\alpha^*) = 2(L+1)\left(\sqrt{L^2 + 2L + 2} - L - 1\right). \tag{2.14}$$

For the operation in the computer, we need to select an $\alpha_0 \in (0, 1/(L+1)^2)$. Since

$$\alpha^* = \frac{1}{(L+1)(\sqrt{(L+1)^2 + 1} + L + 1)} < \frac{1}{2(L+1)^2} < \frac{1}{(L+1)^2}, \tag{2.15}$$

we take $\alpha_0 = \frac{1}{2(L+1)^2}$ so that the estimation of convergent rate of $\{x_n\}$ can be written as

$$\|x_{n+1} - q\| \leq \left[Q(\alpha_0)\right]^{n+1}\|x_0 - q\| = \left(\frac{4L^2 + 8L + 5}{4L^2 + 8L + 6}\right)^{n+1}\|x_0 - q\| \tag{2.16}$$

for any $n \geq 0$.

Example 2.3 If operator A is nonexpansive then $L = 1$. Thus, (2.16) can be written as

$$\|x_{n+1} - q\| \leq \left(\frac{17}{18}\right)^{n+1}\|x_0 - q\| \tag{2.17}$$

for all $n \geq 0$.

In the sequel, we need to discuss the stability of the iterative solution of (1.3).

Proposition 2.4 *If the conditions of Proposition* 2.1 *are satisfied then the Krasnosel'skiĭ-Mann-Opial type iterative solution* $\{x_n\}$ *of* (1.3) *is S-stable.*

Proof Let $\{z_n\}$ is any sequence in X. Setting

$$p_n = (1-\alpha)z_n + \alpha Sz_n \quad (n \geq 0), \tag{2.18}$$

we have

$$\begin{aligned}
\|p_n - q\| &= \left\|p_n - q - \alpha(1+\alpha)^{-1}(Sp_n - Sq)\right\| \\
&= (1+\alpha)^{-1}\left\|(1+\alpha)(p_n - q) - \alpha(Sp_n - q)\right\| \\
&\leq (1+\alpha)^{-1}\left(\|z_n - q\| + \alpha^2\|Sz_n - z_n\| + \alpha\|Sz_n - Sp_n\|\right) \\
&\leq (1+\alpha)^{-1}\left[1 + (L+1)\alpha^2 + L(L+1)\alpha^2\right]\|z_n - q\| \\
&= (1+\alpha)^{-1}\left[1 + (L+1)^2\alpha^2\right]\|z_n - q\|
\end{aligned} \tag{2.19}$$

for all $n \geq 0$. To take

$$\alpha = \frac{1}{2(L+1)^2}, \tag{2.20}$$

we have

$$\|z_{n+1} - q\| \leq \|z_{n+1} - p_n\| + \|p_n - q\| = \epsilon_n + \frac{4L^2 + 8L + 5}{4L^2 + 8L + 6}\|z_n - q\| \tag{2.21}$$

and

$$\epsilon_n = \|z_{n+1} - p_n\| \leq \|z_{n+1} - q\| + \|p_n - q\| \leq \|z_{n+1} - q\| + \frac{4L^2 + 8L + 5}{4L^2 + 8L + 6}\|z_n - q\|. \tag{2.22}$$

It follows from the Lemma 1.4, (2.21), and (2.22) that

$$\lim_{n \to \infty} z_n = q \quad \Leftrightarrow \quad \lim_{n \to \infty} \epsilon_n = 0. \tag{2.23}$$

That is, the Krasnosel'skiĭ-Mann-Opial type iterative solution is S-stable.

This completes the proof. □

Remark 2.5 The corresponding results (see, for example, Theorem 1 of [17], Theorem 1 of [18] and Theorem 3.5 of [19]) are improved in the following senses:

(1) For the Krasnosel'skiĭ-Mann-Opial type iterative solution of $x + Ax = f$, we study simultaneously the existence, uniqueness, convergence, convergence rate estimate, and the stability.

(2) The operator may not be strongly accretive, ϕ-strongly accretive or uniformly accretive. The cases that A is uniformly accretive, ϕ-strongly accretive or strongly accretive, as regards (1.2) and (1.3), have been studied in [17, 19–21] and [22, 23];

(3) By the way, the iterative parameter α does not depend on any geometric structure of space X and on any properties of the operators, but the selection of the parameter is to deal with the convergent rate of the iterative sequence. In this paper, a prototype of the iteration parameter is

$$\alpha = \frac{1}{2(L+1)^2}. \tag{2.24}$$

It is convenient in simplifying the computation and obtaining the convergent acceleration.

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