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Some fixed point theorems for mappings satisfying contractive conditions of integral type

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Abstract

Five fixed point theorems for mappings satisfying contractive conditions of integral type in complete metric spaces are proved. Two examples are added to illustrate the results obtained.

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1 Introduction and preliminaries

Rhoades [1] and Branciari [2] proved the following fixed point theorems for the weakly contraction mapping and contractive mapping of integral type, respectively, which are generalizations of the Banach fixed point theorem.

Theorem 1.1 ([1]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \le d(x, y) - \psi\big(d(x, y)\big), \quad \forall x, y \in X, \tag{1.1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing such that ψ is positive on $\mathbb{R}^+ \setminus \{0\}$, $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$. Then T has a unique fixed point in X.

Theorem 1.2 ([2]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le c \int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x, y \in X, \tag{1.2}$$

where $c \in (0, 1)$ is a constant and $\varphi \in \Phi_1$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Recently several years, the researchers in [3–14] and others continued the study of Rhoades and Branciari, proved some fixed point and common fixed point theorems for various generalized weakly contraction mappings and contractive mappings of integral type in complete metric spaces, Banach spaces, modular spaces and symmetric spaces. Suzuki [15] proved that contractive condition of integral type in complete metric spaces is a special case of Meir-Keeler type.

The objective of this article is both to introduce several mappings satisfying contractive conditions of integral type, one of which extends the mapping (1.1) and is different from the mapping (1.2), and to provide sufficient conditions which ensure the existence of fixed points and convergence of iterative methods for these mappings in complete metric spaces. Two nontrivial examples are given to explain the main results obtained.

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N}$ denotes the set of all positive integers and

$$\begin{aligned}
\Phi_1 = \Big\{ &\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is Lebesgue integrable, summable on each} \\
&\text{compact subset of } \mathbb{R}^+ \text{ and } \int_0^\varepsilon \varphi(t)\,dt > 0 \text{ for each } \varepsilon > 0 \Big\}; \\
\Phi_2 = \big\{ &\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a lower semicontinuous function with } \psi(0) = 0 \\
&\text{and } \psi(t) > 0 \text{ for each } t > 0 \big\}.
\end{aligned}$$

For a self mapping T in a metric space (X, d) and $(x, y, n) \in X^2 \times \mathbb{N}_0$, define

$$\begin{aligned}
&x_n = T^n x, \qquad d_n = d(x_n, x_{n+1}); \\
&M(x,y) = \max\Big\{ d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big] \Big\}; \\
&N(x,y) = \max\big\{ d(x,Tx), d(y,Ty) \big\}; \\
&P(x,y) = \max\big\{ d(x,y), d(x,Tx), d(y,Ty) \big\}; \\
&Q(x,y) = \max\Big\{ d(x,Tx), d(y,Ty), \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big] \Big\}.
\end{aligned}$$

Lemma 1.1 ([10]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = \int_0^a \varphi(t)\,dt.$$

Lemma 1.2 ([10]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = 0$$

if and only if $\lim_{n\to\infty} r_n = 0$.

2 Main results

Now we prove the existence, uniqueness, and iterative approximations of fixed points for the mappings (2.1), (2.8), and (2.19)$\sim$(2.21), respectively.

Theorem 2.1 *Let (φ, ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \int_0^{d(x,y)} \varphi(t)\,dt - \int_0^{\psi(d(x,y))} \varphi(t)\,dt, \quad \forall x, y \in X. \tag{2.1}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Proof Let x be an arbitrary point in X. Suppose that there exists some $n_0 \in \mathbb{N}_0$ with $x_{n_0} = x_{n_0+1}$. Clearly,

$$x_{n_0} = Tx_{n_0} = T^2 x_{n_0} = \cdots = T^m x_{n_0} = \cdots = \lim_{n\to\infty} T^n x_{n_0},$$

that is, x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}_0$. It follows from (2.1) and $(\varphi, \psi) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned}
\int_0^{d_{n+1}} \varphi(t)\,dt &= \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)\,dt \\
&= \int_0^{d(T^{n+1}x,T^{n+2}x)} \varphi(t)\,dt \\
&\le \int_0^{d(T^n x,T^{n+1}x)} \varphi(t)\,dt - \int_0^{\psi(d(T^n x,T^{n+1}x))} \varphi(t)\,dt \\
&= \int_0^{d_n} \varphi(t)\,dt - \int_0^{\psi(d_n)} \varphi(t)\,dt \\
&< \int_0^{d_n} \varphi(t)\,dt, \quad \forall n \in \mathbb{N}_0,
\end{aligned}$$

which yields

$$d_{n+1} < d_n, \quad \forall n \in \mathbb{N}_0,$$

which implies that there exists a constant c with $\lim_{n\to\infty} d_n = c \ge 0$. Suppose that $c > 0$. Put $\liminf_{n\to\infty} \psi(d_n) = \alpha$. It is easy to see that there exists a subsequence $\{d_{n(k)}\}_{n\in\mathbb{N}}$ of $\{d_n\}_{n\in\mathbb{N}_0}$ satisfying $\lim_{k\to\infty} \psi(d_{n(k)}) = \alpha$. Since ψ is lower semicontinuous and $\psi \in \Phi_2$, it follows that $\alpha \ge \psi(c) > 0$. Using (2.1), Lemma 1.1 and $(\varphi, \psi) \in \Phi_1 \times \Phi_2$, we get

$$\begin{aligned}
0 &< \int_0^{c} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d_{n(k)+1}} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d(x_{n(k)+1},x_{n(k)+2})} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d(T^{n(k)+1}x,T^{n(k)+2}x)} \varphi(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le \limsup_{k\to\infty}\left(\int_0^{d(T^{n(k)}x,T^{n(k)+1}x)}\varphi(t)\,dt-\int_0^{\psi(d(T^{n(k)}x,T^{n(k)+1}x))}\varphi(t)\,dt\right)\\
&\le \limsup_{k\to\infty}\left(\int_0^{d_{n(k)}}\varphi(t)\,dt-\int_0^{\psi(d_{n(k)})}\varphi(t)\,dt\right)\\
&\le \limsup_{k\to\infty}\int_0^{d_{n(k)}}\varphi(t)\,dt-\liminf_{k\to\infty}\int_0^{\psi(d_{n(k)})}\varphi(t)\,dt\\
&=\int_0^{c}\varphi(t)\,dt-\int_0^{\alpha}\varphi(t)\,dt\\
&\le\int_0^{c}\varphi(t)\,dt-\int_0^{\psi(c)}\varphi(t)\,dt\\
&<\int_0^{c}\varphi(t)\,dt,
\end{aligned}
$$

which is impossible. Hence $c = 0$ and

$$\lim_{n\to\infty} d_n = 0. \tag{2.2}$$

Now we prove that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. If it is not a Cauchy sequence, then there exist a constant $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}_0}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) > \varepsilon$. It follows that $d(x_{m(k)}, x_{n(k)-1}) \le \varepsilon$. Observe that

$$
\begin{aligned}
\varepsilon &< d(x_{m(k)}, x_{n(k)})\\
&\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})\\
&\le d_{m(k)-1} + d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)-1}) + d_{n(k)-1}\\
&\le 2d_{m(k)-1} + \varepsilon + d_{n(k)-1}, \quad \forall k\in\mathbb{N},
\end{aligned}
\tag{2.3}
$$

and

$$\left|d(x_{m(k)-1}, x_{n(k)-1}) - d(x_{m(k)}, x_{n(k)-1})\right| \le d_{m(k)-1}, \quad \forall k\in\mathbb{N}. \tag{2.4}$$

Letting $k\to\infty$ in (2.3) and (2.4) and using (2.2), we infer that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.5}$$

Put

$$\liminf_{k\to\infty}\psi\big(d(x_{m(k)-1}, x_{n(k)-1})\big) = \beta.$$

Clearly, there exists a subsequence $\{d(x_{m(k_j)-1}, x_{n(k_j)-1})\}_{j\in\mathbb{N}}$ of $\{d(x_{m(k)-1}, x_{n(k)-1})\}_{k\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\psi\big(d(x_{m(k_j)-1}, x_{n(k_j)-1})\big) = \beta. \tag{2.6}$$

Since ψ is lower semicontinuous, it follows from (2.5), (2.6), and $\psi \in \Phi_2$ that $\beta \ge \psi(\varepsilon) > 0$. By means of (2.1), (2.5), (2.6), Lemma 1.1, and $\varphi \in \Phi_1$, we deduce that

$$\begin{aligned}
0 &< \int_0^{\varepsilon} \varphi(t)\,dt \\
&= \limsup_{j\to\infty} \int_0^{d(x_{m(k_j)}, x_{n(k_j)})} \varphi(t)\,dt \\
&= \limsup_{j\to\infty} \int_0^{d(T^{m(k_j)}x, T^{n(k_j)}x)} \varphi(t)\,dt \\
&\le \limsup_{j\to\infty} \left(\int_0^{d(T^{m(k_j)-1}x, T^{n(k_j)-1}x)} \varphi(t)\,dt - \int_0^{\psi(d(T^{m(k_j)-1}x, T^{n(k_j)-1}x))} \varphi(t)\,dt \right) \\
&= \limsup_{j\to\infty} \left(\int_0^{d(x_{m(k_j)-1}, x_{n(k_j)-1})} \varphi(t)\,dt - \int_0^{\psi(d(x_{m(k_j)-1}, x_{n(k_j)-1}))} \varphi(t)\,dt \right) \\
&\le \limsup_{j\to\infty} \int_0^{d(x_{m(k_j)-1}, x_{n(k_j)-1})} \varphi(t)\,dt - \liminf_{j\to\infty} \int_0^{\psi(d(x_{m(k_j)-1}, x_{n(k_j)-1}))} \varphi(t)\,dt \\
&= \int_0^{\varepsilon} \varphi(t)\,dt - \int_0^{\beta} \varphi(t)\,dt \\
&\le \int_0^{\varepsilon} \varphi(t)\,dt - \int_0^{\psi(\varepsilon)} \varphi(t)\,dt \\
&< \int_0^{\varepsilon} \varphi(t)\,dt,
\end{aligned}$$

which is a contradiction. Thus $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since (X,d) is complete, it follows that there exists $a \in X$ such that

$$\lim_{n\to\infty} T^n x = a. \tag{2.7}$$

Next we prove that a is a fixed point of T. In view of (2.1), (2.7), and Lemma 1.2, we obtain

$$\begin{aligned}
0 &\le \int_0^{d(T^{n+1}x, Ta)} \varphi(t)\,dt \le \int_0^{d(T^n x, a)} \varphi(t)\,dt - \int_0^{\psi(d(T^n x, a))} \varphi(t)\,dt \\
&\le \int_0^{d(T^n x, a)} \varphi(t)\,dt \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \int_0^{d(T^{n+1}x, Ta)} \varphi(t)\,dt = 0,$$

which together with Lemma 1.2 gives

$$\lim_{n\to\infty} d\bigl(T^{n+1}x, Ta\bigr) = 0.$$

Consequently, we have

$$d(a,Ta) \le d\bigl(a,T^{n+1}x\bigr) + d\bigl(T^{n+1}x,Ta\bigr) \to 0 \quad \text{as } n \to \infty,$$

that is, $a = Ta$.

Lastly, we prove that a is a unique fixed point of T in X. Suppose that T has another fixed point $b \in X \setminus \{a\}$. It follows from (2.1), $\varphi \in \Phi_1$, and $\psi(d(a,b)) > 0$ that

$$\begin{aligned}
0 &< \int_0^{d(a,b)} \varphi(t)\,dt = \int_0^{d(Ta,Tb)} \varphi(t)\,dt \\
&\le \int_0^{d(a,b)} \varphi(t)\,dt - \int_0^{\psi(d(a,b))} \varphi(t)\,dt \\
&< \int_0^{d(a,b)} \varphi(t)\,dt,
\end{aligned}$$

which is a contradiction. This completes the proof. □

Remark 2.1 In the case $\phi(t) = 1$ for all $t \in \mathbb{R}^+$, Theorem 2.1 reduces to Theorem 1.1. On the other hand, the example below demonstrates that Theorem 2.1 is different from Theorem 1.2.

Example 2.1 Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, $T : X \to X$ and $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Tx = \frac{x}{1+x^2}, \quad \forall x \in X$$

and

$$\varphi(t) = 4t^3, \qquad \psi(t) = \frac{t^2}{\sqrt{1+t^2}}, \quad \forall t \in \mathbb{R}^+.$$

Obviously, $(\varphi, \psi) \in \Phi_1 \times \Phi_2$. Let $x, y \in X$. It is clear that

$$\begin{aligned}
&(1-xy)^2 = 1 - 2xy + x^2y^2 \le 1 + x^2 + y^2 + x^2y^2 = \bigl(1+x^2\bigr)\bigl(1+y^2\bigr), \\
&1 + (x-y)^2 = 1 + x^2 - 2xy + y^2 \le 1 + x^2 + y^2 + x^2y^2 = \bigl(1+x^2\bigr)\bigl(1+y^2\bigr),
\end{aligned}$$

which imply that

$$(1-xy)^4\bigl[1+(x-y)^2\bigr]^2 \le \bigl(1+x^2\bigr)^4\bigl(1+y^2\bigr)^4 \le \bigl[1+2(x-y)^2\bigr]\bigl(1+x^2\bigr)^4\bigl(1+y^2\bigr)^4,$$

which gives

$$\begin{aligned}
\int_0^{d(Tx,Ty)} \varphi(t)\,dt &= \left(\frac{x}{1+x^2} - \frac{y}{1+y^2}\right)^4 = \frac{(x-y)^4(1-xy)^4}{(1+x^2)^4(1+y^2)^4} \\
&\le \frac{(x-y)^4[1+2(x-y)^2]}{[1+(x-y)^2]^2}
\end{aligned}$$

$$= (x-y)^4 - \frac{(x-y)^8}{[1+(x-y)^2]^2}$$
$$= \int_0^{d(x,y)} \varphi(t)\,dt - \int_0^{\psi(d(x,y))} \varphi(t)\,dt,$$

that is, (2.1) holds. Thus the conditions of Theorem 2.1 are satisfied. It follows from Theorem 2.1 that T has a unique fixed point $0 \in X$ and $\lim_{n\to\infty} T^n x = 0$ for each $x \in X$.

In order to verify that Theorem 1.2 is useless in proving the existence of fixed points of T, we need to show that (1.2) does not hold. Otherwise, (1.2) holds, that is, there exists some constant $c \in (0,1)$ satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt = \left(\frac{x}{1+x^2} - \frac{y}{1+y^2}\right)^4 = \frac{(x-y)^4(1-xy)^4}{(1+x^2)^4(1+y^2)^4}$$
$$\le c(x-y)^4 = c\int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x,y \in X,$$

which yields

$$\frac{(1-xy)^4}{(1+x^2)^4(1+y^2)^4} \le c, \quad \forall x,y \in X \text{ with } x \neq y,$$

which means that

$$1 = \lim_{\substack{(x,y)\to(0^+,0^+)\\ x\neq y}} \frac{(1-xy)^4}{(1+x^2)^4(1+y^2)^4} \le c < 1,$$

which is a contradiction.

Theorem 2.2 *Let (φ,ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \int_0^{M(x,y)} \varphi(t)\,dt - \int_0^{\psi(M(x,y))} \varphi(t)\,dt, \quad \forall x,y \in X. \tag{2.8}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Proof Let x be an arbitrary point in X. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then there is nothing to prove. Now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. Note that

$$\begin{aligned} M(x_n,x_{n+1}) &= \max\left\{d(x_n,x_{n+1}), d(x_n,Tx_n), d(x_{n+1},Tx_{n+1}),\right. \\ &\quad \left.\frac{1}{2}\big[d(x_n,Tx_{n+1}) + d(x_{n+1},Tx_n)\big]\right\} \\ &= \max\left\{d_n, d_{n+1}, \frac{1}{2}d(x_n,x_{n+2})\right\} \\ &= \max\{d_n,d_{n+1}\}, \quad \forall n \in \mathbb{N}_0, \end{aligned} \tag{2.9}$$

because

$$\frac{1}{2}d(x_n, x_{n+2}) \leq \frac{1}{2}\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big] \leq \max\{d_n, d_{n+1}\}, \quad \forall n \in \mathbb{N}_0.$$

Now we prove that

$$d_{n+1} < d_n, \quad \forall n \in \mathbb{N}_0. \tag{2.10}$$

Or else there exists some $n_0 \in \mathbb{N}_0$ such that $d_{n_0+1} \geq d_{n_0}$. Making use of (2.8) and (2.9), we know that

$$\begin{aligned}
\int_0^{d_{n_0+1}} \varphi(t)\,dt &= \int_0^{d(x_{n_0+1}, x_{n_0+2})} \varphi(t)\,dt \\
&= \int_0^{d(T^{n_0+1}x, T^{n_0+2}x)} \varphi(t)\,dt \\
&\leq \int_0^{M(T^{n_0}x, T^{n_0+1}x)} \varphi(t)\,dt - \int_0^{\psi(M(T^{n_0}x, T^{n_0+1}x))} \varphi(t)\,dt \\
&= \int_0^{\max\{d_{n_0}, d_{n_0+1}\}} \varphi(t)\,dt - \int_0^{\psi(\max\{d_{n_0}, d_{n_0+1}\})} \varphi(t)\,dt \\
&\leq \int_0^{d_{n_0+1}} \varphi(t)\,dt - \int_0^{\psi(d_{n_0+1})} \varphi(t)\,dt \\
&< \int_0^{d_{n_0+1}} \varphi(t)\,dt,
\end{aligned}$$

which is a contradiction. Note that (2.10) means that there exists a constant c with

$$\lim_{n\to\infty} d_n = c \geq 0. \tag{2.11}$$

Suppose that $c > 0$. Set $\liminf_{n\to\infty} \psi(d_n) = \gamma$. Obviously, there exists a subsequence $\{d_{n(k)}\}_{n\in\mathbb{N}}$ of $\{d_n\}_{n\in\mathbb{N}_0}$ such that $\lim_{k\to\infty} \psi(d_{n(k)}) = \gamma$. Since ψ is lower semicontinuous, it follows from $\psi \in \Phi_2$ that $\gamma \geq \psi(c) > 0$. On account of (2.8)$\sim$(2.11), Lemma 1.1, and $\varphi \in \Phi_1$, we arrive at

$$\begin{aligned}
0 &< \int_0^{c} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d_{n(k)+1}} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d(x_{n(k)+1}, x_{n(k)+2})} \varphi(t)\,dt \\
&= \limsup_{k\to\infty} \int_0^{d(T^{n(k)+1}x, T^{n(k)+2}x)} \varphi(t)\,dt \\
&\leq \limsup_{k\to\infty} \left(\int_0^{M(T^{n(k)}x, T^{n(k)+1}x)} \varphi(t)\,dt - \int_0^{\psi(M(T^{n(k)}x, T^{n(k)+1}x))} \varphi(t)\,dt \right) \\
&\leq \limsup_{k\to\infty} \left(\int_0^{M(x_{n(k)}, x_{n(k)+1})} \varphi(t)\,dt - \int_0^{\psi(M(x_{n(k)}, x_{n(k)+1}))} \varphi(t)\,dt \right)
\end{aligned}$$

$$\begin{aligned}
&= \limsup_{k\to\infty}\left(\int_0^{d_{n_k}} \varphi(t)\,dt - \int_0^{\psi(d_{n_k})} \varphi(t)\,dt\right) \\
&\le \limsup_{k\to\infty}\int_0^{d_{n_k}} \varphi(t)\,dt - \liminf_{k\to\infty}\int_0^{\psi(d_{n_k})} \varphi(t)\,dt \\
&= \int_0^{c} \varphi(t)\,dt - \int_0^{\gamma} \varphi(t)\,dt \\
&\le \int_0^{c} \varphi(t)\,dt - \int_0^{\psi(c)} \varphi(t)\,dt \\
&< \int_0^{c} \varphi(t)\,dt,
\end{aligned}$$

which is absurd. Hence $c = 0$ and (2.2) holds. Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence. It follows that there exist a constant $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}_{k\in\mathbb{N}}$ and $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}_0}$ such that $n(k)$ is minimal in the sense that $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) > \varepsilon$. It follows that (2.5) holds. Observe that (2.2) and (2.5) ensure that

$$\left|d(x_{m(k)-1}, x_{n(k)}) - d(x_{m(k)}, x_{n(k)})\right| \le d_{m(k)-1} \to 0 \quad \text{as } k \to \infty \tag{2.12}$$

and

$$\begin{aligned}
&M(x_{m(k)-1}, x_{n(k)-1}) \\
&\quad= \max\Bigl\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, Tx_{n(k)-1}), \\
&\quad\quad \frac{1}{2}\bigl[d(x_{m(k)-1}, Tx_{n(k)-1}) + d(x_{n(k)-1}, Tx_{m(k)-1})\bigr]\Bigr\} \\
&\quad= \max\Bigl\{d(x_{m(k)-1}, x_{n(k)-1}), d_{m(k)-1}, d_{n(k)-1}, \\
&\quad\quad \frac{1}{2}\bigl[d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)-1}, x_{m(k)})\bigr]\Bigr\} \\
&\quad\to \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon \quad \text{as } k \to \infty.
\end{aligned} \tag{2.13}$$

Put

$$\liminf_{j\to\infty} \psi\bigl(M(x_{m(k)-1}, x_{n(k)-1})\bigr) = \lambda.$$

Clearly, there exists a subsequence $\{M(x_{m(k_j)-1}, x_{n(k_j)-1})\}_{j\in\mathbb{N}}$ of $\{M(x_{m(k)-1}, x_{n(k)-1})\}_{k\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \psi\bigl(M(x_{m(k_j)-1}, x_{n(k_j)-1})\bigr) = \lambda \ge \psi(\varepsilon). \tag{2.14}$$

Combining (2.5), (2.8), (2.12)∼(2.14), Lemma 1.1, and $\varphi \in \Phi_1$, we get

$$\begin{aligned}
0 &< \int_0^{\varepsilon} \varphi(t)\,dt \\
&= \limsup_{j\to\infty} \int_0^{d(x_{m(k_j)}, x_{n(k_j)})} \varphi(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{j\to\infty} \int_0^{d(T^{m(k_j)}x, T^{n(k_j)}x)} \varphi(t)\,dt \\
&\le \limsup_{j\to\infty} \left(\int_0^{M(T^{m(k_j)-1}x, T^{n(k_j)-1}x)} \varphi(t)\,dt - \int_0^{\psi(M(T^{m(k_j)-1}x, T^{n(k_j)-1}x))} \varphi(t)\,dt \right) \\
&= \limsup_{j\to\infty} \left(\int_0^{M(x_{m(k_j)-1}, x_{n(k_j)-1})} \varphi(t)\,dt - \int_0^{\psi(M(x_{m(k_j)-1}, x_{n(k_j)-1}))} \varphi(t)\,dt \right) \\
&\le \limsup_{j\to\infty} \int_0^{M(x_{m(k_j)-1}, x_{n(k_j)-1})} \varphi(t)\,dt - \liminf_{j\to\infty} \int_0^{\psi(M(x_{m(k_j)-1}, x_{n(k_j)-1}))} \varphi(t)\,dt \\
&= \int_0^{\varepsilon} \varphi(t)\,dt - \int_0^{\lambda} \varphi(t)\,dt \\
&\le \int_0^{\varepsilon} \varphi(t)\,dt - \int_0^{\psi(\varepsilon)} \varphi(t)\,dt \\
&< \int_0^{\varepsilon} \varphi(t)\,dt,
\end{aligned}
$$

which is a contradiction. Hence $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Completeness of (X,d) ensures that there exists $a \in X$ satisfying (2.7). Suppose that $d(a, Ta) > 0$. Let

$$
M^* = \max\left\{ d(a, Ta), d\left(Ta, T^2a\right), \frac{1}{2}\left[d\left(a, T^2a\right) + d(a, Ta)\right]\right\}. \tag{2.15}
$$

Note that (2.2) and (2.7) yield

$$
\begin{aligned}
&\lim_{n\to\infty} M(x_{n+1}, Ta) \\
&\quad= \lim_{n\to\infty} \max\left\{ d(x_{n+1}, Ta), d(x_{n+1}, Tx_{n+1}), d\left(Ta, T^2a\right), \frac{1}{2}\left[d\left(x_{n+1}, T^2a\right) + d(Ta, Tx_{n+1})\right]\right\} \\
&\quad= \lim_{n\to\infty} \max\left\{ d(x_{n+1}, Ta), d_{n+1}, d\left(Ta, T^2a\right), \frac{1}{2}\left[d\left(x_{n+1}, T^2a\right) + d(Ta, x_{n+2})\right]\right\} \\
&\quad= \max\left\{ d(a, Ta), 0, d\left(Ta, T^2a\right), \frac{1}{2}\left[d\left(a, T^2a\right) + d(Ta, a)\right]\right\} \\
&\quad= M^*
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} M(x_n, a) \\
&\quad= \lim_{n\to\infty} \max\left\{ d(x_n, a), d(x_n, Tx_n), d(a, Ta), \frac{1}{2}\left[d(x_n, Ta) + d(a, Tx_n)\right]\right\} \\
&\quad= \lim_{n\to\infty} \max\left\{ d(x_n, a), d_n, d(a, Ta), \frac{1}{2}\left[d(x_n, Ta) + d(a, x_{n+1})\right]\right\} \\
&\quad= \max\left\{ 0, 0, d(a, Ta), \frac{1}{2}d(a, Ta)\right\} \\
&\quad= d(a, Ta).
\end{aligned} \tag{2.17}
$$

Put $\liminf_{n\to\infty}\psi(M(x_n,a))=\eta$. Clearly, there exists a subsequence $\{M(x_{n(j)},a)\}_{j\in\mathbb{N}}$ of $\{M(x_n,a)\}_{n\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\psi\bigl(M(x_{n(j)},a)\bigr)=\eta\geq\psi\bigl(d(a,Ta)\bigr). \tag{2.18}$$

In virtue of (2.8), (2.16)∼(2.18), and Lemma 1.1, we conclude that

$$\begin{aligned}
0&<\int_0^{M^*}\varphi(t)\,dt\\
&=\limsup_{j\to\infty}\int_0^{M(x_{n(j)+1},Ta)}\varphi(t)\,dt\\
&=\limsup_{j\to\infty}\int_0^{M(T^{n(j)+1}x,Ta)}\varphi(t)\,dt\\
&\leq\limsup_{j\to\infty}\biggl(\int_0^{M(T^{n(j)}x,a)}\varphi(t)\,dt-\int_0^{\psi(M(T^{n(j)}x,a))}\varphi(t)\,dt\biggr)\\
&=\limsup_{j\to\infty}\biggl(\int_0^{M(x_{n(j)},a)}\varphi(t)\,dt-\int_0^{\psi(M(x_{n(j)},a))}\varphi(t)\,dt\biggr)\\
&\leq\limsup_{j\to\infty}\int_0^{M(x_{n(j)},a)}\varphi(t)\,dt-\liminf_{j\to\infty}\int_0^{\psi(M(x_{n(j)},a))}\varphi(t)\,dt\\
&=\int_0^{d(a,Ta)}\varphi(t)\,dt-\int_0^{\eta}\varphi(t)\,dt\\
&\leq\int_0^{d(a,Ta)}\varphi(t)\,dt-\int_0^{\psi(d(a,Ta))}\varphi(t)\,dt\\
&<\int_0^{d(a,Ta)}\varphi(t)\,dt,
\end{aligned}$$

which together with (2.15) means that

$$d(a,Ta)\leq M^*<d(a,Ta),$$

which is impossible. Consequently, $a=Ta$ is a fixed point of T in X. Suppose that T has another fixed point $b\in X\setminus\{a\}$. Notice that

$$\begin{aligned}
M(a,b)&=\max\biggl\{d(a,b),d(a,Ta),d(b,Tb),\frac{1}{2}\bigl[d(a,Tb)+d(b,Ta)\bigr]\biggr\}\\
&=\max\bigl\{d(a,b),0,0,d(a,b)\bigr\}\\
&=d(a,b),
\end{aligned}$$

which together with $\varphi\in\Phi_1$, (2.8), and $\psi(d(a,b))>0$ means that

$$\begin{aligned}
0&<\int_0^{d(a,b)}\varphi(t)\,dt=\int_0^{d(Ta,Tb)}\varphi(t)\,dt\\
&\leq\int_0^{M(a,b)}\varphi(t)\,dt-\int_0^{\psi(M(a,b))}\varphi(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{d(a,b)} \varphi(t)\,dt - \int_0^{\psi(d(a,b))} \varphi(t)\,dt \\
&< \int_0^{d(a,b)} \varphi(t)\,dt,
\end{aligned}
$$

which is a contradiction. Consequently, T possesses a unique fixed point $a \in X$. This completes the proof. □

Remark 2.2 The below example is an application of Theorem 2.2.

Example 2.2 Let $X = [0,2] \cup \{6\}$ be endowed with the Euclidean metric $d = |\cdot|$, $T : X \to X$ and $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$
Tx = \begin{cases} \frac{x}{2}, & \forall x \in [0,2], \\ 2, & x = 6 \end{cases}
$$

and

$$
\varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+, \qquad \psi(t) = \begin{cases} \frac{\sqrt{3}}{4}t, & \forall t \in [0,2], \\ \sqrt{1 + \frac{1}{1+t}}, & \forall t \in (2,+\infty). \end{cases}
$$

Clearly, $(\varphi,\psi) \in \Phi_1 \times \Phi_2$. For $x, y \in X$ with $y \le x$, we consider the following four cases.

Case 1. Let $x, y \in [0,2]$ with $y \le \frac{x}{2}$. It is easy to verify that

$$
M(x,y) = \max\left\{ |x-y|, \frac{x}{2}, \frac{y}{2}, \frac{1}{2}\left(\left| x - \frac{y}{2} \right| + \left| y - \frac{x}{2} \right| \right) \right\} = x - y \le 2,
$$

which yields

$$
\begin{aligned}
\int_0^{d(Tx,Ty)} \varphi(t)\,dt &= \frac{1}{4}(x-y)^2 \le (x-y)^2 - \frac{3}{16}(x-y)^2 \\
&= \int_0^{M(x,y)} \varphi(t)\,dt - \int_0^{\psi(M(x,y))} \varphi(t)\,dt.
\end{aligned}
$$

Case 2. Let $x, y \in [0,2]$ with $\frac{x}{2} < y \le x$. It is clear that

$$
M(x,y) = \max\left\{ |x-y|, \frac{x}{2}, \frac{y}{2}, \frac{1}{2}\left(\left| x - \frac{y}{2} \right| + \left| y - \frac{x}{2} \right| \right) \right\} = \frac{x}{2} \le 1,
$$

which gives

$$
\begin{aligned}
\int_0^{d(Tx,Ty)} \varphi(t)\,dt &= \frac{1}{4}(x-y)^2 = \frac{1}{4}x^2 - \left(\frac{1}{2}xy - \frac{1}{4}y^2 \right) \\
&= \frac{1}{4}x^2 - \left(-\frac{1}{4}(y-x)^2 + \frac{1}{4}x^2 \right) \\
&\le \frac{1}{4}x^2 - \frac{3}{16}x^2 \\
&= \int_0^{M(x,y)} \varphi(t)\,dt - \int_0^{\psi(M(x,y))} \varphi(t)\,dt.
\end{aligned}
$$

Case 3. Let $y \in [0,2]$ and $x = 6$. Obviously, we have

$$M(x,y) = \max\left\{6-y, 4, \frac{y}{2}, \frac{1}{2}\left(6-\frac{y}{2}+2-y\right)\right\} = 6-y \geq 4,$$

which implies that

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)\,dt &= \left(2-\frac{1}{2}y\right)^2 \leq 4 < 4^2 - \left(1+\frac{1}{1+4}\right) \\ &\leq (6-y)^2 - \left(1+\frac{1}{1+6-y}\right) \\ &= \int_0^{M(x,y)} \varphi(t)\,dt - \int_0^{\psi(M(x,y))} \varphi(t)\,dt.\end{aligned}$$

Case 4. Let $x = y = 6$. It follows that

$$M(x,y) = \max\left\{0, 6-2, 6-2, \frac{1}{2}(6-2+6-2)\right\} = 4,$$

which means that

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)\,dt &= 0 < 4^2 - \left(1+\frac{1}{1+4}\right) \\ &= \int_0^{M(x,y)} \varphi(t)\,dt - \int_0^{\psi(M(x,y))} \varphi(t)\,dt.\end{aligned}$$

That is, (2.8) holds. Thus the conditions of Theorem 2.2 are satisfied. It follows from Theorem 2.2 that T has a unique fixed point $0 \in X$ and $\lim_{n\to\infty} T^n x = 0$ for every $x \in X$.

Similar to the proofs of Theorems 2.1 and 2.2, we have the following results and we omit their proofs.

Theorem 2.3 *Let (φ, ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \leq \int_0^{N(x,y)} \varphi(t)\,dt - \int_0^{\psi(N(x,y))} \varphi(t)\,dt, \quad \forall x,y \in X. \tag{2.19}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Theorem 2.4 *Let (φ, ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \leq \int_0^{P(x,y)} \varphi(t)\,dt - \int_0^{\psi(P(x,y))} \varphi(t)\,dt, \quad \forall x,y \in X. \tag{2.20}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Theorem 2.5 *Let (φ, ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \int_0^{Q(x,y)} \varphi(t)\,dt - \int_0^{\psi(Q(x,y))} \varphi(t)\,dt, \quad \forall x, y \in X. \tag{2.21}$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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