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Correction: Fixed point theorems of contractive mappings in cone b-metric spaces and applications

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Correction

In this note we correct some errors that appeared in the article (Huang and Xu in Fixed Point Theory Appl. 2013:112, 2013) by modifying some conditions in the main theorems and examples.

After examining the proofs of the main results in [1], we can find that there is something wrong with the proof of the Cauchy sequence in [1, Theorem 2.1]. This leads to subsequent errors in Theorem 2.3 and related examples in [1]. We also find that it is not rigorous to use the corresponding lemmas, and so the proof is inaccurate. The detailed reasons are given in the following.

On p.5 in [1], we conclude that

$$\frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0)+s^{p-1}\lambda^m d(x_1,x_0)\to\theta$$

as $m\to\infty$ for any $p\geq 1$. This is incorrect. Indeed, note that taking $\lambda=\frac{1}{\sqrt{s}}>\frac{1}{s}$ and $p=m+1$ leads to

$$\frac{s^p\lambda^{m+1}}{s-\lambda}d(x_1,x_0)+s^{p-1}\lambda^m d(x_1,x_0)=\frac{s^{\frac{m+2}{2}}}{s^{\frac{3}{2}}-1}d(x_1,x_0)+s^{\frac{m}{2}}d(x_1,x_0)\nrightarrow\theta$$

as $m\to\infty$. Therefore, it is impossible to utilize [1, Lemma 1.8, 1.9] and demonstrate that $\{x_n\}$ is a Cauchy sequence.

In this note, we would like to slightly modify only one of the used conditions to achieve our claim.

The following theorem is a modification to [1, Theorem 2.1]. The proof is the same as that in [1] except the proof of the Cauchy sequence. We will attain the desired goal by using the new modified condition $\lambda\in[0,\frac{1}{s})$ instead of $\lambda\in[0,1)$.

Theorem 2.1 *Let (X,d) be a complete cone b-metric space with the coefficient $s\geq 1$. Suppose that the mapping $T:X\to X$ satisfies the contractive condition*

$$d(Tx,Ty)\leq\lambda d(x,y)\quad \textit{for } x,y\in X,$$

where $\lambda \in [0, \frac{1}{s})$ *is a constant. Then* T *has a unique fixed point in* X. *Furthermore, the iterative sequence* $\{T^n x\}$ *converges to the fixed point.*

Proof In order to show that $\{x_n\}$ is a Cauchy sequence, we only need the following calculations. For any $m \geq 1$, $p \geq 1$, it follows that

$$\begin{aligned} d(x_m, x_{m+p}) &\leq s\big[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})\big] \\ &\leq sd(x_m, x_{m+1}) + s^2\big[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})\big] \\ &\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\ &\quad + \cdots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\ &\leq s\lambda^m d(x_1, x_0) + s^2\lambda^{m+1} d(x_1, x_0) + s^3\lambda^{m+2} d(x_1, x_0) \\ &\quad + \cdots + s^{p-1}\lambda^{m+p-2} d(x_1, x_0) + s^p\lambda^{m+p-1} d(x_1, x_0) \\ &= s\lambda^m\big[1 + s\lambda + s^2\lambda^2 + \cdots + (s\lambda)^{p-1}\big] d(x_1, x_0) \leq \frac{s\lambda^m}{1 - s\lambda} d(x_1, x_0). \end{aligned}$$

Let $\theta \ll c$ be given. Notice that $\frac{s\lambda^m}{1-s\lambda} d(x_1, x_0) \to \theta$ as $m \to \infty$ for any p. Making full use of [1, Lemma 1.8], we find $m_0 \in \mathbb{N}$ such that

$$\frac{s\lambda^m}{1 - s\lambda} d(x_1, x_0) \ll c$$

for each $m > m_0$. Thus,

$$d(x_m, x_{m+p}) \leq \frac{s\lambda^m}{1 - s\lambda} d(x_1, x_0) \ll c$$

for all $m \geq 1$, $p \geq 1$. So, by [1, Lemma 1.9], $\{x_n\}$ is a Cauchy sequence in (X, d). The proof is completed. □

As is indicated in the reviewer's comments, [1, Example 2.2] is too trivial. Therefore, [1, Example 2.2] is withdrawn. Now we give another example as follows.

Example 2.2 Let $X = [0, 0.48]$, $E = \mathbb{R}^2$ and let $1 \leq p \leq 6$ be a constant. Take $P = \{(x, y) \in E : x, y \geq 0\}$. We define $d : X \times X \to E$ as

$$d(x, y) = \big(|x - y|^p, |x - y|^p\big) \quad \text{for all } x, y \in X.$$

Then (X, d) is a complete cone b-metric space with $s = 2^{p-1}$. Let us define $T : X \to X$ as

$$Tx = \frac{1}{2}\left(\cos\frac{x}{2} - \left|x - \frac{1}{2}\right|\right) \quad \text{for all } x \in X.$$

Thus, for all $x, y \in X$, we have

$$\begin{aligned} d(Tx, Ty) &= \big(|Tx - Ty|^p, |Tx - Ty|^p\big) \\ &= \frac{1}{2^p}\left(\left|\left(\cos\frac{x}{2} - \cos\frac{y}{2}\right) - \left(\left|x - \frac{1}{2}\right| - \left|y - \frac{1}{2}\right|\right)\right|^p, \end{aligned}$$

$$\begin{aligned}
&\left|\left(\cos\frac{x}{2}-\cos\frac{y}{2}\right)-\left(\left|x-\frac{1}{2}\right|-\left|y-\frac{1}{2}\right|\right)\right|^p\Bigg) \\
&\leq \frac{1}{2^p}\left(\left(\left|\cos\frac{x}{2}-\cos\frac{y}{2}\right|+|x-y|\right)^p,\left(\left|\cos\frac{x}{2}-\cos\frac{y}{2}\right|+|x-y|\right)^p\right) \\
&\leq \frac{1}{2^p}\left(\left(\frac{|x+y|}{8}|x-y|+|x-y|\right)^p,\left(\frac{|x+y|}{8}|x-y|+|x-y|\right)^p\right) \\
&\leq 0.56^p\left(|x-y|^p,|x-y|^p\right) < \frac{1}{2^{p-1}}\left(|x-y|^p,|x-y|^p\right).
\end{aligned}$$

Hence, by Theorem 2.1, there exists $x_0 \in X$ (in fact, it satisfies $0.472251591454 < x_0 < 0.472251591479$) such that x_0 is the unique fixed point of T.

For the same reason, we need to use the new condition $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \frac{2}{1+s}$ instead of the original condition $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, \frac{2}{s}\}$ in [1, Theorem 2.3]. The correct statement is as follows.

Theorem 2.3 *Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose that the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \leq \lambda_1 d(x,Tx) + \lambda_2 d(y,Ty) + \lambda_3 d(x,Ty) + \lambda_4 d(y,Tx) \quad \textit{for } x,y \in X,$$

where the constant $\lambda_i \in [0,1)$ and $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \frac{2}{1+s}$, $i = 1,2,3,4$. Then T has a unique fixed point in X. Moreover, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof Following an identical argument that is given in [1, Theorem 2.3] except substituting $0 \leq \lambda \leq 1$ for $0 \leq \lambda \leq \frac{1}{s}$ in line 26 of p.6 in [1], we obtain the proof of Theorem 2.3. □

In addition, based on the changes of Theorem 2.1, we need to change the condition $h^2 < \min\{\frac{\delta}{M^2}, \frac{1}{L^2}\}$ into $h^2 < \min\{\frac{\delta}{M^2}, \frac{1}{2L^2}\}$ for [1, Example 3.1]. Let us give the corrected example.

We now apply Theorem 2.1 to the first-order periodic boundary problem

$$\begin{cases} \frac{dx}{dt} = F(t,x(t)), \\ x(0) = \xi, \end{cases} \tag{2.1}$$

where $F : [-h,h] \times [\xi - \delta, \xi + \delta]$ is a continuous function.

Example 2.4 Consider boundary problem (2.1) with the continuous function F, and suppose that $F(x,y)$ satisfies the local Lipschitz condition, *i.e.*, if $|x| \leq h$, $y_1, y_2 \in [\xi - \delta, \xi + \delta]$, it induces

$$\left|F(x,y_1) - F(x,y_2)\right| \leq L|y_1 - y_2|.$$

Set $M = \max_{[-h,h]\times[\xi-\delta,\xi+\delta]} |F(x,y)|$ such that $h^2 < \min\{\frac{\delta}{M^2}, \frac{1}{2L^2}\}$, then there exists a unique solution of (2.1).

Proof Let $X = E = C([-h,h])$ and $P = \{u \in E : u \geq 0\}$. Put $d : X \times X \to E$ as $d(x,y) = f(t)\max_{-h\leq t\leq h} |x(t) - y(t)|^2$ with $f : [-h,h] \to \mathbb{R}$ such that $f(t) = e^t$. It is clear that (X,d) is a complete cone b-metric space with $s = 2$.

Note that (2.1) is equivalent to the integral equation

$$x(t) = \xi + \int_0^t F(\tau, x(\tau))\,\mathrm{d}\tau.$$

Define a mapping $T : C([-h,h]) \to \mathbb{R}$ by $Tx(t) = \xi + \int_0^t F(\tau, x(\tau))\,\mathrm{d}\tau$. If

$$x(t), y(t) \in B(\xi, \delta f) \triangleq \{\varphi(t) \in C([-h,h]) : d(\xi, \varphi) \leq \delta f\},$$

then from

$$\begin{aligned} d(Tx, Ty) &= f(t) \max_{-h\leq t\leq h} \left| \int_0^t F(\tau, x(\tau))\,\mathrm{d}\tau - \int_0^t F(\tau, y(\tau))\,\mathrm{d}\tau \right|^2 \\ &= f(t) \max_{-h\leq t\leq h} \left| \int_0^t [F(\tau, x(\tau)) - F(\tau, y(\tau))]\,\mathrm{d}\tau \right|^2 \\ &\leq h^2 f(t) \max_{-h\leq \tau\leq h} \left| F(\tau, x(\tau)) - F(\tau, y(\tau)) \right|^2 \\ &\leq h^2 L^2 f(t) \max_{-h\leq \tau\leq h} \left| x(\tau) - y(\tau) \right|^2 \\ &= h^2 L^2 d(x, y), \end{aligned}$$

and

$$d(Tx, \xi) = f(t) \max_{-h\leq t\leq h} \left| \int_0^t F(\tau, x(\tau))\,\mathrm{d}\tau \right|^2 \leq h^2 f \max_{-h\leq \tau\leq h} \left| F(\tau, x(\tau)) \right|^2 \leq h^2 M^2 f \leq \delta f,$$

we speculate that $T : B(\xi, \delta f) \to B(\xi, \delta f)$ is a contractive mapping.

Finally, we prove that $(B(\xi, \delta f), d)$ is complete. In fact, suppose that $\{x_n\}$ is a Cauchy sequence in $B(\xi, \delta f)$. Then $\{x_n\}$ is also a Cauchy sequence in X. Since (X, d) is complete, there is $x \in X$ such that $x_n \to x$ ($n \to \infty$). So, for each $c \in \operatorname{int} P$, there exists N, whenever $n > N$, we obtain $d(x_n, x) \ll c$. Thus, it follows from

$$d(\xi, x) \leq d(x_n, \xi) + d(x_n, x) \leq \delta f + c$$

and Lemma 1.12 in [1] that $d(\xi, x) \leq \delta f$, which means $x \in B(\xi, \delta f)$, that is, $(B(\xi, \delta f), d)$ is complete. □

Owing to the above statement, all conditions of Theorem 2.1 are satisfied. Hence T has a unique fixed point $x(t) \in B(\xi, \delta f)$. That is to say, there exists a unique solution of (2.1).

Remark 2.5 Theorem 2.1 and Theorem 2.3 generalize and improve the corresponding results in [2–4].

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