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Iterative algorithm of common solutions for a constrained convex minimization problem, a quasi-variational inclusion problem and the fixed point problem of a strictly pseudo-contractive mapping

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Abstract

In this paper, a new iterative algorithm is proposed for finding a common solution to a constrained convex minimization problem, a quasi-variational inclusion problem and the fixed point problem of a strictly pseudo-contractive mapping in a real Hilbert space. It is proved that the sequence generated by the proposed algorithm converges strongly to a common solution of the three above described problems. By applying this result to some special cases, several interesting results can be obtained.
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1 Introduction

Variational inequalities, introduced by Hartman and Stampacchia [1] in the early sixties, are one of the most interesting and intensively studied classes of mathematical problems. They are a very powerful tool of the current mathematical technology and have been extended to study a considerable amount of problems arising in mechanics, physics, optimization and control, nonlinear programming, transportation equilibrium and engineering sciences (see, *e.g.*, [2–4]). As a useful and important generalization of variational inequalities, quasi-variational inclusion problems are also further studied (see, *e.g.*, [5–14] and the references contained therein).

Throughout this paper, we assume that H is a real Hilbert space with the inner product $\langle \cdot , \cdot \rangle$ and the induced norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. $\mathcal{F}(T)$ denotes a fixed point set of the mapping T.

Let Φ be a single-valued mapping of C into H and M be a multi-valued mapping with domain $\mathbb{D}(M) = C$. The quasi-variational inclusion problem is to find $u \in C$ such that

$$0 \in \Phi(u) + Mu. \tag{1.1}$$

The solution set of quasi-variational inclusion problem (1.1) is denoted by VI(C, Φ, M). In particular, if $M = \partial \delta_C$, where $\delta_C : H \to [0, \infty]$ is the indicator function of C, *i.e.*,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the variational inclusion problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle \Phi(u), v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

This problem is called the Hartman-Stampacchia variational inequality problem [1]. The solution set of problem (1.2) is denoted by VI(C, Φ).

Recall that $T : C \to C$ is called a k-strictly pseudo-contractive mapping if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C, \tag{1.3}$$

and T is called a pseudo-contractive mapping if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

It is obvious that $k = 0$, then the mapping T is nonexpansive, that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is well known that finding fixed points of nonexpansive mappings is an important topic in the theory of nonexpansive mappings and has wide applications in a number of applied areas, such as the convex feasibility problem [15, 16], the split feasibility problem [17], image recovery and signal processing [18]. After that, as an important generalization of nonexpansive mappings, strictly pseudo-contractive mappings become one of the most interesting studied classes of nonexpansive mappings (see, *e.g.*, [19–22]). In fact, strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings do such as in solving an inverse problem [23].

In order to find a common element of the solution set of quasi-variational inclusion problem (1.1) and the fixed point set of k-strictly pseudo-contractive mapping (1.3), which is also a solution of the following constrained convex minimization problem:

$$\min_{x \in C} f(x), \tag{1.4}$$

where $f : C \to R$ is a real-valued convex function and assumes that problem (1.4) is consistent (*i.e.*, its solution set is nonempty), let Ω denote its solution set. Ceng *et al.* [24]

studied the following algorithm: take $x_0 = x \in C$ arbitrarily and

$$
\begin{cases}
y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\
t_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\
z_n = (1 - \alpha_n - \widehat{\alpha}_n)x_n + \alpha_n J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)) \\
\quad + \widehat{\alpha}_n S J_{M,\mu_n}(t_n - \mu_n \Phi(t_n)), \\
C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x, \quad n \geq 0.
\end{cases}
$$

Under appropriate conditions they obtained one strong convergence theorem.

In this paper, motivated and inspired by the above facts, we propose a new algorithm as follows: take $x_0 \in C$ arbitrarily, set $C_0 = C$, and

$$
\begin{cases}
y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\
z_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\
t_n = J_{M,\mu_n}(z_n - \mu_n \Phi(z_n)), \\
w_n = \alpha_n t_n + (1 - \alpha_n)S t_n, \\
C_{n+1} = \{w \in C_n : \|w_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,
\end{cases}
$$

and also get a strong convergence theorem under certain conditions.

The remainder of this paper is organized as follows. In Section 2, some definitions and lemmas are provided to get the main results of this paper. In Section 3, we give and prove one strong convergence theorem about our proposed algorithm. Finally, in Section 4, we apply our conclusion to some special cases.

2 Preliminaries

Let H be a real Hilbert space. It is well known that

$$
\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2
$$

and

$$
\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2
$$

for all $x, y \in H$ and $t \in [0, 1]$.

Now, we recall some definitions and useful results which will be used in the next section.

Definition 2.1 Let $T : C \to H$ be a nonlinear operator.

(1) T is Lipschitz continuous if there exists a constant $L > 0$ such that

$$
\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.
$$

(2) T is monotone if

$$
\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C.
$$

(3) T is ρ-strongly monotone if there exists a number $\rho > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \rho \|x - y\|^2, \quad \forall x, y \in C.$$

(4) T is η-inverse-strongly monotone if there exists a number $\eta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \eta \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

It is easy to see that the following results hold: (i) strongly monotone is monotone; (ii) an η-inverse-strongly monotone mapping is monotone and $\frac{1}{\eta}$-Lipschitz continuous; (iii) T is k-strictly pseudo-contractive, then $I - T$ is $\frac{1-k}{2}$-inverse strongly monotone.

Definition 2.2 A multi-valued mapping $M : \mathbb{D}(M) \subset H \to 2^H$ is called monotone if its graph $G(M) = \{(x, f) \in H \times H : x \in \mathbb{D}(M), f \in Mx\}$ is a monotone set in $H \times H$, that is, M is monotone if and only if

$$(x, f), (y, g) \in G(M) \quad \Rightarrow \quad \langle x - y, f - g \rangle \geq 0.$$

A monotone multi-valued mapping M is called maximal if for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Remark 2.1 [24] The following results are equivalent:
  (1) A multi-valued mapping M is maximal monotone;
  (2) M is monotone and $(I + \lambda M)\mathbb{D}(M) = H$ for each $\lambda > 0$;
  (3) M is monotone and its graph $G(M)$ is not properly contained in the graph of any other monotone mapping in H.

Definition 2.3 $P_C : H \to C$ is called a metric projection if for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Lemma 2.1 *Let C be a nonempty closed convex subset of H and let $P_C : H \to C$ be a metric projection, then*
  (1) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle, \forall x, y \in H$;
  (2) *moreover, P_C is a nonexpansive mapping, i.e., $\|P_C x - P_C y\| \leq \|x - y\|, \forall x, y \in H$;*
  (3) $\langle x - P_C x, y - P_C x \rangle \leq 0, \forall x \in H, y \in C$;
  (4) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, y \in C$.

Definition 2.4 Let $M : \mathbb{D}(M) \subset H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\mu} : H \to H$ defined by

$$J_{M,\mu} x = (I + \mu M)^{-1} x, \quad \forall x \in H$$

is called the resolvent operator associated with M, where μ is any positive number and I is the identity mapping.

Lemma 2.2 [5, 24, 25] *The resolvent operator $J_{M,\mu}$ associated with M is single-valued and firmly nonexpansive, i.e.,*

$$\|J_{M,\mu}x - J_{M,\mu}y\|^2 \leq \langle J_{M,\mu}x - J_{M,\mu}y, x - y \rangle, \quad \forall x, y \in H.$$

Consequently, $J_{M,\mu}$ is nonexpansive and monotone.

Lemma 2.3 [24] *Let M be a multi-valued maximal monotone mapping with $\mathbb{D}(M) = C$. Then, for any given $\mu > 0$, $u \in C$ is a solution of problem (1.1) if and only if $u \in C$ satisfies*

$$u = J_{M,\mu}\big(u - \mu\Phi(u)\big).$$

Lemma 2.4 [24] *Let M be a multi-valued maximal monotone mapping with $\mathbb{D}(M) = C$ and let $\Phi : C \to H$ be a monotone, continuous and single-valued mapping. Then $M + \Phi$ is maximal monotone.*

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Lemma 2.5 [26] *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a k-strictly pseudocontractive mapping, then the following results hold:*

(1) *inequality (1.3) is equivalent to*

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1-k}{2}\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

(2) *T is Lipschitz continuous with a constant $\frac{1+k}{1-k}$, i.e.,*

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C.$$

(3) *(Demi-closedness principle) $I - T$ is demi-closed on C, that is,*

$$\text{if } x_n \rightharpoonup x^* \in C \text{ and } (I - T)x_n \to 0, \text{ then } x^* = Tx^*.$$

Lemma 2.6 *Let C be a nonempty closed convex subset of H and let $T : C \to H$ be an η-inverse-strongly monotone mapping, then for all $x, y \in C$ and $\eta > 0$, we have*

$$\begin{aligned}
\big\|(I - \lambda T)x - (I - \lambda T)y\big\|^2 &= \big\|(x - y) - \lambda(Tx - Ty)\big\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle Tx - Ty, x - y \rangle + \lambda^2\|Tx - Ty\|^2 \\
&\leq \|x - y\|^2 + \lambda(\lambda - 2\eta)\|Tx - Ty\|^2.
\end{aligned}$$

So, if $0 < \lambda \leq 2\eta$, then $I - \lambda T$ is a nonexpansive mapping from C to H.

Lemma 2.7 [24, 27] *Let $A : C \to H$ be a monotone, Lipschitz continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \big\{z \in H : \langle v - u, z \rangle \geq 0, \forall u \in C\big\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$.

For the minimization problem (1.4), if f is (Frechet) differentiable, then we have the following lemma.

Lemma 2.8 [28] (Optimality condition) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem* (1.4) *is that x^* solves the variational inequality*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{2.1}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(x^* - \lambda \nabla f(x^*))$$

for every constant $\lambda > 0$. If, in addition, f is convex, then the optimality condition (2.1) *is also sufficient.*

3 Main results

In this section, we prove a strong convergence theorem by an iterative algorithm for finding a solution of the constrained convex minimization problem (1.4), which is also a common solution of the quasi-variational inclusion problem (1.1) and the fixed point problem of a k-strictly pseudo-contractive mapping (1.3) in a real Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For the minimization problem* (1.4), *assume that f is (Frechet) differentiable and the gradient ∇f is a ρ-inverse-strongly monotone mapping. Let $\Phi : C \to H$ be an η-inverse-strongly monotone mapping and M be a maximal monotone mapping with $\mathbb{D}(M) = C$, and let $S : C \to C$ be a k-strictly pseudo-contractive mapping such that $\mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) \neq \emptyset$. Pick any $x_0 \in C$ and set $C_0 = C$. Let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\ z_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\ t_n = J_{M,\mu_n}(z_n - \mu_n \Phi(z_n)), \\ w_n = \alpha_n t_n + (1 - \alpha_n) S t_n, \\ C_{n+1} = \{w \in C_n : \|w_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases} \tag{3.1}$$

where the following conditions hold:

(i) $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < \rho$;

(ii) $\epsilon \leq \mu_n \leq 2\eta$ *for some $\epsilon \in (0, 2\eta]$*;

(iii) $k < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof It is obvious that C_n is closed for each $n \in \mathbb{N}$. Since $\|w_n - w\| \le \|x_n - w\|$ is equivalent to

$$\|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - w \rangle \le 0,$$

we have that C_n is convex for each $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined. We divide the proof into five steps.

Step 1. We show by induction that $\mathcal{F} \subset C_n$ for each $n \in \mathbb{N}$.

It is obvious that $\mathcal{F} \subset C_0 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \in \mathbb{N}$. Let $p \in \mathcal{F}$, we have

$$
\begin{aligned}
\|w_n - p\|^2 &= \left\| \alpha_n t_n + (1 - \alpha_n) St_n - p \right\|^2 \\
&= \left\| \alpha_n (t_n - p) + (1 - \alpha_n)(St_n - p) \right\|^2 \\
&= \alpha_n \|t_n - p\|^2 + (1 - \alpha_n)\|St_n - p\|^2 - \alpha_n(1 - \alpha_n)\|t_n - St_n\|^2 \\
&\le \alpha_n \|t_n - p\|^2 + (1 - \alpha_n)\big[\|t_n - p\|^2 + k\|t_n - St_n\|^2\big] - \alpha_n(1 - \alpha_n)\|t_n - St_n\|^2 \\
&= \|t_n - p\|^2 + (1 - \alpha_n)(k - \alpha_n)\|t_n - St_n\|^2 \\
&\le \|t_n - p\|^2.
\end{aligned}
\tag{3.2}
$$

According to Lemma 2.3, Lemma 2.2 and Lemma 2.6, we get

$$
\begin{aligned}
\|t_n - p\|^2 &= \left\| J_{M, \mu_n}\big(z_n - \mu_n \Phi(z_n)\big) - J_{M, \mu_n}\big(p - \mu_n \Phi(p)\big) \right\|^2 \\
&\le \left\| z_n - \mu_n \Phi(z_n) - \big(p - \mu_n \Phi(p)\big) \right\|^2 \\
&\le \|z_n - p\|^2 + \mu_n(\mu_n - 2\eta)\left\| \Phi(z_n) - \Phi(p) \right\|^2 \\
&\le \|z_n - p\|^2.
\end{aligned}
\tag{3.3}
$$

Since the gradient ∇f is a ρ-inverse-strongly monotone mapping and $p \in \mathcal{F} \subset \Omega$, from Lemma 2.8, we have

$$\langle \nabla f(y_n) - \nabla f(p), y_n - p \rangle \ge 0 \quad \text{and} \quad \langle \nabla f(p), y_n - p \rangle \ge 0. \tag{3.4}$$

From Lemma 2.1(4) and (3.4), we obtain

$$
\begin{aligned}
\|z_n - p\|^2 &= \left\| P_C\big(x_n - \lambda_n \nabla f(y_n)\big) - p \right\|^2 \\
&\le \left\| x_n - \lambda_n \nabla f(y_n) - p \right\|^2 - \left\| x_n - \lambda_n \nabla f(y_n) - z_n \right\|^2 \\
&= \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle \nabla f(y_n), p - z_n \rangle \\
&= \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \big[\langle \nabla f(y_n) - \nabla f(p), p - y_n \rangle \\
&\quad + \langle \nabla f(p), p - y_n \rangle + \langle \nabla f(y_n), y_n - z_n \rangle \big] \\
&\le \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle \nabla f(y_n), y_n - z_n \rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - z_n \rangle \\
&\quad - \|y_n - z_n\|^2 + 2\lambda_n \langle \nabla f(y_n), y_n - z_n \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n \nabla f(y_n) - y_n, z_n - y_n \rangle.
\end{aligned}
\tag{3.5}
$$

It is easy to see that ρ-inverse-strongly monotone mapping ∇f is $\frac{1}{\rho}$-Lipschitz continuous. Further, since $y_n = P_C(x_n - \lambda_n \nabla f(x_n))$ and by Lemma 2.1(3) we have

$$
\begin{aligned}
&\langle x_n - \lambda_n \nabla f(y_n) - y_n, z_n - y_n \rangle \\
&= \langle x_n - \lambda_n \nabla f(x_n) - y_n, z_n - y_n \rangle + \langle \lambda_n \nabla f(x_n) - \lambda_n \nabla f(y_n), z_n - y_n \rangle \\
&\leq \langle \lambda_n \nabla f(x_n) - \lambda_n \nabla f(y_n), z_n - y_n \rangle \\
&\leq \lambda_n \|\nabla f(x_n) - \nabla f(y_n)\| \|z_n - y_n\| \\
&\leq \frac{\lambda_n}{\rho} \|x_n - y_n\| \|z_n - y_n\|.
\end{aligned}
\tag{3.6}
$$

Substituting (3.6) into (3.5), we obtain

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n \nabla f(y_n) - y_n, z_n - y_n \rangle \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \frac{2\lambda_n}{\rho} \|x_n - y_n\| \|z_n - y_n\| \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \frac{\lambda_n^2}{\rho^2} \|x_n - y_n\|^2 + \|z_n - y_n\|^2 \\
&= \|x_n - p\|^2 + \left(\frac{\lambda_n^2}{\rho^2} - 1 \right) \|x_n - y_n\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned}
\tag{3.7}
$$

From (3.2), (3.3) and (3.7), we have

$$
\|w_n - p\| \leq \|x_n - p\|.
\tag{3.8}
$$

Hence $p \in C_{n+1}$. This implies that $p \in C_n$ for all $n \in \mathbb{N}$.

Step 2. We prove that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \to \infty} \|x_n - w_n\| = 0$.

Let $x^* = P_{\mathcal{F}} x_0$. From $x_n = P_{C_n} x_0$ and $x^* \in \mathcal{F} \subset C_n$, we obtain

$$
\|x_n - x_0\| \leq \|x^* - x_0\|.
\tag{3.9}
$$

Then $\{x_n\}$ is bounded. This implies that $\{z_n\}$, $\{t_n\}$ and $\{w_n\}$ are also bounded. Since $x_n = P_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$
\|x_n - x_0\| \leq \|x_{n+1} - x_0\|.
$$

Therefore $\lim_{n \to \infty} \|x_n - x_0\|$ exists. From $x_n = P_{C_n} x_0$, $x_{n+1} \in C_{n+1} \subset C_n$ and Lemma 2.1(4), we obtain

$$
0 \leq \|x_{n+1} - x_n\|^2 \leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2,
$$

which implies

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|w_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, and hence

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \leq 2\|x_n - x_{n+1}\|. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.12}$$

Step 3. We show that $\lim_{n\to\infty} \|t_n - St_n\| = 0$, $\lim_{n\to\infty} \|x_n - z_n\| = 0$ and $\lim_{n\to\infty} \|x_n - t_n\| = 0$.

For $p \in \mathcal{F}$, from (3.2), (3.3) and (3.7), we have

$$\begin{aligned}
\|w_n - p\|^2 &\leq \|t_n - p\|^2 + (1-\alpha_n)(k-\alpha_n)\|t_n - St_n\|^2 \\
&\leq \|z_n - p\|^2 + (1-\alpha_n)(k-\alpha_n)\|t_n - St_n\|^2 \\
&\leq \|x_n - p\|^2 + \left(\frac{\lambda_n^2}{\rho^2} - 1\right)\|x_n - y_n\|^2 + (1-\alpha_n)(k-\alpha_n)\|t_n - St_n\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\left(1 - \frac{\lambda_n^2}{\rho^2}\right)&\|x_n - y_n\|^2 + (1-\alpha_n)(\alpha_n - k)\|t_n - St_n\|^2 \\
&\leq \|x_n - p\|^2 - \|w_n - p\|^2 \\
&\leq \big(\|x_n - p\| + \|w_n - p\|\big)\|x_n - w_n\|. \tag{3.13}
\end{aligned}$$

Since $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \rho$ and $k < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, from (3.12) and (3.13) we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \tag{3.14}$$

and

$$\lim_{n\to\infty} \|t_n - St_n\| = 0. \tag{3.15}$$

As ∇f is $\frac{1}{\rho}$-Lipschitz continuous, we have

$$\begin{aligned}
\|z_n - y_n\| &= \left\|P_C\big(x_n - \lambda_n \nabla f(y_n)\big) - P_C\big(x_n - \lambda_n \nabla f(x_n)\big)\right\| \\
&\leq \left\|x_n - \lambda_n \nabla f(y_n) - \big(x_n - \lambda_n \nabla f(x_n)\big)\right\| \\
&= \lambda_n \left\|\nabla f(y_n) - \nabla f(x_n)\right\| \\
&\leq \frac{\lambda_n}{\rho} \|x_n - y_n\|.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.16}$$

From (3.14) and (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.17}$$

We observe

$$\begin{aligned}
\|w_n - t_n\| &= \|\alpha_n t_n + (1 - \alpha_n)St_n - t_n\| \\
&= (1 - \alpha_n)\|St_n - t_n\| \\
&\leq \|St_n - t_n\|.
\end{aligned} \tag{3.18}$$

From (3.15), we get

$$\lim_{n\to\infty} \|w_n - t_n\| = 0. \tag{3.19}$$

Combining (3.12) and (3.19), we have

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.20}$$

Step 4. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to u. We show that

$$u \in \mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M).$$

Indeed, firstly, we show $u \in \mathcal{F}(S)$. Since $\|x_n - t_n\| \to 0$ and $x_{n_i} \rightharpoonup u$, we have $t_{n_i} \rightharpoonup u$. From $\|St_n - t_n\| \to 0$, we obtain $\|St_{n_i} - t_{n_i}\| \to 0$ as $i \to \infty$. By Lemma 2.5 (Demi-closedness principle), we can conclude that $u \in \mathcal{F}(S)$.

Secondly, we show $u \in \Omega$. Since $z_n = P_C(x_n - \lambda_n \nabla f(y_n))$ and by Lemma 2.1(3), we have

$$\langle x_n - \lambda_n \nabla f(y_n) - z_n, z_n - v \rangle \geq 0,$$

that is,

$$\left\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + \nabla f(y_n) \right\rangle \geq 0.$$

Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, from Lemma 2.7, we know that T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$. Let $G(T)$ be the graph of T and let $(v, h) \in G(T)$. Then we have $h \in Tv =$

$\nabla f(v) + N_C v$. Hence $h - \nabla f(v) \in N_C v$. So, we have

$$\langle v - z, h - \nabla f(v) \rangle \geq 0, \quad \forall z \in C.$$

Therefore,

$$
\begin{aligned}
\langle v - z_{n_i}, h \rangle & \\
& \geq \langle v - z_{n_i}, \nabla f(v) \rangle \\
& \geq \langle v - z_{n_i}, \nabla f(v) \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + \nabla f(y_{n_i}) \right\rangle \\
& = \langle v - z_{n_i}, \nabla f(v) - \nabla f(z_{n_i}) \rangle + \langle v - z_{n_i}, \nabla f(z_{n_i}) - \nabla f(y_{n_i}) \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
& \geq \langle v - z_{n_i}, \nabla f(z_{n_i}) - \nabla f(y_{n_i}) \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned}
\tag{3.21}
$$

Since $\|x_n - y_n\| \to 0$ and $\|x_n - z_n\| \to 0$, we have $y_{n_i} \rightharpoonup u$ and $z_{n_i} \rightharpoonup u$. Then, from (3.21), we obtain $\langle v - u, h - 0 \rangle = \langle v - u, h \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $0 \in Tu$ and hence $u \in \text{VI}(C, \nabla f)$. According to Lemma 2.8, we obtain $u \in \Omega$.

Finally, let us show $u \in \text{VI}(C, \Phi, M)$. Since $\Phi : C \to H$ is η-inverse-strongly monotone and M is maximal monotone, by Lemma 2.4 we know that $M + \Phi$ is maximal monotone. Take a fixed $(y, g) \in G(M + \Phi)$ arbitrarily. Then we have $g \in \Phi(y) + My$, that is,

$$g - \Phi(y) \in My.$$

Since $t_{n_i} = J_{M, \mu_{n_i}}(z_{n_i} - \mu_{n_i} \Phi(z_{n_i}))$, then

$$\frac{1}{\mu_{n_i}} \left(z_{n_i} - \mu_{n_i} \Phi(z_{n_i}) - t_{n_i} \right) \in Mt_{n_i}.$$

Therefore,

$$\left\langle y - t_{n_i}, g - \Phi(y) - \frac{1}{\mu_{n_i}} \left(z_{n_i} - \mu_{n_i} \Phi(z_{n_i}) - t_{n_i} \right) \right\rangle \geq 0,$$

which hence yields

$$
\begin{aligned}
\langle y - t_{n_i}, g \rangle & \\
& \geq \left\langle y - t_{n_i}, \Phi(y) + \frac{1}{\mu_{n_i}} \left(z_{n_i} - \mu_{n_i} \Phi(z_{n_i}) - t_{n_i} \right) \right\rangle \\
& = \langle y - t_{n_i}, \Phi(y) - \Phi(z_{n_i}) \rangle + \left\langle y - t_{n_i}, \frac{z_{n_i} - t_{n_i}}{\mu_{n_i}} \right\rangle \\
& \geq \eta \left\| \Phi(y) - \Phi(t_{n_i}) \right\|^2 + \langle y - t_{n_i}, \Phi(t_{n_i}) - \Phi(z_{n_i}) \rangle + \left\langle y - t_{n_i}, \frac{z_{n_i} - t_{n_i}}{\mu_{n_i}} \right\rangle \\
& \geq \langle y - t_{n_i}, \Phi(t_{n_i}) - \Phi(z_{n_i}) \rangle + \left\langle y - t_{n_i}, \frac{z_{n_i} - t_{n_i}}{\mu_{n_i}} \right\rangle.
\end{aligned}
\tag{3.22}
$$

Observe that

$$\left| \left\langle y - t_{n_i}, \Phi(t_{n_i}) - \Phi(z_{n_i}) \right\rangle + \left\langle y - t_{n_i}, \frac{z_{n_i} - t_{n_i}}{\mu_{n_i}} \right\rangle \right|$$

$$\leq \|y - t_{n_i}\| \|\Phi(t_{n_i}) - \Phi(z_{n_i})\| + \frac{1}{\mu_{n_i}} \|y - t_{n_i}\| \|z_{n_i} - t_{n_i}\|$$

$$\leq \frac{1}{\eta} \|y - t_{n_i}\| \|t_{n_i} - z_{n_i}\| + \frac{1}{\epsilon} \|y - t_{n_i}\| \|z_{n_i} - t_{n_i}\|$$

$$= \left(\frac{1}{\eta} + \frac{1}{\epsilon} \right) \|y - t_{n_i}\| \|z_{n_i} - t_{n_i}\|.$$

By $\|x_n - z_n\| \to 0$ and $\|x_n - t_n\| \to 0$, we have $\|z_n - t_n\| \to 0$. Then

$$\lim_{i \to \infty} \left| \left\langle y - t_{n_i}, \Phi(t_{n_i}) - \Phi(z_{n_i}) \right\rangle + \left\langle y - t_{n_i}, \frac{z_{n_i} - t_{n_i}}{\mu_{n_i}} \right\rangle \right| = 0.$$

Let $i \to \infty$, from (3.22) we get

$$\langle y - u, g - 0 \rangle = \langle y - u, g \rangle \geq 0.$$

This implies that $0 \in \Phi(u) + Mu$. Hence $u \in \mathrm{VI}(C, \Phi, M)$. Therefore,

$$u \in \mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M).$$

Step 5. We show that $x_n \to x^*$, where $x^* = P_{\mathcal{F}} x_0$.

Indeed, from $x^* = P_{\mathcal{F}} x_0$, $u \in \mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M)$ and (3.9), we have

$$\|x^* - x_0\| \leq \|u - x_0\| \leq \liminf_{i \to \infty} \|x_{n_i} - x_0\| \leq \limsup_{i \to \infty} \|x_{n_i} - x_0\| \leq \|x^* - x_0\|.$$

Then

$$\lim_{i \to \infty} \|x_{n_i} - x_0\| = \|u - x_0\|.$$

From $x_{n_i} - x_0 \rightharpoonup u - x_0$ and the Kadec-Klee property of H, we have $x_{n_i} - x_0 \to u - x_0$, and hence $x_{n_i} \to u$. Since $x_{n_i} = P_{C_{n_i}} x_0$ and $x^* \in \mathcal{F} \subset C_{n_i}$, we have

$$-\left\| x^* - x_{n_i} \right\|^2 = \left\langle x^* - x_{n_i}, x_{n_i} - x_0 \right\rangle + \left\langle x^* - x_{n_i}, x_0 - x^* \right\rangle \geq \left\langle x^* - x_{n_i}, x_0 - x^* \right\rangle.$$

Let $i \to \infty$, by $u \in \mathcal{F}$ and $x^* = P_{\mathcal{F}} x_0$, we have

$$-\left\| x^* - u \right\|^2 \geq \left\langle x^* - u, x_0 - x^* \right\rangle \geq 0.$$

Hence $u = x^*$, which implies that $x_n \to x^*$. This completes the proof. □

4 Applications

From Theorem 3.1, we can obtain the following theorems.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For the minimization problem* (1.4), *assume that f is* (Frechet) *differentiable and the gradient ∇f is a ρ-inverse-strongly monotone mapping. Let $S : C \to C$ be a k-strictly pseudo-contractive mapping such that $\mathcal{F} = \mathcal{F}(S) \cap \Omega \neq \emptyset$. Pick any $x_0 \in C$ and set $C_0 = C$. Let $\{x_n\} \subset C$ be a sequence generated by*

$$
\begin{cases}
y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\
z_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\
w_n = \alpha_n z_n + (1 - \alpha_n) S z_n, \\
C_{n+1} = \{w \in C_n : \|w_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,
\end{cases}
\tag{4.1}
$$

where the following conditions hold:

  (i) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \rho$;
  (ii) $k < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof Let $\Phi = M = 0$ in Theorem 3.1, we have $\text{VI}(C, 0, 0) = C$ and $\mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \text{VI}(C, 0, 0) = \mathcal{F}(S) \cap \Omega$. Let η be any positive number in the interval $(0, \infty)$ and take any sequence $\{\mu_n\} \subset [\epsilon, 2\eta]$ for some $\epsilon \in (0, 2\eta]$. In addition, we have

$$
t_n = J_{M,\mu_n}\big(z_n - \mu_n \Phi(z_n)\big) = (I + \mu_n M)^{-1} z_n = z_n.
$$

Therefore, by Theorem 3.1 we obtain the expected result. $\qquad \square$

Theorem 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping such that $\mathcal{F}(S) \neq \emptyset$. Pick any $x_0 \in C$ and set $C_0 = C$. Let $\{x_n\} \subset C$ be a sequence generated by*

$$
\begin{cases}
w_n = \alpha_n x_n + (1 - \alpha_n) S x_n, \\
C_{n+1} = \{w \in C_n : \|w_n - w\| \leq \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0,
\end{cases}
\tag{4.2}
$$

where the following condition holds: $k < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.*

Proof Let $\nabla f = \Phi = M = 0$ and $k = 0$ in Theorem 3.1. Let ρ, η be any positive number in the interval $(0, \infty)$. Take any sequence $\{\lambda_n\}$ which satisfies $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \rho$ and take any sequence $\{\mu_n\} \subset [\epsilon, 2\eta]$ for some $\epsilon \in (0, 2\eta]$. In this case, we have

$$
\begin{cases}
y_n = P_C(x_n - \lambda_n \nabla f(x_n)) = x_n, \\
z_n = P_C(x_n - \lambda_n \nabla f(y_n)) = x_n, \\
t_n = J_{M,\mu_n}(z_n - \mu_n \Phi(z_n)) = z_n, \\
w_n = \alpha_n t_n + (1 - \alpha_n) S t_n = \alpha_n x_n + (1 - \alpha_n) S x_n.
\end{cases}
\tag{4.3}
$$

Therefore, by Theorem 3.1 we obtain the expected result. $\qquad \square$

Theorem 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. For the minimization problem* (1.4), *assume that f is (Frechet) differentiable and the gradient ∇f is a ρ-inverse-strongly monotone mapping. Let $\Gamma : C \to C$ be γ-strictly pseudo-contractive and let $S : C \to C$ be a k-strictly pseudo-contractive mapping such that $\mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathcal{F}(\Gamma) \neq \emptyset$. Pick any $x_0 \in C$ and set $C_0 = C$. Let $\{x_n\} \subset C$ be a sequence generated by*

$$
\begin{cases}
y_n = P_C(x_n - \lambda_n \nabla f(x_n)), \\
z_n = P_C(x_n - \lambda_n \nabla f(y_n)), \\
t_n = (1 - \mu_n)z_n + \mu_n \Gamma(z_n), \\
w_n = \alpha_n t_n + (1 - \alpha_n)St_n, \\
C_{n+1} = \{w \in C_n : \|w_n - w\| \le \|x_n - w\|\}, \\
x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 0,
\end{cases}
\tag{4.4}
$$

where the following conditions hold:
   (i) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \rho$;
  (ii) $\epsilon \le \mu_n \le 1 - \gamma$ *for some $\epsilon \in (0, 1 - \gamma]$*;
 (iii) $k < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.
Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}x_0$.

Proof Let $\Phi = I - \Gamma$ and $M = 0$ in Theorem 3.1, then we have that Φ is η-inverse strongly monotone with $\eta = \frac{1-\gamma}{2}$. Now, we show that $\mathrm{VI}(C, \Phi, M) = \mathcal{F}(\Gamma)$. In fact, since $\Phi = I - \Gamma$ and $M = 0$, we obtain

$$
\begin{aligned}
0 \in \mathrm{VI}(C, \Phi, M) \quad &\Leftrightarrow \quad 0 \in \Phi(u) + Mu \\
&\Leftrightarrow \quad 0 = \Phi(u) \\
&\Leftrightarrow \quad 0 = u - \Gamma(u) \\
&\Leftrightarrow \quad u \in \mathcal{F}(\Gamma).
\end{aligned}
$$

Thus,

$$
\mathcal{F} = \mathcal{F}(S) \cap \Omega \cap \mathrm{VI}(C, \Phi, M) = \mathcal{F}(S) \cap \Omega \cap \mathcal{F}(\Gamma).
$$

Note that $\mu_n \in [\epsilon, 1 - \gamma] \subset [0, 1]$, hence $(1 - \mu_n)z_n + \mu_n \Gamma(z_n) \in C$. In this case, we have

$$
\begin{aligned}
t_n &= J_{M,\mu_n}\big(z_n - \mu_n \Phi(z_n)\big) \\
&= (I + \mu_n M)^{-1}\big(z_n - \mu_n \Phi(z_n)\big) \\
&= z_n - \mu_n \Phi(z_n) = z_n - \mu_n(I - \Gamma)(z_n) \\
&= (1 - \mu_n)z_n + \mu_n \Gamma(z_n).
\end{aligned}
$$

Therefore, by Theorem 3.1 we obtain the expected result. $\qquad\square$

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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