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Generalized viscosity approximation methods for nonexpansive mappings

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Abstract

We combine a sequence of contractive mappings $\{f_n\}$ and propose a generalized viscosity approximation method. One side, we consider a nonexpansive mapping S with the nonempty fixed point set defined on a nonempty closed convex subset C of a real Hilbert space H and design a new iterative method to approximate some fixed point of S, which is also a unique solution of the variational inequality. On the other hand, using similar ideas, we consider N nonexpansive mappings $\{S_i\}_{i=1}^N$ with the nonempty common fixed point set defined on a nonempty closed convex subset C. Under reasonable conditions, strong convergence theorems are proven. The results presented in this paper improve and extend the corresponding results reported by some authors recently.

MSC: 47H09; 47H10; 47J20; 47J25

Keywords: nonexpansive mapping; contractive mapping; variational inequality; fixed point; viscosity approximation method

1 Introduction

Let H be a real Hilbert space with an inner product $\langle , \rangle$ and norm $\| \cdot \|$, and C be a nonempty closed convex subset of H.

Let $S : C \to C$ be a nonlinear mapping, we use $\mathrm{Fix}(S)$ to denote the set of fixed points of S (*i.e.*, $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$). A mapping is called nonexpansive if the following inequality holds:

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$.

In 1967, Halpern [1] used contractions to approximate a nonexpansive mapping and considered the following explicit iterative process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n, \quad \forall n \geq 0,$$

where u is a given point and $S : C \to C$ is nonexpansive. He proved the strong convergence of $\{x_n\}$ to a fixed point of S provided that $\alpha_n = n^{-\theta}$ with $\theta \in (0,1)$. In 2000, Moudafi [2] introduced the viscosity approximation method for nonexpansive mappings. Until now, in many references, viscosity approximation methods still are used and studied, which formally generates the sequence $\{x_n\}$ by the recursive formula:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n,$$

where f is a contraction and $\alpha_n \subset (0,1)$ is a slowly vanishing sequence. See, for instance, [3–6]. In fact, Yamada's hybrid steepest descent algorithm is also a kind of viscosity approximation method (see [7]).

The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

In recent years, the theory of variational inequality has been extended to the study of a large variety of problems arising in structural analysis, economics, engineering sciences, and so on. See [8–10] and the references cited therein.

Recently, Zhou and Wang [11] proposed a simpler explicit iterative algorithm for finding a solution of variational inequality over the set of common fixed points of a finite family nonexpansive mappings. They introduced an explicit scheme as follows.

Theorem 1.1 *Let H be a real Hilbert space and $F : H \to H$ be an L-Lipschitz continuous and η-strongly monotone mapping. Let $\{S_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $C = \bigcap_{i=1}^N \operatorname{Fix}(S_i) \neq \emptyset$. For any point $x_0 \in H$, define a sequence $\{x_n\}$ in the following manner*:

$$x_{n+1} = (I - \lambda_n \mu F)S_N^n S_{N-1}^n \cdots S_1^n x_n, \quad n \geq 0, \tag{1.2}$$

where $\mu \in (0, 2\eta/L^2)$ and $S_i^n := (1 - \beta_n^i)I + \beta_n^i S_i$ for $i = 1, 2, \ldots, N$. When the parameters satisfy appropriate conditions, the sequence converges strongly to the unique solution of the variational inequality (1.1).

In this paper, motivated by the above works, we introduce a more generalized iterative method like viscosity approximation. In Section 3, we combine a sequence of contractive mappings and obtain strong convergence theorem for approximating fixed point of a nonexpansive mapping. In Section 4, we propose a new iterative algorithm for finding some common fixed point of a finite family nonexpansive mappings, which is also a unique solution for the variational inequality over the set of fixed point of these mappings on Hilbert spaces.

2 Preliminaries

In order to prove our results, we collect some facts and tools in a real Hilbert space H, which are listed as below.

Lemma 2.1 *Let H be a real Hilbert space. We have the following inequalities*:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle x + y, y\rangle$, $\forall x, y \in H$.
(ii) $\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2$, $\forall t \in [0,1]$, $\forall x, y \in H$.

Lemma 2.2 [12] *Let $\{S_i\}_{i=1}^2$ be γ_i-averaged on C such that $\operatorname{Fix}(S_1) \cap \operatorname{Fix}(S_2) \neq \emptyset$. Then the following conclusions hold*:

(i) *both* S_1S_2 *and* S_2S_1 *are* γ*-averaged, where* $\gamma = \gamma_1 + \gamma_2 - \gamma_1\gamma_2$;
(ii) $\mathrm{Fix}(S_1) \cap \mathrm{Fix}(S_2) = \mathrm{Fix}(S_1S_2) = \mathrm{Fix}(S_2S_1)$.

Recall that given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x - P_Cx\| \leq \|x - y\|$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C.

Lemma 2.3 [13] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given* $x \in H$ *and* $z \in C$, *then* $y = P_Cx$ *if and only if we have the relation*

$$\langle x - y, y - z\rangle \geq 0 \quad \textit{for all } z \in C.$$

Lemma 2.4 [10] *Let H be a Hilbert space and C be a nonempty closed convex subset of H, and* $T : C \to C$ *a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If* $\{x_n\}$ *is a sequence in C weakly converging to x and if* $\{(I - T)x_n\}$ *converges strongly to y, then* $(I - T)x = y$.

Lemma 2.5 [5] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $$\sum_{n=1}^{\infty} \gamma_n = \infty;$$

(ii) $$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0 \quad \textit{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [14] *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space and* $\{\beta_n\}$ *be a sequence of real numbers such that* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *for all* $n = 0,1,2,\ldots$. *Suppose that* $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ *for all* $n = 0,1,2,\ldots$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

3 Generalized viscosity approximation method combining with a nonexpansive mapping

In this section, we combine a sequence of contractive mappings and apply a more generalized iterative method like viscosity approximation to approximate some fixed point of a nonexpansive mapping defined on a closed convex subset C of a Hilbert space H, which is also the solution of the variational inequality

$$\langle f(x^*) - x^*, p - x^*\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(S). \tag{3.1}$$

Suppose the contractive mapping sequence $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C. The uniform convergence of $\{f_n(x)\}$ on D is denoted by $f_n(x) \rightrightarrows f(x)$ $(n \to \infty)$, $x \in D$.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{f_n\}$ be a sequence of ρ_n-contractive self-maps of C with $0 \le \rho_l = \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Let $S : C \to C$ be a nonexpansive mapping. Assume the set $\operatorname{Fix}(S) \neq \emptyset$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C. Given $x_1 \in C$, let $\{x_n\}$ be generated by the following algorithm*:

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n)Sx_n. \tag{3.2}$$

If the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then the sequence $\{x_n\}$ converges strongly to a point $x^* \in \operatorname{Fix}(S)$, which is also the unique solution of the variational inequality (3.1).

Proof The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is bounded.

For any $q \in \operatorname{Fix}(S)$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \big\|\alpha_n f_n(x_n) + (1-\alpha_n)Sx_n - q\big\| \\
&\le \alpha_n \big\|f_n(x_n) - q\big\| + (1-\alpha_n)\|Sx_n - Sq\| \\
&\le \alpha_n\rho_n\|x_n - q\| + (1-\alpha_n)\|x_n - q\| + \alpha_n\big\|f_n(q) - q\big\| \\
&\le \big(1 - \alpha_n(1-\rho_n)\big)\|x_n - q\| + \alpha_n(1-\rho_n)\frac{\|f_n(q) - q\|}{1-\rho_n} \\
&\le \max\left\{\|x_n - q\|, \frac{\|f_n(q) - q\|}{1-\rho_n}\right\}.
\end{aligned}$$

From the uniform convergence of $\{f_n\}$ on D, it is easy to get the boundedness of $\{f_n(q)\}$. Thus there exists a positive constant M_1, such that $\|f_n(q) - q\| \le M_1$. By induction, we obtain $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{M_1}{1-\rho_u}\}$. Hence, $\{x_n\}$ is bounded, so are $\{Sx_n\}$ and $\{f_n(x_n)\}$.

Step 2. Show that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Indeed, observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|\alpha_n f_n(x_n) + (1-\alpha_n)Sx_n - \alpha_{n-1}f_{n-1}(x_{n-1}) - (1-\alpha_{n-1})Sx_{n-1}\big\| \\
&= \big\|\alpha_n\big(f_n(x_n) - f_n(x_{n-1})\big) + \alpha_n\big(f_n(x_{n-1}) - f_{n-1}(x_{n-1})\big) \\
&\quad + (\alpha_n - \alpha_{n-1})\big(f_{n-1}(x_{n-1}) - Sx_{n-1}\big) + (1-\alpha_n)(Sx_n - Sx_{n-1})\big\| \\
&\le \alpha_n\rho_n\|x_n - x_{n-1}\| + \alpha_n\big\|f_n(x_{n-1}) - f_{n-1}(x_{n-1})\big\|
\end{aligned}$$

$$\begin{aligned}&+ |\alpha_n - \alpha_{n-1}|\big(\|Sx_n\| + \|f_{n-1}(x_{n-1})\|\big) + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &= \big(1 - \alpha_n(1-\rho_n)\big)\|x_n - x_{n-1}\| + \alpha_n \|f_n(x_{n-1}) - f_{n-1}(x_{n-1})\| \\ &+ |\alpha_n - \alpha_{n-1}|\big(\|Sx_n\| + \|f_{n-1}(x_{n-1})\|\big).\end{aligned}$$

By the conditions (i)-(iii) and the uniform convergence of $f_n(x)$, we have

$$\frac{\alpha_n\|f_n(x_{n-1}) - f_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|Sx_n\| + \|f_{n-1}x_{n-1}\|)}{\alpha_n(1-\rho_n)} \to 0$$

as $n \to \infty$. By Lemma 2.5, (3.3) holds.

Step 3. Show that

$$\|Sx_n - x_n\| \to 0. \tag{3.4}$$

Since

$$\|Sx_n - x_n\| \le \|x_{n+1} - x_n\| + \|x_{n+1} - Sx_n\|.$$

By the condition (i), we have $\|x_{n+1} - Sx_n\| = \alpha_n\|f_n(x_n) - Sx_n\| \to 0$. Combining with (3.3), it is easy to get (3.4).

Step 4.

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{3.5}$$

where $x^* = P_{\mathrm{Fix}(S)}f(x^*)$ is a unique solution of the variational inequality (3.1).

Since $f_n(x)$ is uniformly convergent on D, we have $\lim_{n\to\infty}(f_n(x^*) - x^*) = f(x^*) - x^*$.

Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle. \tag{3.6}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we can assume $x_{n_j} \rightharpoonup \hat{x}$. From (3.4), we obtain $Sx_{n_j} \rightharpoonup \hat{x}$. Using Lemma 2.4, we have $\hat{x} \in \mathrm{Fix}(S)$. Since $x^* = P_{\mathrm{Fix}(S)}f(x^*)$, we get

$$\lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle = \langle f(x^*) - x^*, \hat{x} - x^*\rangle \le 0.$$

Combining with (3.6), the inequality (3.5) holds.

Step 5. Show that

$$x_n \to x^*, \tag{3.7}$$

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad= \|\alpha_n f_n(x_n) + (1-\alpha_n)Sx_n - x^*\|^2 \\ &\quad\le (1-\alpha_n)^2\|Sx_n - x^*\|^2 + 2\alpha_n\langle x_{n+1} - x^*, f_n(x_n) - x^*\rangle\end{aligned}$$

$$\begin{aligned}
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle x_{n+1} - x^*, f_n(x_n) - f_n(x^*)\rangle + 2\alpha_n \langle x_{n+1} - x^*, f_n(x^*) - x^*\rangle \\
&\le (1-\alpha_n)^2 \|x_n - x^*\|^2 + \alpha_n \rho_n \left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + 2\alpha_n \langle x_{n+1} - x^*, f_n(x^*) - x^*\rangle.
\end{aligned}$$

Transform the inequality into another form, we obtain

$$\|x_{n+1} - x^*\|^2 \le \left(1 - \frac{\alpha_n(2-\alpha_n-2\rho_n)}{1-\alpha_n\rho_n}\right)\|x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\rho_n}\langle x_{n+1} - x^*, f_n(x^*) - x^*\rangle.$$

By Schwartz's inequality, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle x_{n+1} - x^*, f_n(x^*) - x^*\rangle \\
&\quad \le \lim_{n\to\infty} \|x_{n+1} - x^*\| \|f_n(x^*) - f(x^*)\| + \limsup_{n\to\infty} \langle x_{n+1} - x^*, f(x^*) - x^*\rangle.
\end{aligned}$$

By the boundedness of $\{x_n\}$, $f_n(x) \rightrightarrows f(x)$, (3.3) and (3.5), we have

$$\limsup_{n\to\infty} \langle x_{n+1} - x^*, f_n(x^*) - x^*\rangle \le 0.$$

It follows from Lemma 2.5 that (3.7) holds. □

Remark 3.2 In [2], Moudafi proposed the viscosity iterative algorithm as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Sx_n, \tag{3.8}$$

where f is a contraction on H. It is a special case of (3.2) in this paper when $f_1 = f_2 = \cdots = f_n = \cdots = f$, $\forall n \in \mathbb{N}$ and $C = H$. Of course, Halpern's iteration method is also a special case of (3.2) when $f_1 = f_2 = \cdots = f_n = \cdots = u$, $\forall n \in \mathbb{N}$.

Remark 3.3 In [7], the following iterative process was introduced:

$$x_{n+1} = Sx_n - \mu\alpha_n F(Sx_n).$$

Rewriting the equation, we get

$$\begin{aligned}
x_{n+1} &= \alpha_n (I - \mu F)Sx_n + (1-\alpha_n)Sx_n \\
&= \alpha_n f(x_n) + (1-\alpha_n)Sx_n.
\end{aligned} \tag{3.9}$$

It is easily to verify $f := (I - \mu F)S$ is a contractive mapping on H when $0 < \mu < 2\eta/L^2$. That is, Yamada's method is a kind of viscosity approximation method. Of course it is also a special case of Theorem 3.1.

4 Generalized viscosity approximation method combining with a finite family of nonexpansive mappings

In this section, we apply a more generalized iterative method like viscosity approximation to approximate a common element of the set of fixed points of a finite family of nonexpansive mappings on Hilbert spaces.

Let $\{f_n\}$ be a sequence of ρ_n-contractive self-maps of C with $0 < \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$ and $\{S_i\}_{i=1}^N$ be N nonexpansive self-mapping of C. Assume the common fixed point set $F = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$ and $\{f_n(q)\}$ is convergent for any $q \in F$. Put $f(q) := \lim_{n\to\infty} f_n(q)$, since every f_n is ρ_n-contractive, we have

$$\|f_n(p) - f_n(q)\| \leq \rho_n \|p-q\| \leq \rho_u \|p-q\|$$

for any $p, q \in F$. Further we obtain $\|f(p) - f(q)\| \leq \rho_u \|p-q\|$. Next we prove the sequence $\{x_n\}$ converges strongly to a point $x^* \in F = \bigcap_{i=1}^N \mathrm{Fix}(S_i)$, which also solves the variational inequality

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0, \quad \forall p \in F. \tag{4.1}$$

As we know, it is equivalent to the fixed point equation $x^* = P_F f(x^*)$.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{f_n\}$ be a sequence of ρ_n-contractive self-maps of C with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Let, for each $1 \leq i \leq N$ ($N \geq 1$ be an integer), $S_i : C \to C$ be a nonexpansive mapping. Assume the set $F = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$ and $\{f_n(q)\}$ is convergent for any $q \in F$. Given $x_1 \in C$, let $\{x_n\}$ be generated by the following algorithm*:

$$\begin{cases} x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n) S_N^n S_{N-1}^n \cdots S_1^n x_n, \\ S_i^n = (1-\lambda_i^n) I + \lambda_i^n S_i, \quad i = 1, 2, \ldots, N. \end{cases} \tag{4.2}$$

If the parameters $\{\alpha_n\}$ and $\{\lambda_i^n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\lambda_i^n \in (\lambda_l, \lambda_u)$ *for some* $\lambda_l, \lambda_u \in (0,1)$ *and* $\lim_{n\to\infty} |\lambda_i^n - \lambda_i^{n+1}| = 0$, $\forall i = 1, 2, \ldots, N$,

then the sequence $\{x_n\}$ converges strongly to a point $x^* \in F$, which is also the unique solution of the variational inequality (4.1).

Proof We will prove the theorem in the case of $N = 2$. The proof is divided into several steps.

Step 1. We show first that $\{x_n\}$ is bounded.

For any $q \in F$, we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n f_n(x_n) + (1-\alpha_n) S_2^n S_1^n x_n - q\| \\ &\leq \alpha_n \|f_n(x_n) - q\| + (1-\alpha_n) \|S_2^n S_1^n x_n - S_2^n S_1^n q\| \\ &\leq \alpha_n \rho_n \|x_n - q\| + (1-\alpha_n)\|x_n - q\| + \alpha_n \|f_n(q) - q\| \\ &\leq \big(1 - \alpha_n(1-\rho_n)\big)\|x_n - q\| + \alpha_n(1-\rho_n) \frac{\|f_n(q) - q\|}{1-\rho_n} \\ &\leq \max\left\{\|x_n - q\|, \frac{\|f_n(q) - q\|}{1-\rho_n}\right\}. \end{aligned}$$

From the convergence of $\{f_n(q)\}$, it is easy to get the boundness of $\{f_n(q)\}$. Thus there exists a positive constant M_1, such that $\|f_n(q) - q\| \leq M_1$. By induction, we obtain $\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{M_1}{1-\rho_u}\}$. Hence, $\{x_n\}$ is bounded, and so are $\{S_1 x_n\}$ and $\{S_2^n S_1^n x_n\}$.

Step 2. We show that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.3}$$

Since both S_2^n and S_1^n are averaged nonexpansive mappings, by Lemma 2.2, $S_2^n S_1^n$ is also averaged. Rewrite $S_2^n S_1^n = (1-\beta_n)I + \beta_n V_n$, where $\beta_n = \lambda_1^n + \lambda_2^n - \lambda_1^n \lambda_2^n$. Then we have

$$\begin{aligned}
x_{n+1} &= \alpha_n f_n(x_n) + (1-\alpha_n)\big[(1-\beta_n)I + \beta_n V_n\big]x_n \\
&= \alpha_n f_n(x_n) + (1-\beta_n)x_n - \alpha_n(1-\beta_n)x_n + (1-\alpha_n)\beta_n V_n x_n \\
&= (1-\beta_n)x_n + \beta_n\left[\alpha_n \frac{f_n(x_n) - (1-\beta_n)x_n}{\beta_n} + (1-\alpha_n)V_n x_n\right] \\
&= (1-\beta_n)x_n + \beta_n z_n.
\end{aligned}$$

Further we obtain

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad = \left\| \frac{\alpha_{n+1}}{\beta_{n+1}}\big[f_{n+1}(x_{n+1}) - (1-\beta_{n+1})x_{n+1}\big] + (1-\alpha_{n+1})V_{n+1}x_{n+1} \right. \\
&\qquad \left. - \frac{\alpha_n}{\beta_n}\big[f_n(x_n) - (1-\beta_n)x_n\big] - (1-\alpha_n)V_n x_n \right\| \\
&\quad = \|V_{n+1}x_{n+1} - V_n x_n\| + \left\| \left[\frac{\alpha_{n+1}}{\beta_{n+1}} f_{n+1}(x_{n+1}) - \frac{\alpha_n}{\beta_n} f_n(x_n)\right] \right. \\
&\qquad \left. - \left[\frac{\alpha_{n+1}(1-\beta_{n+1})}{\beta_{n+1}}x_{n+1} - \frac{\alpha_n(1-\beta_n)}{\beta_n}x_n\right] - \alpha_{n+1}V_{n+1}x_{n+1} + \alpha_n V_n x_n \right\| \\
&\quad \le \|x_{n+1} - x_n\| + \|V_{n+1}x_n - V_n x_n\| + \left| \frac{\alpha_{n+1}}{\beta_{n+1}} f_{n+1}(x_{n+1}) - \frac{\alpha_n}{\beta_n} f_n(x_n) \right| \\
&\qquad + \left\| \frac{\alpha_{n+1}(1-\beta_{n+1})}{\beta_{n+1}}x_{n+1} - \frac{\alpha_n(1-\beta_n)}{\beta_n}x_n \right\| \\
&\qquad + \|\alpha_{n+1}V_{n+1}x_{n+1} - \alpha_n V_n x_n\|.
\end{aligned} \tag{4.4}$$

Write $\lambda_1 = 2\lambda_l - \lambda_l^2$, $\lambda_2 = 2\lambda_u - \lambda_u^2$. From the condition (iii), it is easily to get $0 < \lambda_1 \le \beta_n \le \lambda_2$ and $\beta_{n+1} - \beta_n \to 0$ as $n \to \infty$. We have

$$\begin{aligned}
\|V_{n+1}x_n - V_n x_n\| &= \left\| \frac{S_2^{n+1}S_1^{n+1} - (1-\beta_{n+1})I}{\beta_{n+1}}x_n - \frac{S_2^n S_1^n - (1-\beta_n)I}{\beta_n}x_n \right\| \\
&\le \left\| \frac{S_2^{n+1}S_1^{n+1}}{\beta_{n+1}}x_n - \frac{S_2^n S_1^n}{\beta_n}x_n \right\| + \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n+1}} \right| \|x_n\| \\
&\le \frac{1}{\beta_n}\left\| S_2^{n+1}S_1^{n+1}x_n - S_2^n S_1^n x_n \right\| + \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n+1}} \right| \left(\left\| S_2^{n+1}S_1^{n+1}x_n \right\| + \|x_n\| \right) \\
&\le \frac{1}{\lambda_1}\left(\left\| S_1^{n+1}x_n - S_1^n x_n \right\| + \left\| S_2^{n+1}S_1^n x_n - S_2^n S_1^n x_n \right\| \right) \\
&\quad + \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n+1}} \right| \left(\left\| S_2^{n+1}S_1^{n+1}x_n \right\| + \|x_n\| \right)
\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{\lambda_1}\left(\left\|S_1^{n+1}x_n - S_1^n x_n\right\| + \left\|S_2^{n+1}S_1^n x_n - S_2^n S_1^n x_n\right\|\right)\\ &\quad + \left|\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right| M_2,\end{aligned} \tag{4.5}$$

where $M_2 = \sup_n\{\|S_2^{n+1}S_1^{n+1}x_n\| + \|x_n\|\}$. Since $|\lambda_i^{n+1} - \lambda_i^n| \to 0$, $i = 1, 2$, we can deduce

$$\left\|S_1^{n+1}x_n - S_1^n x_n\right\| \leq \left|\lambda_1^{n+1} - \lambda_1^n\right|\left(\|x_n\| + \|S_1 x_n\|\right) \to 0 \tag{4.6}$$

and

$$\left\|S_2^{n+1}S_1^n x_n - S_2^n S_1^n x_n\right\| \leq \left|\lambda_2^{n+1} - \lambda_2^n\right|\left(\left\|S_1^n x_n\right\| + \left\|S_2 S_1^n x_n\right\|\right) \to 0. \tag{4.7}$$

Substituting (4.5) into (4.4), we have

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\\ &\quad \leq \frac{1}{\lambda_1}\left(\left\|S_1^{n+1}x_n - S_1^n x_n\right\| + \left\|S_2^{n+1}S_1^n x_n - S_2^n S_1^n x_n\right\|\right) + \frac{|\beta_n - \beta_{n+1}|}{\beta_n\beta_{n+1}} M_2\\ &\qquad + \left\|\frac{\alpha_{n+1}}{\beta_{n+1}} f_{n+1}(x_{n+1}) - \frac{\alpha_n}{\beta_n} f_n(x_n)\right\| + \left\|\frac{\alpha_{n+1}(1-\beta_{n+1})}{\beta_{n+1}} x_{n+1} - \frac{\alpha_n(1-\beta_n)}{\beta_n} x_n\right\|\\ &\qquad + \|\alpha_{n+1}V_{n+1}x_{n+1} - \alpha_n V_n x_n\|.\end{aligned}$$

Combining (4.6), (4.7), and condition (i), we get

$$\limsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \leq 0.$$

By Lemma 2.6, we conclude that $\lim_{n\to\infty}\|z_n - x_n\| \to 0$. Further we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|z_n - x_n\| \to 0.$$

Step 3. We show that

$$\left\|S_2^n S_1^n x_n - x_n\right\| \to 0. \tag{4.8}$$

By (4.2), we get

$$\left\|x_{n+1} - S_2^n S_1^n x_n\right\| = \alpha_n\left\|f_n(x_n) - S_2^n S_1^n x_n\right\| \to 0.$$

We have

$$\left\|x_n - S_2^n S_1^n x_n\right\| \leq \left\|x_{n+1} - S_2^n S_1^n x_n\right\| + \|x_n - x_{n+1}\|.$$

Combining with (4.3), (4.8) holds.

Since $\{\lambda_i^n\} \subset (\lambda_l, \lambda_u)$, we can assume that $\lambda_i^{n_j} \to \lambda_i^0$ as $n \to \infty$. It is easy to get $0 < \lambda_i^0 < 1$ for $i = 1, 2$. Write $S_i^0 = (1 - \lambda_i^0)I + \lambda_i^0 S_i$, $i = 1, 2$. Then we have $\mathrm{Fix}(S_i^0) = \mathrm{Fix}(S_i)$, $i = 1, 2$ and

$$\lim_{j\to\infty}\sup_{x\in D}\left\|S_i^{n_j}x - S_i^0 x\right\| = 0, \tag{4.9}$$

where D is an arbitrary bounded subset including $\{x_{n_j}\}$. By using (4.8) and (4.9), we obtain $\|S_2^0 S_1^0 x_n - x_n\| \to 0$.

Step 4. We have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{4.10}$$

where $x^* = P_F f(x^*)$ is a unique solution of the variational inequality (4.1).

Since $f_n(q)$ is convergent, we have $\lim_{n\to\infty}(f_n(x^*) - x^*) = f(x^*) - x^*$.

The proof of Step 4 is similar to that of Theorem 3.1.

Step 5. We show that

$$x_n \to x^*. \tag{4.11}$$

The proof of Step 5 is similar to that of Theorem 3.1. □

Remark 4.2 In [11], put $S_n = S_N^n S_{N-1}^n \cdots S_1^n$, and we rewrite Zhou and Wang's iterative algorithm as follows:

$$\begin{aligned} x_{n+1} &= (I - \alpha_n \mu F) S_n x_n \\ &= \alpha_n (I - \mu F) S_n x_n + (1 - \alpha_n) S_n x_n \\ &= \alpha_n f_n(x_n) + (1 - \alpha_n) S_n x_n. \end{aligned} \tag{4.12}$$

It is easily to verify $(I - \mu F)S_n$ is a contractive mapping on H when $0 < \mu < 2\eta/L^2$. Thus it is a special case of Theorem 4.1 when $f_n := (I - \mu F)S_n$, $\forall n \in N$ and $C = H$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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