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On inclusion of F-contractions in (δ, k)-weak contractions

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Abstract

We obtain a certain property of F-contractions, which enables us to generalize and extend Wardowski's (Fixed Point Theory and Applications 2012:94, 2012) result and other fixed point theorems to the class of nonexpansive operators in real Banach spaces. We give illustrative examples to demonstrate nontrivial applicability of the property and use it to prove that F-contractions are closely related to (δ, k)-weak contractions introduced by Berinde (Carpath. J. Math. 19(1):7-22, 2003; Nonlinear Anal. Forum 9(1):43-53, 2004).

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1 Introduction and preliminaries

This work is concerned with contractive maps defined on real Banach spaces. Let (X, d) be a metric space and let $T : X \to X$ be a mapping such that there exists at least a constant $L > 0$ with $d(Tx, Ty) \le Ld(x, y)$ for all $x, y \in X$, then T is called L-*Lipschitzian* operator. T is called a *contraction* if $L \in [0, 1)$ and for $L = 1$, T is called a *nonexpansive* operator. T is called *contractive* if $d(Tx, Ty) < d(x, y)$, while T is referred to as *expansive* if $L > 1$. A point $p \in X$ is called a *fixed point* of an operator T if $p = Tp$ and the collection of all fixed points of an operator T is denoted by $\mathrm{Fix}(T)$. The study of fixed points of contractive and expansive maps still attracts attention of numerous researchers studying extensions and generalizations of the Banach fixed point result. The famous Banach fixed point theorem, called *contraction mapping principle*, remains a basic result in fixed point theory because of the simplicity in the conditional requirements of the theorem on one hand and because of the simple nature of the Picard iteration scheme called *successive approximations*, which gives approximate fixed points with error estimates, on the other hand. The Banach contraction principle in a Banach space $(E, \|\cdot\|)$ asserts that if $T : E \longrightarrow E$ is a contraction (*i.e.*, for all $x, y \in E$, $\|Tx - Ty\| \le L\|x - y\|$ for some $L \in (0, 1)$), then T has a unique fixed point given by $\lim_{n\to\infty} T^n x_0$, where x_0 is any initial point in E. The contraction condition in a complete metric space (X, d) is given by $d(Tx, Ty) \le Ld(x, y)$. There have been many generalizations and extensions of the Banach fixed point theorem, and the approaches used in these generalizations and extensions include (a) replacing the contraction constant L by a suitable function $\phi : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ of $d(x, y)$ to obtain contractive maps [1, 2]; (b) modifying $d(x, y)$ on the *RHS* with displacements of the forms $d(x, Tx)$ and $d(y, Tx)$ [3–6].

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Recently, Wardowski [7] introduced a new type of contraction called F-contraction in his studies of contractive maps and proved a new fixed point theorem concerning F-contractions, for which the Banach contraction principle and some other known contractive conditions in the literature can be obtained as special cases.

Definition 1 [7] Let $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a mapping satisfying:

- F_1. F is strictly increasing, *i.e.*, for all $s, t \in \mathbb{R}_+$ such that $s < t$, $F(s) < F(t)$;
- F_2. For each sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$.
- F_3. There exists $k \in (0,1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

A mapping $T : X \longrightarrow X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\text{for all } x, y \in X, \quad d(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big). \tag{1}$$

The aim of this article is to prove that all F-contractions satisfy condition (2) as follows:

$$\begin{aligned} &\|y - Tx\| \leq M\|x - y\| \\ &\quad \text{whenever } \|x - y\| \leq \|y - Tx\| \end{aligned} \tag{2}$$

for some $M \geq 1$ and for all x and y in a certain subset of a closed convex and bounded subset of a Banach space where $x \neq y$. Examples of F-contractions and some particular types of the function F, which yield various known contractive conditions, are given in [7] where it is also proved that F-contractions have unique fixed points given by the limit of the Picard successive approximation $\lim_{n\to\infty} T^n x_0$ mentioned earlier. The main result proved by Wardowski in [7] is stated below.

Theorem 1.1 [7] *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.*

Examples of contractive conditions making use of the displacements $d(x, Tx)$ and $d(y, Tx)$ is the class of (δ, k)-*weak contractions* introduced and used by Berinde [4] to obtain fixed point and uniqueness theorems for a large class of weakly Picard operators. We give the definition of weak contraction below.

Definition 2 [4] Let X be a metric space, $\delta \in (0,1)$ and $k \geq 0$, then a mapping $T : X \longrightarrow X$ is called (δ, k)-weak contraction (or a weak contraction) if and only if

$$d(Tx, Ty) \leq \delta d(x, y) + k d(y, Tx) \quad \text{for all } x, y \in X. \tag{3}$$

The purpose of this article is to prove that if $T : E \longrightarrow E$ is an F-contraction on a Banach space E, then T satisfies condition (2) which enables us to prove that T is a (δ, k)-weak contraction. In addition we prove as an extension that the averaging operator $S_\lambda = \lambda I + (1 - \lambda)T$ for a nonexpansive mapping T is an example of (δ, k)-weak contraction. It should be recalled that $\mathrm{Fix}(T) = \mathrm{Fix}(S_\lambda)$ for $\lambda \in [0,1)$. We give illustrative examples below to give an insight concerning property (2).

Example 1.1 Let E be real numbers $\mathbb{R}$, let $K \subset \mathbb{R}$ be a closed interval $[0,1]$ and define $T : K \to K$ by $Tx = e^{-x}$. Clearly, T is contractive since, by a mean value theorem, $|e^{-x} - e^{-y}| = e^{-c_{xy}}|x-y|$ for some constant $c_{xy} \in (0,1)$ for each $x, y \in K$. This yields $|e^{-x} - e^{-y}| < e^0|x-y|$ giving the desired result $|e^{-x} - e^{-y}| < |x-y|$. But T does not satisfy condition (2) as shown below: For all $x, y \in K = [0,1]$, we have $|x-y| \leq |y - e^{-x}|$ and, given any constant $M \geq 1$, we can find an open neighborhood of zero K_0 such that $\frac{|y-e^{-x}|}{|x-y|} > M$ for some distinct $x, y \in K_0$. By our hypothesis, T may not have a fixed point in K_0 which is in concrete agreement with reality.

On the other hand, let X and K be, respectively, $\mathbb{R}$ and $[0,1]$, while $T : K \to K$ is defined by $Tx = xe^{-x}$. Similarly, T is contractive since, by a mean value theorem, $|xe^{-x} - ye^{-y}| = (1-c_{xy})e^{-c_{xy}}|x-y| < |x-y|$. In this case, T satisfies our hypothesis and also has a fixed point, as shown below, in an appropriate open neighborhood K_0 of zero: $|x-y| \leq |y - xe^{-x}|$ for all $x, y \in K$, but we can find $M \geq 1$ such that $|y - xe^{-x}| \leq M|x-y|$. We observe that since $|x-y| > 0$, $\frac{|y-xe^{-x}|}{|x-y|} \leq \frac{|x-y|}{|x-y|} + \frac{x(1-e^{-x})}{|x-y|} < M$ for all x and y in an appropriate neighborhood of zero K_0. It is remarkable that in agreement with our hypothesis, T has a unique fixed point $0 \in K_0$. We shall prove in this work that this holds for all contractive maps, *i.e.*, all contractive operators that satisfy condition (2) have unique fixed points, by proving that it is true for nonexpansive operators.

Other examples of (δ, k)-weak contractions are given in [4, 8]. It is shown in [3, 4] that a lot of well-known contractive conditions in literature are special cases of weak contraction condition (3) as it does not require that $\delta + k$ be less than 1, which is assumed in almost all fixed point theorems based on the contractive condition which involves displacements of the forms $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, $d(y,Tx)$ (see, for example, Berinde [3], Kannan [5] and Zamfirescu [6]). In [4] Berinde proved the theorem below.

Theorem 1.2 *Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a* (δ,k)*-weak contraction. Then*

(1) $\operatorname{Fix}(T) = \{x \in X : Tx = x\} \neq \emptyset$.

(2) *For any* $x_0 \in X$, *the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$ *converges to some* $x^* \in \operatorname{Fix}(T)$.

(3) *The estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0,1,2,\ldots, \tag{4}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1,2,\ldots, \tag{5}$$

hold, where δ *is the constant appearing in* (3).

(4) *Under the additional condition that there exist* $\theta \in (0,1)$ *and some* $k_1 \geq 0$ *such that*

$$d(Tx, Ty) \leq \theta . d(x,y) + k_1 . d(x, Tx) \quad \text{for all } x, y \in X, \tag{6}$$

the fixed point x^* *is unique and the Picard iteration converges at the rate* $d(x_n, x^*) \leq \theta d(x_{n-1}, x^*)$, $n \in \mathbb{N}$.

In the sequel we shall make use of the following result in connection with metric and normed linear spaces.

Proposition 1.1 *Let Y be a subset of a complete metric space (X,d). Then (Y,d) is a metric space and (Y,d) is complete if and only if Y is closed in X.*

2 Main results

We now present our main results. It should be mentioned here that in our extension of Wardowski result on F-contractions we no longer guarantee the uniqueness of fixed points. Our first result is concerned with the inclusion of F-contractions in (δ,k)-weak contractions, while the second result is an extension of the Wardowski result. First, we prove the following important and supporting lemma.

Lemma 2.1 *Let V be a real normed linear space and let $T : V \longrightarrow V$ be any map. If $\|x - y\| \leq \|y - Tx\|$, then $Tx \neq Ty$ for any distinct $x,y \in V$ satisfying $x,y \notin \mathrm{Fix}(T)$.*

Proof Given a self-mapping T of a real normed linear space V with $\|x - y\| \leq \|y - Tx\|$, where $x,y \in V$ satisfy $x \neq y$; $x,y \notin \mathrm{Fix}(T)$, let $z = \lambda_0 x + (1 - \lambda_0)Ty$ be a projection of the point y onto the line segment $[x, Ty]$ for some $\lambda_0 \in [0,1]$, then $\|y - z\| \leq \|y - Ty\|$. This yields $\|y - (\lambda_0 x + (1 - \lambda_0)Ty)\| \leq \|y - Ty\|$. When y, z and Ty are colinear (in particular when V is one-dimensional), we have

$$\|y - Ty\| + \lambda_0\|x - Ty\| \leq \|y - Ty\|. \tag{7}$$

It is clear that $Tx = Ty$ in (7) implies $x \in \mathrm{Fix}(T)$, which is a contradiction; therefore $Tx \neq Ty$. The end of the proof. □

Theorem 2.1 *Let E be a real Banach space, let K be a bounded closed and convex subset of E and $T : K \to K$. If T is an F-contraction, then*

i. *There exists an open subset $K_1 \subset K$ such that T satisfies the following condition:*

$$\|y - Tx\| \leq M\|x - y\| \quad \textit{whenever } \|x - y\| \leq \|y - Tx\|$$

for some $M \geq 1$ and for all $x,y \in K_1$; $x \neq y$, $x,y \notin \mathrm{Fix}(T)$.

ii. *The F-contraction T is a (δ,k)-weak contraction.*

iii. *The averaged operator $S_\lambda = \lambda I + (1 - \lambda)T$ is a (δ,k)-weak contraction and T and S_λ have a common unique fixed point in K for $\lambda \in (0,1)$.*

Proof Given a bounded closed convex subset K of a real Banach space E. Let $T : K \longrightarrow K$ be an F-contraction and let Z denote the collection of elements of K such that if $x,y \in Z$ then $\|x - y\| \leq \|y - Tx\|$, $x \neq y$, $x,y \notin \mathrm{Fix}(T)$. We shall show that (2) is satisfied whenever and $\|x - y\| \leq \|y - Tx\|$ in an open set $K_1 \subset K$ to be derived shortly. Combining these inequalities, we obtain the following:

$$\|Tx - Ty\| < \|x - y\|, \tag{8}$$

$$\|x - y\| \leq \|y - Tx\|$$

$$\Longrightarrow \quad \|Tx - Ty\| \leq \|y - Tx\|. \tag{9}$$

Adding (8) and (9) yields

$$\begin{aligned} &2\|Tx - Ty\| \le \|x - y\| + \|y - Tx\| \\ &\quad\Longrightarrow \quad \|y - Tx\| - \|y - Ty\| \le \frac{1}{2}\|x - y\| + \frac{1}{2}\|y - Tx\| \\ &\quad\Longrightarrow \quad \|y - Tx\| \le \|x - y\| + 2\|y - Ty\|. \end{aligned} \tag{10}$$

Set $K_o = Z \cap \{T^n x_0\}_{n\ge1}$, where $\{T^n x_0\}_{n\ge1}$ is a sequence of successive approximations starting from an arbitrary $x_0 \in K$. Clearly, $K_o \neq \emptyset$ since we can always find $n, m \in \mathbb{N}$ such that $\|T^n x_0 - T^m x_0\| \le \|T^m x_0 - T^{n+1} x_0\|$ for any $x_0 \in K$ and by Lemma 2.1 $T^m x_0 \neq T^{n+1} x_0$ if $x_0, Tx_0 \notin \mathrm{Fix}(T)$. It follows, in this case, that $\|y - Ty\|$ takes the form $\|x_m - x_{m+1}\|$, while $\|x - y\| \le \|y - Tx\|$ takes the form $\|x_n - x_m\| \le \|x_m - x_{n+1}\|$, where $x_k = T^k x_0$, $k = 1, 2, \ldots$. Clearly, $\|x_m - x_{m+1}\| \le \|x_n - x_m\|$ since the condition $\|x_n - x_m\| \le \|x_m - x_{n+1}\|$ implies that $n \ge m$. This means that $\|y - Ty\| \le \|x - y\|$ in K_o so, in K_o, (10) yields $\|y - Tx\| \le 3\|x - y\|$. Let K_1 be the smallest open set in K containing K_o. Using the continuity of T, we conclude that if T is an F-contraction, then condition (2) is implied in K_1. This proves i.

ii. By Theorem 1.1, F-contractions have unique fixed points x^*. In this case we can take K_1 to be the open ball $B(x^*; r)$ centered at x^* with radius r. Clearly, $T : B(x^*; r) \to B(x^*; r)$ for a suitable value of r. In $B(x^*; r)$ we have $\|Tx - Ty\| < \|x - y\|$ since F-contractions are contractive operators. We shall show that this contractive condition and boundedness of $B(x^*; r)$ imply that F-contractions are (δ, k)-weak contractions. This follows from the fact that $\|Tx - Ty\| < \delta\|x - y\| + (1 - \delta)(\|y - Tx\| + \|y - Ty\|) \le \delta\|x - y\| + (1 - \delta)(M + 1)\|y - Tx\|$.

The proof of iii. is an application of Theorem 1.2 and Proposition 1.1. Recall that Proposition 1.1 asserts that $(K, \|\cdot\|)$ is a complete metric space since E is a Banach space and K is a closed subset of E. It suffices to show that the averaging operator S_λ (for a contractive map T) is a (δ, k)-weak contraction on K; the existence of a fixed point in K follows via Theorem 1.2.

Next, let S_λ, $\lambda \in (0, 1)$ denote an averaging operator, *i.e.*, $S_\lambda = \lambda I + (1 - \lambda)T$, we obtain

$$\begin{aligned} \|S_\lambda x - S_\lambda y\| &= \bigl\|\lambda x + (1 - \lambda)Tx - \bigl[\lambda y + (1 - \lambda)Ty\bigr]\bigr\| \\ &\le \bigl\|y - \bigl[\lambda x + (1 - \lambda)Tx\bigr]\bigr\| + (1 - \lambda)\|y - Ty\| \\ &\le \|y - S_\lambda x\| + (1 - \lambda)\|y - Tx\| + (1 - \lambda)\|Tx - Ty\|. \end{aligned} \tag{11}$$

When $\|y - Tx\| \le \|x - y\|$, then (11) yields $\|S_\lambda x - S_\lambda y\| \le 2(1 - \lambda)\|x - y\| + \|y - S_\lambda x\|$ and we are done. It is easy to show, in K_1 derived above, that $\|x_n - x_m\| \longrightarrow 0$ as $n, m \longrightarrow \infty$. But in K_1, when $\|x - y\| \le \|y - Tx\|$, then by (2) equation (11) yields $4(1 - \lambda)\|x - y\| + \|y - S_\lambda x\|$. So, choosing $\lambda \in (0, 1)$ such that $(1 - \lambda) < \min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}$, we conclude that S_λ is a (δ, k)-weak contraction with $\delta \ge \frac{1-\lambda}{4}$ and $k = 1$, where λ satisfies $4(1 - \lambda) < 1$. Therefore, by Theorem 1.2 T has a fixed point in K, and successive approximations for S_λ converge to a fixed point of T in K. The end of the proof. □

From the above method of the proof of Theorem 2.1, we obtain a fixed point theorem stated below for contractive mappings satisfying condition (2).

Theorem 2.2 *Let E be a real Banach space, let K be a bounded closed and convex subset of E and $T : K \to K$. If T is a nonexpanive map such that there exist a constant $M \ge 1$ and* *an open subset* $K_1 \subset K$ *such that* T *satisfies the following condition*:

$$\|y - Tx\| \leq M\|x - y\| \quad \textit{whenever } \|x - y\| \leq \|y - Tx\|$$

for all $x, y \in K_1$; *with* $x \neq y$, $x, y \notin \mathrm{Fix}(T)$. *Then the averaged operator* $S_\lambda = \lambda I + (1 - \lambda)T$ *is a* (δ, k)-*weak contraction and* T *has a fixed point in* K.

Remark 2.1 We conclude with the following observations:

1. The uniqueness of fixed points of nonexpansive maps (whenever they exist) is guaranteed if condition (6) in Theorem 1.2 is satisfied for a corresponding averaged operator S_λ.
2. Condition (2), Lemma 2.1 and Theorem 2.1 together with other properties of (δ, k)-weak contractions constitute the properties of F-contractions and the properties of nonexpansive operators with a nonempty fixed point set $\mathrm{Fix}(T)$.
3. The bounded closed convex subset K can be replaced with a closed convex subset K, with emphasis on the boundedness of the open set K_1.

Competing interests

The author declares that he has no competing interests.

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References

1. Boyd, DW, Wong, JSW: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458-464 (1969)
2. Geraghty, MA: On contractive mappings. Proc. Am. Math. Soc. **40**(2), 604-608 (1973)
3. Berinde, V: On the approximation of fixed point of weak contraction mappings. Carpath. J. Math. **19**(1), 7-22 (2003)
4. Berinde, V: Approximating fixed point of weak contractions using Picard's iteration. Nonlinear Anal. Forum **9**(1), 43-53 (2004)
5. Kannan, R: Some result on fixed points. Bull. Calcutta Math. Soc. **10**, 71-76 (1968)
6. Zamfirescu, T: Fixed point theorem in metric spaces. Arch. Math. **23**, 292-298 (1992)
7. Wardowski, D: Fixed points of a new contractive type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
8. Berinde, M, Berinde, V: On a general class of multivalued weakly Picard mappings. J. Math. Anal. Appl. **326**, 772-782 (2007)

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