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Contractive multivalued maps in terms of Q-functions on complete quasimetric spaces

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Abstract

In this paper we prove the existence of a fixed point for multivalued maps satisfying a contraction condition in terms of Q-functions, and via Bianchini-Grandolfi gauge functions, for complete T_0-quasipseudometric spaces. Our results extend, improve, and generalize some recent results in the literature. We present some examples to validate and illustrate our results.

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1 Introduction and preliminaries

The notion of metric space, introduced by Fréchet [1], is one of the cornerstones of both applied and pure mathematics. The metric space is indispensable in many branches of mathematics. For example, in these days, one of the core topics in group theory is to construct a metric on a given group under the certain conditions. Due to its wide application areas in all quantitative sciences, this notion has been generalized and extended in various way, such as quasimetrics, symmetrics, b-metrics, G-metrics, fuzzy metrics, *etc.* Among all, we attract attention to the notion of Q-function, introduced by Al-Homidan *et al.* [2] in the framework of quasimetric space as an extension of the concept of w-distance defined by Kada *et al.* [3]. In fact, the authors of [2] proved, among other results, a quasimetric version of the celebrated Nadler fixed point theorem [4]. Recently, Marín *et al.* [5] generalized some results of [2] by using Bianchini-Grandolfi gauge functions. Almost simultaneously, Latif and Al-Mezel [6] obtained a quasimetric generalization of a well-known fixed point theorem of Mizoguchi and Takahashi [7, Theorem 5] (see also [8, 9]) for multivalued maps on complete metric spaces.

In this paper we prove the existence of fixed point for a lower semicontinuous multivalued map satisfying certain contraction condition in terms of Q-functions via Bianchini-Grandolfi gauge functions on a complete T_0-quasipseudometric space. We also prove a weaker version of that theorem by removing the lower semicontinuity assumption. We state some examples to show the validity of the conditions and to indicate our generalizations have worth, and finally give applications to the case of contractive multivalued maps on complete partial metric spaces. Our results improve, generalize, and extend several known results in this direction.

Let $\mathbb{N}$ denote the set of positive integer numbers, while ω denotes the set of nonnegative integer numbers.

For the sake of completeness of the paper, we recall several pertinent notions and fundamental results.

Let X be nonempty set and $d : X \times X \to [0,\infty)$ be a function such that

(qpm$_1$) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and
(qpm$_2$) $d(x,z) \leq d(x,y) + d(y,z)$,

for all $x,y,z \in X$. Then d is called a T_0-quasipseudometric on a set X. The pair (X,d) is said to be a T_0-quasipseudometric space.

If one replaces the condition (qpm$_1$) with the stronger condition

(qpm$_1$)* $d(x,y) = 0 \Leftrightarrow x = y$,

then d is called a quasimetric on X. In this case, the pair (X,d) is said to be a quasimetric space.

In the sequel we will use the abbreviation T_0-qpm (respectively, T_0-qpm space) instead of T_0-quasipseudometric (respectively, T_0-quasipseudometric space).

Given a T_0-qpm d on a set X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ is also a T_0-qpm, called the conjugate of d. It is clear that the function d^s defined by $d^s(x,y) = \max\{d^{-1}(x,y), d(x,y)\}$ is a metric on X. (Note that if d is a metric on X, then $d = d^s$.)

Consequently, every T_0-qpm d on X induces three topologies defined as follows.

(τ_1) The first topology, τ_d which has as a base the family of open balls $\{B_d(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.
(τ_2) The second topology, $\tau_{d^{-1}}$ which has as a base the family of open balls $\{B_{d^{-1}}(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_{d^{-1}}(x,\varepsilon) = \{y \in X : d^{-1}(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.
(τ_3) The last topology induced by the metric d^s and denoted by τ_{d^s}.

Notice that both τ_d and $\tau_{d^{-1}}$ are T_0 topologies on X. Furthermore, if d is a quasimetric on X, then d^{-1} is also a quasimetric on X and hence, both τ_d and $\tau_{d^{-1}}$ are T_1 topologies on X.

It immediately follows that a sequence $(x_n)_{n\in\mathbb{N}}$ in a T_0-qpm space (X,d) is τ_d-convergent to $x \in X$ if and only if $\lim_{n\to\infty} d(x,x_n) = 0$. Analogously, a sequence $(x_n)_{n\in\mathbb{N}}$ in a T_0-qpm space (X,d) is $\tau_{d^{-1}}$-convergent to $x \in X$ if and only if $\lim_{n\to\infty} d(x_n,x_n) = 0$.

In the literature, the notion of completeness for quasimetric spaces can be varied; see *e.g.* [5, 10, 11]. In the context of our paper we shall use the following very general notion: A T_0-qpm space (X,d) is said to be complete if every Cauchy sequence in the metric space (X,d^s) is $\tau_{d^{-1}}$-convergent.

Now, we recall the definition of Q-function, as introduced by Al-Homidan-Ansari-Yao [2].

Definition 1 Let (X,d) be a T_0-qpm space and $q : X \times X \to [0,\infty)$ be a function which satisfies

(Q$_1$) $q(x,z) \leq q(x,y) + q(y,z)$, for all $x,y,z \in X$,
(Q$_2$) if $x \in X$, $M > 0$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\tau_{d^{-1}}$-converges to a point $y \in X$, and satisfies $q(x,y_n) \leq M$, for all $n \in \mathbb{N}$, then $q(x,y) \leq M$,
(Q$_3$) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

Then q is called a Q-function on (X,d).

If q satisfies conditions (Q_1) and (Q_3), and

(Q_2') for each $x, y \in X$ the function $q(x,\cdot) : X \to [0,\infty)$ is $\tau_{d^{-1}}$-lower semicontinuous on (X,d),

then q is called a w-distance on (X,d). Note that every w-distance is a Q-function.

Remark 1 It is evident that d is a w-distance on (X,d) if d is a metric on X. Note also that if (X,d) is a T_0-qpm space then d is not necessarily a Q-function on (X,d) [2, Example 2.3] (see also [5, Proposition 2.3]).

We conclude this section with the following simple fact which will be useful in the rest of the paper.

Lemma 1 [5] *Let q be a Q-function on a T_0-qpm space (X,d), let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon) > 0$ for which condition* (Q_3) *holds. If $q(x,y) \le \delta$ and $q(x,z) \le \delta$ then $d^s(y,z) \le \varepsilon$.*

2 Main results

Let (X,d) be a T_0-qpm space. The collection of all nonempty subsets (respectively, τ_{d^s}-closed subsets) of X will be denoted by 2^X (respectively, $\mathrm{Cl}_{d^s}(X)$).

Let Ψ be the family of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(φ_1) φ is nondecreasing;
(φ_2) $\sum_{n=1}^{+\infty} \varphi^n(t) < \infty$ for all $t > 0$, where φ^n is the nth iterate of φ.

These functions are known in the literature as Bianchini-Grandolfi gauge functions in some sources (see *e.g.* [12–14]) and as (c)-comparison functions in some other sources (see *e.g.* [15]). It is easily proved that if $\varphi \in \Psi$, then $\varphi(t) < t$ for any $t > 0$ (see *e.g.* [15]).

The following lemma will be crucial to prove our first theorem.

Lemma 2 *Let (X,d) be a T_0-qpm space, q a Q-function on (X,d), $\varphi : [0,\infty) \to [0,\infty)$ a Bianchini-Grandolfi gauge function and $T : X \to 2^X$ a multivalued map such that for each $x, y \in X$ and $u \in Tx$; there is $v \in Ty$ satisfying*

$$q(u,v) \le \varphi\bigl(\max\bigl\{q(x,y), q(x,u), q(y,v)\bigr\}\bigr). \tag{1}$$

Then, for each $x_0 \in X$ there is a sequence $(x_n)_{n\in\omega}$ satisfying the following three conditions:

(a) *$x_{n+1} \in Tx_n$ for all $n \in \omega$.*
(b) *For each $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that $q(x_n, x_m) < \delta$ whenever $m > n \ge n_\delta$.*
(c) *$(x_n)_{n\in\omega}$ is a Cauchy sequence in the metric space (X, d^s).*

Proof Fix $x_0 \in X$. Let $x_1 \in Tx_0$. By hypothesis, there exists $x_2 \in Tx_1$ such that

$$q(x_1,x_2) \le \varphi\bigl(\max\bigl\{q(x_0,x_1), q(x_1,x_2)\bigr\}\bigr).$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$q(x_2,x_3) \le \varphi\bigl(\max\bigl\{q(x_1,x_2), q(x_2,x_3)\bigr\}\bigr).$$

Following this process we construct a sequence $(x_n)_{n\in\omega}$ in X such that $x_{n+1} \in Tx_n$ and

$$q(x_{n+1}, x_{n+2}) \leq \varphi\bigl(\max\bigl\{q(x_n, x_{n+1}), q(x_{n+1}, x_{n+2})\bigr\}\bigr), \tag{2}$$

for all $n \in \omega$.

Now we distinguish two cases.

Case 1. There exists $k \in \omega$ such that $q(x_k, x_{k+1}) = 0$. Then, by condition (2) and the fact that $\varphi(t) < t$ for all $t > 0$, we deduce that $q(x_{k+1}, x_{k+2}) = 0$. Repeating this argument, we obtain $q(x_{k+j}, x_{k+j+1}) = 0$ for all $j \in \omega$, so, by condition (Q_1), $q(x_n, x_m) = 0$ whenever $m > n \geq k$. It follows from Lemma 1 that for each $\varepsilon > 0$, $d^s(x_n, x_m) \leq \varepsilon$ whenever $n, m > k$, and thus $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d^s). Thus we have shown that conditions (a), (b), and (c) are satisfied.

Case 2. $q(x_n, x_{n+1}) > 0$ for all $n \in \omega$. Then, by condition (2) and the fact that $\varphi(t) < t$ for all $t > 0$, we deduce that $q(x_n, x_{n+1}) > q(x_{n+1}, x_{n+2})$ for all $n \in \omega$, so

$$q(x_{n+1}, x_{n+2}) \leq \varphi\bigl(q(x_n, x_{n+1})\bigr) < q(x_n, q_{n+1}),$$

for all $n \in \omega$. Therefore

$$q(x_n, x_{n+1}) \leq \varphi^n\bigl(q(x_0, x_1)\bigr), \tag{3}$$

for all $n \in \omega$. Now choose an arbitrary $\varepsilon > 0$. Let $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ for which condition (Q_3) holds. We shall show that conditions (b) and (c) hold. Indeed, since $q(x_0, x_1) > 0$, $\sum_{n=0}^{\infty} \varphi^n(q(x_0, x_1)) < \infty$, so there is $n_\delta \in \omega$ such that

$$\sum_{n=n_\delta}^{\infty} \varphi^n\bigl(q(x_0, x_1)\bigr) < \delta. \tag{4}$$

Then, for $m > n \geq n_\delta$, we obtain

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m_{-1}}, x_m) \\ &\leq \varphi^n\bigl(q(x_0, x_1)\bigr) + \varphi^{n+1}\bigl(q(x_0, x_1)\bigr) + \cdots + \varphi^{m-1}\bigl(q(x_0, x_1)\bigr) \\ &\leq \sum_{j=n_\delta}^{\infty} \varphi^j\bigl(q(x_0, x_1)\bigr) < \delta. \end{aligned} \tag{5}$$

In particular, $q(x_{n_\delta}, x_n) \leq \delta$ and $q(x_{n_\delta}, x_m) \leq \delta$ whenever $n, m > n_\delta$. Thus, by Lemma 1, $d^s(x_n, x_m) \leq \varepsilon$ whenever $n, m > n_\delta$. Hence $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d^s). This concludes the proof. □

We also need the following notion.

Definition 2 Let q be a Q-function on a T_0-qpm space (X, d). We say that a multivalued map $T : X \to 2^X$ is q-lower semicontinuous (q-l.s.c. in short) if the function $x \mapsto q(x, Tx)$ is lower semicontinuous on the metric space (X, d^s), where $q(x, Tx) = \inf\{q(x, y) : y \in Tx\}$.

Remark 2 An antecedent of the above concept can be found in Theorem 3.3 of the paper by Daffer and Kaneko [8], where they proved a generalization of Nadler's fixed point theorem for a multivalued map T on a complete metric space (X,d) by assuming that the function $x \mapsto d(x, Tx)$ is lower semicontinuous on (X,d).

Before establishing our first fixed point result we recall that a point $z \in X$ is said to be a fixed point of a multivalued map $T: X \to 2^X$ if $z \in Tz$.

Theorem 1 *Let (X,d) be a complete T_0-qpm space, q a Q-function on (X,d), $\varphi: [0,\infty) \to [0,\infty)$ a Bianchini-Grandolfi gauge function and $T: X \to \mathrm{Cl}_{d^s}(X)$ a q-l.s.c. multivalued map such that for each $x,y \in X$ and $u \in Tx$, there is $v \in Ty$ satisfying*

$$q(u,v) \leq \varphi\bigl(\max\bigl\{q(x,y), q(x,u), q(y,v)\bigr\}\bigr). \tag{6}$$

Then T has a fixed point $z \in X$ such that $q(z,z) = 0$.

Proof Fix $x_0 \in X$. Then there is a sequence $(x_n)_{n\in\omega}$ satisfying the three conditions (a), (b) and (c) of Lemma 2. Since (X,d) is complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$.

We shall prove that $z \in Tz$. To this end, first we prove that $\lim_{n\to\infty} q(x_n,z) = 0$. Indeed, given $\varepsilon > 0$ take $\delta = \delta(\varepsilon) < \varepsilon$ for which condition (Q_3) holds. Fix $n \geq n_\delta$. By condition (b), we have $q(x_n,x_m) \leq \delta$ whenever $m > n$, so from condition (Q_2) we deduce that $q(x_n,z) \leq \delta < \varepsilon$ whenever $n \geq n_\delta$.

Next we show that $\lim_{n\to\infty} d^s(x_n,z) = 0$. Indeed, given $\varepsilon > 0$ take $\delta = \delta(\varepsilon) < \varepsilon$ for which condition (Q_3) holds. Since $q(x_{n_\delta},z) \leq \delta$ and $q(x_{n_\delta},x_n) \leq \delta$ whenever $n > n_\delta$, it follows from Lemma 1 that $d^s(z,x_n) \leq \varepsilon$ whenever $n > n_\delta$.

Now we prove that there is a sequence $(z_k)_{k\in\mathbb{N}}$ in Tz such that $\lim_{k\to\infty} q(z,z_k) = 0$. Indeed, since the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies (b) and, by assumption, T is q-l.s.c., we deduce that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$q(x_{n_k},x_{n_k+1}) < \frac{1}{k} \quad \text{and} \quad q(z, Tz) < q(x_{n_k}, Tx_{n_k}) + \frac{1}{k},$$

for all $k \in \mathbb{N}$. Therefore, there exists a sequence $(z_k)_{k\in\mathbb{N}}$ in Tz satisfying

$$q(z,z_k) < q(z, Tz) + \frac{1}{k} < q(x_{n_k}, Tx_{n_k}) + \frac{2}{k} \leq q(x_{n_k},x_{n_k+1}) + \frac{2}{k},$$

for all $k \in \mathbb{N}$. Hence

$$\lim_{k\to\infty} q(z,z_k) = 0. \tag{7}$$

Then, by (Q_1) and the fact that $\lim_{n\to\infty} q(x_{n_k},z) = 0$, we deduce that $\lim_{k\to\infty} q(x_{n_k},z_k) = 0$. So, by (Q_3) and Lemma 1, we obtain

$$\lim_{n\to\infty} d^s(z,z_k) = 0. \tag{8}$$

Consequently $z \in \mathrm{Cl}_{d^s} Tz = Tz$. Finally, $q(z,z) = 0$ by (7), (8), and condition (Q_2). □

Next we give an example which shows that q-lower semicontinuity of T cannot be omitted in Theorem 1 not even when (X,d) is a complete metric space.

Example 1 Let $X = \{0\} \cup \mathbb{N} \cup A$, where $A = \{1/n : n \in \mathbb{N}\backslash\{1\}\}$, and let d be the restriction to X of the usual metric on the set of real numbers. It is clear that (X, d) is a complete metric space.

Now let $q : X \times X \to [0, \infty)$ be given by

$q(x, x) = 0$ for all $x \in X$,

$q(0, x) = 2$ for all $x \in \mathbb{N} \cup A$,

$q(x, y) = q(y, x) = 1$ for all $x \in \mathbb{N}$, $y \in A$,

$q(x, y) = 1$ for all $x, y \in \mathbb{N}$,

$q(x, 0) = 1$ for all $x \in \mathbb{N}$,

$q(x, y) = |x - y|$ for all $x, y \in A$, and

$q(x, 0) = x$ for all $x \in A$.

It is easy to check that q is a Q-function (actually it is a w-distance) on (X, d).

Define $T : X \to \mathrm{Cl}_d(X)$ as

$T0 = \mathbb{N}$,

$Tx = 1/2x$ for all $x \in \mathbb{N}$, and

$Tx = x/2$ for all $x \in A$.

Since $q(0, T0) = 2$ and $q(x, Tx) = x/2$ for all $x \in A$, we deduce that T is not q-l.s.c. Moreover, it is obvious that T has no fixed point. However, we shall show that the contraction condition (6) is satisfied for the Bianchini-Grandolfi gauge function φ defined as $\varphi(t) = t/2$ for all $t \geq 0$.

To this end, we first note that for $x = 0$, $y \in \mathbb{N} \cup A$, and $u \in Tx$, we have $u \in \mathbb{N}$, $Ty = \{v\}$ with $v \in A$, and hence

$$q(u, v) = 1 = \varphi(2) = \varphi\big(q(x, y)\big).$$

Similarly, if $x \in \mathbb{N}$, $y = 0$ and $u \in Tx$, we take $v = x \in Ty$, and thus

$$q(u, v) = 1 = \varphi(2) = \varphi\big(q(x, y)\big).$$

If $x \in A$, $y = 0$, and $u \in Tx$, we have $u = x/2$ and taking $v = 1 \in Ty$, we deduce

$$q(u, v) = 1 = \varphi(2) = \varphi\big(q(y, v)\big).$$

Now, if $x, y \in A$ and $u \in Tx$, we have $u = x/2$ and $Ty = \{v\}$ where $v = y/2$, so that

$$q(u, v) = \frac{1}{2}|x - y| = \frac{1}{2}q(x, y) = \varphi\big(q(x, y)\big).$$

Similarly, if $x, y \in \mathbb{N}$, with $x \neq y$, and $u \in Tx$, we have $u = 1/2x \in A$ and $Ty = \{v\}$ where $v = 1/2y$, so that

$$q(u, v) = \frac{1}{2}\left|\frac{1}{x} - \frac{1}{y}\right| < \frac{1}{2} = \varphi(1) = \varphi\big(q(x, y)\big).$$

Finally, for $x \in \mathbb{N}$, $y \in A$, and $u \in Tx$, we have $u \in A$ and $Ty = \{v\}$, with $v \in A$, so that

$$q(u, v) = |u - v| < \frac{1}{2} = \varphi(1) = \varphi\big(q(x, y)\big).$$

The case that $x \in A$ and $y \in \mathbb{N}$ is similar, and hence it is omitted.

Our next fixed point result shows that q-lower semicontinuity of T can be removed if the contraction condition (6) is replaced with $q(u,v) \leq \varphi(\max\{q(x,y), q(x,u)\})$.

Theorem 2 *Let (X,d) be a complete T_0-qpm space, q a Q-function on (X,d), $\varphi : [0,\infty) \to [0,\infty)$ a Bianchini-Grandolfi gauge function and $T : X \to \mathrm{Cl}_{d^s}(X)$ a multivalued map such that for each $x, y \in X$ and $u \in Tx$, there is $v \in Ty$ satisfying*

$$q(u,v) \leq \varphi\bigl(\max\bigl\{q(x,y), q(x,u)\bigr\}\bigr). \tag{9}$$

Then T has a fixed point.

Proof Fix $x_0 \in X$. Then there is a sequence $(x_n)_{n\in\omega}$ satisfying the three conditions (a), (b), and (c) of Lemma 2. Since (X,d) is complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$.

Now, as in the proof of Theorem 1, we obtain $\lim_{n\to\infty} q(x_n, z) = 0$.

For each $n \in \omega$, take $z_n \in Tz$ such that

$$q(x_n, z_n) \leq \varphi\bigl(\max\bigl\{q(x_{n-1}, z), q(x_{n-1}, x_n)\bigr\}\bigr). \tag{10}$$

We show that $\lim_{n\to\infty} q(x_n, z_n) = 0$. Indeed, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_{n-1}, z) < \varepsilon$ and $q(x_{n-1}, x_n) < \varepsilon$ for all $n > n_0$. Take any $n > n_0$. If $q(x_{n-1}, z) = q(x_{n-1}, x_n) = 0$, then $q(x_n, z_n) = 0$. Otherwise, we have

$$0 < \max\bigl\{q(x_{n-1}, z), q(x_{n-1}, x_n)\bigr\} < \varepsilon,$$

so, by (10) and the fact that $\varphi(t) < t$ for all $t > 0$, we deduce that $q(x_n, z_n) < \varepsilon$.

Consequently

$$\lim_{n\to\infty} d^s(z, z_n) = 0,$$

by Lemma 1. We conclude that $z \in Tz$. □

The following consequences of Theorem 2, which are also illustrated by Example 4 below, improve and generalize in several directions the Banach contraction principle.

Corollary 1 *Let (X,d) be a complete T_0-qpm space, q a Q-function on (X,d), $\varphi : [0,\infty) \to [0,\infty)$ a Bianchini-Grandolfi gauge function and $T : X \to \mathrm{Cl}_{d^s}(X)$ a multivalued map such that for each $x, y \in X$ and $u \in Tx$, there is $v \in Ty$ satisfying*

$$q(u,v) \leq \varphi\bigl(q(x,y)\bigr).$$

Then T has a fixed point.

If we take $\varphi(t) = rt$ where $r \in [0,1)$ we get one of the main results in [2].

Corollary 2 *Let (X,d) be a complete T_0-qpm space, q a Q-function on (X,d), $T : X \to \mathrm{Cl}_{d^s}(X)$ a multivalued map and $r \in [0,1)$ such that for each $x, y \in X$ and $u \in Tx$, there is*

$v \in Ty$ satisfying

$$q(u,v) \le rq(x,y).$$

Then T has a fixed point.

Corollary 1 was proved in [5, Theorem 3.3]. In fact, it was showed that there is a fixed point $z \in X$ of T such that $q(z,z) = 0$. This suggests the following question that remains open: Under the conditions of Theorem 2, is there is a fixed point $z \in X$ of T such that $q(z,z) = 0$?

Remark 3 Example 1 shows that Theorem 2 is not true when the contraction condition (9) is replaced with $q(u,v) \le \varphi(\max\{q(x,y), q(y,v)\})$. Indeed, take in that example, $x \in A$, $y = 0$ and $u \in Tx$. Then we have $u = x/2$, and hence $q(u,v) = 1 > x = \max\{q(x,y), q(x,u)\}$.

Theorems 1 and 2 are independent from each other. The following two examples show this fact.

Example 2 Let $X = \omega$, *i.e.*, $X = \{0\} \cup \mathbb{N}$, and let d be the quasimetric on X defined as
$d(x,x) = 0$ for all $x \in X$,
$d(x,y) = x$ if $x > y$, and
$d(x,y) = x + y$ if $x < y$.
Clearly (X,d) is a complete quasimetric space and τ_d is the discrete topology on X, so $\tau_d = \tau_{d^s}$. Furthermore, it is almost obvious that d is a w-distance on (X,d).

Now let $T : X \to \mathrm{Cl}_{d^s}(X)$ given as
$T0 = 0$,
$T1 = \{x \in \mathbb{N} : x > 1\}$, and
$Tx = \{0\} \cup \{y \in \mathbb{N} : y > x\}$ for all $x \in \mathbb{N}\backslash\{1\}$.

Since τ_{d^s} is the discrete topology on X it immediately follows that T is d-l.s.c.

Consider the Bianchini-Grandolfi gauge function φ given by
$\varphi(t) = t/2$ if $0 \le t < 2$, and
$\varphi(t) = n$ if $t \in [n+1, n+2)$, $n \in \mathbb{N}$.

An easy computation of the different cases shows that the contraction condition (6) is satisfied. Indeed, let $x, y \in X$ and $u \in Tx$. In the cases where for $u = 0$ we can choose $v = 0 \in Ty$, the conclusion is obvious. Therefore, we briefly discuss the rest of the cases.

If $x = 0$, $y = 1$, we have $u = 0$, and taking $v = 2 \in Ty$ we deduce

$$d(u,v) = 2 = \varphi(3) = \varphi\bigl(d(y,v)\bigr).$$

If $x = 1$, $y = 0$ and $u \in Tx$, we have $v = 0$ and thus

$$d(u,v) = u = \varphi(u+1) = \varphi\bigl(d(x,u)\bigr).$$

If $x \in \mathbb{N}\backslash\{1\}$, $y = 0$ and $u \in Tx$, with $u \neq 0$, we deduce

$$d(u,v) = u = \varphi(u+1) \le \varphi\bigl(d(x,u)\bigr).$$

If $x = 1$, $y \in \mathbb{N}\backslash\{1\}$ and $u \in Tx$, take $v = 0 \in Ty$, and, as in the preceding case,

$$d(u,v) = u = \varphi(u+1) \leq \varphi\bigl(d(x,u)\bigr).$$

If $x \in \mathbb{N}\backslash\{1\}$, $y = 1$ and $u \in Tx$, take $v = 2 \in Ty$ and thus (recall that $u = 0$ or $u > x$)

$$d(u,v) = \max\{u,v\} \leq \max\{u+x-1,v\} = \varphi\bigl(\max\bigl\{d(x,u),d(y,v)\bigr\}\bigr).$$

Finally, if $x, y \in \mathbb{N}\backslash\{1\}$ and $u \in Tx$, with $u \neq 0$, take $v = 0 \in Ty$ and thus

$$d(u,v) = u < u + x - 1 = \varphi(u+x) = \varphi\bigl(d(x,u)\bigr).$$

Hence, all conditions of Theorem 1 are satisfied. However, we cannot apply Theorem 2 because for $x = 0$, $y = 1$, $u = 0$ and any $v \in Ty$, we have

$$d(u,v) = v > 1 = \max\bigl\{d(x,y),d(x,u)\bigr\} > \varphi(1) = \varphi\bigl(\max\bigl\{d(x,y),d(x,u)\bigr\}\bigr).$$

Example 3 Let $X = \{0,1\} \cup A$, where $A = \{1 - 1/n : n \in \mathbb{N}\backslash\{1\}\}$, and let d be the restriction to X of the usual metric on the set of real numbers. It is clear that (X,d) is a complete metric space.

Now let $q : X \times X \to [0,\infty)$ be given by
$q(x,x) = 0$ for all $x \in X\backslash\{1\}$,
$q(1,1) = 1$,
$q(0,x) = q(x,0) = 1/2$ for all $x \in X\backslash\{0\}$,
$q(1,x) = x$ for all $x \in A$,
$q(x,1) = 1 - x$ for all $x \in A$, and
$q(x,y) = |x - y|$ for all $x, y \in A$.

It is not difficult to check that q is a w-distance on (X,d).

Define $T : X \to \mathrm{Cl}_d(X)$ as
$T0 = 0$,
$T1 = \{0,1\}$, and
$Tx = \{0,(1+x)/2\}$ for all $x \in A$,
and let φ be the Bianchini-Grandolfi gauge function given by $\varphi(t) = t/2$ for all $t \geq 0$.

Notice that T is not q-l.s.c. because $q(1,T1) = q(1,0) = 1/2$, but for each $x \in A$,

$$q(x,Tx) = q\left(x,\frac{1+x}{2}\right) = \frac{1-x}{2} \leq \frac{1}{4}.$$

Hence, we cannot apply Theorem 1 to this example. We show that, nevertheless, the contraction condition (9) is satisfied and consequently the conditions of Theorem 2 hold.

Indeed, let $x, y \in X$ and $u \in Tx$. In the cases where for $u = 0$ we can choose $v = 0 \in Ty$, the conclusion is obvious. Therefore we discuss the rest of the cases.

If $x = y = 1$ and $u = 1$, take $v = 0$, and thus

$$q(u,v) = \frac{1}{2} = \varphi(1) = \varphi\bigl(q(x,u)\bigr).$$

If $x = 1$, $y = 0$ and $u = 1$, we have $v = 0$, and, as in the preceding case, $q(u,v) = \varphi(q(x,u))$.

If $x = 1$, $y \in A$ and $u = 1$, take $v = 0$, and, as in the preceding case, $q(u,v) = \varphi(q(x,u))$.

If $x \in A$, $y = 1$ and $u = (1+x)/2$, take $v = 1$, and thus

$$q(u,v) = 1 - \frac{1+x}{2} = \frac{1-x}{2} = \varphi(1-x) = \varphi\big(q(x,y)\big).$$

Finally, if $x, y \in A$ and $u = (1+x)/2$, take $v = (1+y)/2$, and thus

$$q(u,v) = \frac{1}{2}|x-y| = \varphi\big(|x-y|\big) = \varphi\big(q(x,y)\big).$$

We conclude this section with an example where the conditions of Theorems 1 and 2 are satisfied, but for which we cannot apply Corollary 1.

Example 4 Let $X = \{0, 1/4, 2\} \cup [5/12, 3/4]$ and let d be the T_0-qpm on X defined as

$d(x,x) = 0$ for all $x \in X$,

$d(x,0) = 0$ for all $x \in X$, and

$d(x,y) = 1$ otherwise.

It is clear that d is complete. In fact, d^s is the discrete metric on X.

Moreover, it is easy to check that the function $q : X \times X \to [0,\infty)$ defined as

$$q(x,y) = x + y,$$

for all $x, y \in X$, is a w-distance on (X,d).

Now let $T : X \to \mathrm{Cl}_{d^s}(X)$ given as

$T0 = T\frac{1}{4} = \{0\}$,

$Tx = \{0, 1/4\}$ if $x \in [5/12, 3/4]$, and

$T2 = [5/12, 3/4]$,

and let $\varphi : [0,\infty) \to [0,\infty)$ given as $\varphi(t) = t/(2+t)$ if $0 \le t < 1$, and $\varphi(t) = t/3$ if $t \ge 1$.

Clearly φ is a Bianchini-Grandolfi gauge function (note that for $0 < t < 1$ and $n \in \mathbb{N}$ we have $\varphi^n(t) < t/2^n$).

We shall show that the conditions of Theorem 2 are satisfied. Note that it suffices to check (9). To this end, and due to the facts that $T0 = T1/4 = \{0\}$, $q(0,0) = 0$, and that q is symmetric we only consider the following cases.

- Case 1. $5/12 \le x \le 3/4$, $y \in \{0, 1/4\}$ and $u \in Tx$. Take $v = 0 \in Ty$. If $u = 0$, $q(u,v) = 0$, and the inequality (1) is trivially satisfied. If $u = 1/4$ we deduce that

$$q(u,v) = \frac{1}{4} \le \frac{x+u}{2+x+u} = \varphi(x+u) = \varphi\big(q(x,u)\big).$$

- Case 2. $5/12 \le x, y \le 1$ and $u \in Tx$. Take $v = 0 \in Ty$. Then the conclusion follows exactly as in Case 1.
- Case 3. $x \in \{0, 1/4\}$, $y = 2$ and $u \in Tx$. Then $u = 0$. Taking $v = 5/12 \in Ty$, we deduce that

$$q(u,v) = \frac{5}{12} < \frac{2}{3} = \varphi(2) \le \varphi\big(q(x,y)\big).$$

- Case 4. $5/12 \le x \le 3/4$, $y = 2$ and $u \in Tx$. Taking $v = 5/12 \in Ty$, we deduce that

$$q(u,v) \le \frac{1}{4} + \frac{5}{12} = \frac{2}{3} = \varphi(2) \le \varphi\big(q(x,y)\big).$$

- Case 5. $x = 2$, $y \in \{0, 1/4\}$ and $u \in Tx$. Take $v = 0 \in Ty$. Then

$$q(u,v) = u < \frac{x+u}{3} = \varphi\bigl(q(x,u)\bigr).$$

- Case 6. $x = 2$, $5/12 \le y \le 3/4$ and $u \in Tx$. Taking $v = 0 \in Ty$, we deduce that

$$q(u,v) = u \le \frac{3}{4} \le \frac{2 + 5/12}{3} \le \frac{x+u}{3} = \varphi\bigl(q(x,u)\bigr).$$

- Case 7. $x = y = 2$ and $u \in Tx$. Taking $v = 5/12 \in Ty$, we deduce that

$$q(u,v) = u + v \le \frac{3}{4} + \frac{5}{12} < \frac{4}{3} = \varphi\bigl(q(x,y)\bigr).$$

Moreover, the conditions of Theorem 1 are also satisfied because T is trivially q-l.s.c.

Observe that, nevertheless, we cannot apply Corollary 1 to this example, because for $x = 2$, $y = 0$ and $u = 3/4$, we only have $v = 0 \in Ty$, and thus

$$q(u,v) = q\left(\frac{3}{4}, 0\right) = \frac{3}{4} > \frac{2}{3} = \varphi(2) = \varphi\bigl(q(x,y)\bigr).$$

Furthermore, it cannot be applied to the complete metric space (X, d^s) with $q = d^s$ because for $x = 0$, $y = 2$, $u = 0$ and any $v \in Ty$ we deduce that

$$\begin{aligned} d^s(u,v) &= d^s(0,v) = d(0,v) = 1 > \frac{1}{3} = \varphi(1) \\ &= \varphi\bigl(\max\bigl\{d(x,y), d(x,u), d(y,v)\bigr\}\bigr) = \varphi\bigl(\max\bigl\{d^s(x,y), d^s(x,u), d^s(y,v)\bigr\}\bigr). \end{aligned}$$

Finally, note that the preceding relations also show that condition (9) does not follow for the T_0-qpm d.

3 Application to partial metric spaces

Matthews introduced in [16] (see also [17]) the 'equivalent' notions of a weightable T_0-qpm space and of a partial metric space in order to construct a consistent topological model for certain programming languages.

Let us recall that a T_0-qpm space (X, d) is weightable if there is a function $w : X \to [0, \infty)$ such that

$$d(x,y) + w(x) = d(y,x) + w(y),$$

for all $x, y \in X$. In this case, we say that the pair (X, d) is a weightable T_0-qpm space. The function w is called a weighting function for (X, d).

Note that Matthews used the term 'weightable quasimetric spaces' for such spaces.

Now, we state the definition of partial metric space as given by Matthews [16, 17].

Definition 3 A partial metric on a set X is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$,

(P2) $p(x,x) \le p(x,y)$,

(P3) $p(x,y) = p(y,x)$,
(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then the pair (X,p) is called a partial metric space.

Observe that from (P1) and (P2) it follows that if $p(x,y) = 0$ then $x = y$.

Each partial metric p on a set X induces a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A typical example of a partial metric space is the pair (X,p) where $X = [0,\infty)$ and p is the partial metric on X given by $p(x,y) = \max\{x,y\}$ for all $x,y \in X$.

A partial metric p on a set X induces, in a natural way, three metrics on X, denoted by p^S, p_M and p_0, respectively, that are defined, for each $x,y \in X$, as follows:

$p^S(x,y) = 2p(x,y) - p(x,x) - p(y,y)$,
$p_M(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}$, and
$p_0(x,x) = 0$, and $p_0(x,y) = p(x,y)$ if $x \neq y$.

It is easy to show (see *e.g.* [18]) that $\tau_p \subseteq \tau_{p^S} = \tau_{p_M} \subseteq \tau_{p_0}$.

Matthews proved [17, Theorems 4.1 and 4.2] that the concepts of weightable T_0-qpm space and partial metric space are equivalent in the following sense.

Proposition 1 (i) *Let (X,d) be a weightable T_0-qpm space with weighting function w. Then the function $p_d : X \times X \to [0,\infty)$ defined by $p_d(x,y) = d(x,y) + w(x)$ for all $x,y \in X$ is a partial metric on X. Furthermore, $\tau_d = \tau_{p_d}$.*

(ii) *Conversely, let (X,p) be a partial metric space. Then the function $d_p : X \times X \to [0,\infty)$ defined by $d_p(x,y) = p(x,y) - p(x,x)$ for all $x,y \in X$ is a weightable T_0-qpm space on X. Furthermore, $\tau_d = \tau_{d_p}$.*

It is clear from the above proposition that for each partial metric p on X one has $p = p_{d_p}$, and that for each weightable T_0-qpm on X one has $d = d_{p_d}$.

Since Matthews proved in [17, Theorem 5.3] his well-known partial metric generalization of the Banach contraction principle several authors have investigated the problem of obtaining fixed points for a variety of contractive conditions for self-maps and multivalued maps on partial metric spaces. This research has been specially intensive in the last five years (see *e.g.* [19, 20] and the bibliographic references contained in them). In connection with our approach it is interesting to note that the partial metric p_d induced by a weightable T_0-qpm space (X,d) allows us to construct some nice Q-functions on (X,d). This is stated in the next two lemmas. The first one was proved in [5, Proposition 2.10].

Lemma 3 [5] *Let (X,d) be a weightable T_0-qpm space with weighting function w. Then the induced partial metric p_d is a Q-function on (X,d).*

A slight modification of the proof of the above lemma allows us to state the following.

Lemma 4 *Let (X,d) be a weightable T_0-qpm space with weighting function w. Then the function $q_d : X \times X \to [0,\infty)$ defined by $q_d(x,y) = p_d(x,y) + p_d(y,y)$ for all $x,y \in X$, is a Q-function on (X,d).*

Then, and as a natural consequence of Theorems 1 and 2, we obtain the following results that generalize and improve, among other results, [5, Theorem 3.3] and [17, Theorem 3.5].

Corollary 3 *Let (X,p) be a partial metric space such that the induced weightable T_0-qpm d_p is complete, let $\varphi : [0,\infty) \to [0,\infty)$ be a Bianchini-Grandolfi gauge function and let $T : X \to \mathrm{Cl}_{(d_p)^s}(X)$ be a multivalued map. If one of the following two conditions is satisfied, then T has a fixed point.*

(A) *T is p-l.s.c. and for each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v) \le \varphi\bigl(\max\bigl\{p(x,y),p(x,u),p(y,v)\bigr\}\bigr).$$

(B) *T is q_{d_p}-l.s.c. and for each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v)+p(v,v) \le \varphi\bigl(\max\bigl\{p(x,y)+p(y,y),p(x,u)+p(u,u),p(y,v)+p(v,v)\bigr\}\bigr).$$

Corollary 4 *Let (X,p) be a partial metric space such that the induced weightable T_0-qpm d_p is complete, let $\varphi : [0,\infty) \to [0,\infty)$ be a Bianchini-Grandolfi gauge function and let $T : X \to \mathrm{Cl}_{(d_p)^s}(X)$ be a multivalued map. If one of the following two conditions is satisfied, then T has a fixed point.*

(A) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v) \le \varphi\bigl(\max\bigl\{p(x,y),p(x,u)\bigr\}\bigr).$$

(B) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v)+p(v,v) \le \varphi\bigl(\max\bigl\{p(x,y)+p(y,y),p(x,u)+p(u,u)\bigr\}\bigr).$$

Corollary 5 *Let (X,p) be a partial metric space such that the induced weightable T_0-qpm d_p is complete, let $\varphi : [0,\infty) \to [0,\infty)$ be a Bianchini-Grandolfi gauge function and let $T : X \to \mathrm{Cl}_{(d_p)^s}(X)$ be a multivalued map. If one of the following two conditions is satisfied then T has a fixed point.*

(A) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v) \le \varphi\bigl(p(x,y)\bigr).$$

(B) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v)+p(v,v) \le \varphi\bigl(p(x,y)+p(y,y)\bigr).$$

Corollary 6 *Let (X,p) be a partial metric space such that the induced weightable T_0-qpm d_p is complete, let $T : X \to \mathrm{Cl}_{(d_p)^s}(X)$ be a multivalued map and $r \in [0,1)$. If one of the following two conditions is satisfied, then T has a fixed point.*

(A) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v) \le rp(x,y).$$

(B) *For each $x,y \in X$ and $u \in Tx$, there is $v \in T(y)$ satisfying*

$$p(u,v)+p(v,v) \le r\bigl(p(x,y)+p(y,y)\bigr).$$

Remark 4 Since the T_0-qpm space (X,d) of Example 4 is weightable (with weighting function w given by $w(0)=0$ and $w(x)=1$ otherwise), we deduce that Corollary 3(A) cannot be applied to the partial metric p_d induced by d. Indeed, take $x=0$, $y=2$ and $u=0$ in Example 4. Then for any $v\in Ty$, we have

$$p_d(u,v)=d(0,v)+w(0)=1>\varphi(1)=\varphi\bigl(p_d(x,y)\bigr)=\varphi\bigl(\max\bigl\{p_d(x,y),p_d(x,u)\bigr\}\bigr).$$

We conclude the paper with a simple example where we can apply the part (B) of the above corollaries but not the part (A) of them.

Example 5 Let $X=\{0,1\}$ and let p be the partial metric on X given by $p(x,y)=\max\{x,y\}$ for all $x,y\in X$. Obviously d_p is a complete T_0-qpm on X. Let $T:X\to \mathrm{Cl}_{(d_p)^s}(X)$ such that $T0=X$ and $T1=0$. Since $(d_p)^s$ is the discrete metric on X, it follows that T is p-l.s.c. on (X,d_p).

Observe that we cannot apply Corollary 3(A) for any Bianchini-Grandolfi gauge function because for $x=y=0$ and $u=1\in Tx$, we have $v=0$, and thus

$$p(u,v)=1=p(x,u)=\max\bigl\{p(x,y),p(x,u),p(y,v)\bigr\}.$$

Now we show that we can apply Corollary 6(B), and hence Corollaries 5(B), 4(B), and 3(B).

Let $r=1/2$, $x,y\in X$ and $u\in Tx$. In the cases where for $u=0$ we can choose $v=0\in Ty$, the conclusion is obvious. Therefore we only consider the following two cases.

Case 1. $x=y=0$ and $u=1$. Then $v=0$ and hence

$$p(u,v)+p(v,v)=1=r\bigl(p(x,u)+p(u,u)\bigr).$$

Case 2. $x=0$, $y=1$ and $u=1$. Then $v=0$, and as in Case 1, $p(u,v)+p(v,v)=r(p(x,u)+p(u,u))$.

Finally, note that we cannot apply Corollary 3 to any of the complete metrics p^S, p_M, and p_0, since it is clear that these metrics coincide with the discrete metric on X, and for $x=y=0$, $u=1$, and $v=0$, we have

$$p^S(u,v)=1=p^S(x,u)=\max\bigl\{p^S(x,y),p^S(x,u),p^S(y,v)\bigr\}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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