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Strong convergence theorem for the modified generalized equilibrium problem and fixed point problem of strictly pseudo-contractive mappings

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Abstract

The purpose of this paper is to modify the generalized equilibrium problem introduced by Ceng *et al.* (J. Glob. Optim. 43:487-502, 2012) and to introduce the K-mapping generated by a finite family of strictly pseudo-contractive mappings and finite real numbers modifying the results of Kangtunyakarn and Suantai (Nonlinear Anal. 71:4448-4460, 2009). Then we prove the strong convergence theorem for finding a common element of the set of fixed points of a finite family of strictly pseudo-contractive mappings and a finite family of the set of solutions of the modified generalized equilibrium problem. Moreover, using our main result, we obtain the additional results related to the generalized equilibrium problem.

Keywords: strictly pseudo-contractive mapping; K-mapping; the modified generalized equilibrium problem; equilibrium-like function

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. A mapping $f: C \to C$ is contractive if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

We now recall some well-known concepts and results as follows.

Definition 1.1 Let $B : C \to C$ be a mapping. Then B is called

(i) monotone if

$$\langle Bx - By, x - y\rangle \ge 0, \quad \forall x, y \in C,$$

(ii) υ-strongly monotone if there exists a positive real number υ such that

$$\langle Bx - By, x - y\rangle \ge \upsilon\|x - y\|^2, \quad \forall x, y \in C,$$

(iii) ξ-inverse strongly monotone if there exists a positive real number ξ such that

$$\langle x - y, Bx - By\rangle \ge \xi\|Bx - By\|^2, \quad \forall x, y \in C,$$

(iv) μ-Lipschitz continuous if there exists a nonnegative real number $\mu \geq 0$ such that

$$\|Bx - By\| \leq \mu\|x - y\|, \quad \forall x, y \in C.$$

Definition 1.2 Let $T : C \to C$ be a mapping. Then:

(i) An element $x \in C$ is said to be a fixed point of T if $Tx = x$ and $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T.

(ii) Mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(iii) T is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Note that the class of κ-strictly pseudo-contractions strictly includes the class of nonexpansive mappings, that is, nonexpansive mapping is a 0-strictly pseudo-contraction mapping. In a real Hilbert space H (1.1) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2}\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in C.$$

Remark 1.1 $T : C \to C$ is a κ-strictly pseudo-contraction if and only if $I - T$ is $\frac{1-\kappa}{2}$-inverse strongly monotone.

In the last decades, many researcher have studied fixed point theorems associated with various types of nonlinear mapping; see, for instance, [1–4]. Fixed point problems arise in many fields such as the vibration of masses attached to strings or nets [5] and a network bandwidth allocation problem [6] which is one of the central issues in modern communication networks. For applications to neural networks, fixed point theorems can be used to design dynamic neural network in order to solve steady state solutions [7]. For general information on neural networks, see for instance, [8, 9].

Let $F : C \times C \to \mathbb{R}$ be bifunction. *The equilibrium problem* for F is to determine its equilibrium point, *i.e.*, the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}. \tag{1.2}$$

Equilibrium problems were introduced by [10] in 1994 where such problems have had a significant impact and influence in the development of several branches of pure and applied sciences. Various problems in physics, optimization, and economics are related to seeking some elements of $EP(F)$; see [10, 11]. Many authors have been investigating iterative algorithms for the equilibrium problems; see, for example, [11–15].

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(\cdot, \cdot)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = \max\left\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v)\right\}, \quad \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v \in V} d(u, v)$, $d(U, v) = \inf_{u \in U} d(u, v)$ and $d(u, v) = \|u - v\|$.

Let C be a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $T : C \to CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \to \mathbb{R}$ an equilibrium-like function, that is, $\Phi(w,u,v) + \Phi(w,v,u) = 0$ for all $(w,u,v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued mapping $T : C \to CB(H)$.

(H1) For each fixed $v \in C$, $(w,u) \mapsto \Phi(w,u,v)$ is an upper semicontinuous function from $H \times C \to \mathbb{R}$, that is, for $(w,u) \in H \times C$, whenever $w_n \to w$ and $u_n \to u$ as $n \to \infty$,

$$\limsup_{n\to\infty} \Phi(w_n,u_n,v) \le \Phi(w,u,v).$$

(H2) For each fixed $(w,v) \in H \times C$, $u \mapsto \Phi(w,u,v)$ is a concave function.

(H3) For each fixed $(w,u) \in H \times C$, $v \mapsto \Phi(w,u,v)$ is a convex function.

In 2009, Ceng *et al.* [16] introduced *the generalized equilibrium problem* (*GEP*) as follows:

$$(GEP) \qquad \begin{cases} \text{Find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \Phi(w,u,v) + \varphi(v) - \varphi(u) \ge 0, \quad \forall v \in C. \end{cases} \tag{1.3}$$

The set of such solutions $u \in C$ of (GEP) is denoted by $(GEP)_s(\Phi,\varphi)$. In the case of $\varphi = 0$ and $\Phi(w,u,v) \equiv G(u,v)$, then $(GEP)_s(\Phi,\varphi)$ is denoted by $EP(G)$.

By using Nadler's theorem [17], they introduced the following algorithm:

Let $x_1 \in C$ and $w_1 \in T(x_1)$, there exist sequences $\{w_n\} \subseteq H$ and $\{x_n\}, \{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \quad \|w_n - w_{n+1}\| \le (1 + \frac{1}{n})\mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n,u_n,v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, v - u_n\rangle \ge 0, \quad \forall v \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \quad n = 1,2,\ldots. \end{cases} \tag{1.4}$$

They proved the strong convergence theorem of the sequence $\{x_n\}$ generated by (1.4) as follows.

Theorem 1.2 ([16]) *Let C be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying* (H1)-(H3) *and S be a nonexpansive mapping of C into itself such that $F(S) \cap (GEP)_s(\Phi,\varphi) \ne \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}$, $\{w_n\}$, and $\{u_n\}$ be sequences generated by* (1.4), *where $\{\alpha_n\} \subseteq [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

If there exists a constant $\lambda > 0$ such that

$$\Phi\big(w_1, T_{r_1}(x_1), T_{r_2}(x_2)\big) + \Phi\big(w_2, T_{r_2}(x_2), T_{r_1}(x_1)\big) \le -\lambda \big\| T_{r_1}(x_1) - T_{r_2}(x_2)\big\|^2,$$

for all $(r_1, r_2) \in \Xi \times \Xi$, $(x_1, x_2) \in C \times C$ *and* $w_i \in T(x_i)$, $i = 1, 2$, *where* $\Xi = \{r_n : n \geq 1\}$, *then for* $\hat{x} = P_{F(S)\cap(GEP)_s(\Phi,\varphi)} f(\hat{x})$, *there exists* $\hat{w} \in T(\hat{x})$ *such that* $(\hat{x}, \hat{w})$ *is a solution of* (GEP) *and*

$$x_n \to \hat{x}, \qquad w_n \to \hat{w} \quad \text{and} \quad u_n \to \hat{x} \quad \text{as } n \to \infty.$$

In 2012, Kangtunyakarn [12] introduced the iterative algorithm as follows.

Algorithm 1.3 ([12]) Let $T_i : i = 1, 2, \ldots, N$, be κ_i-pseudo-contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \ldots, N\}$ and let S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I$, $I = [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \ldots, N-1$, $\kappa \leq \alpha_1^{n,N} \leq 1$, $\kappa \leq \alpha_3^{n,N} \leq d < 1$, $\kappa \leq \alpha_2^{n,N} \leq e < 1$ for all $j = 1, 2, \ldots, N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1)$, $w_1^2 \in D(x_1)$, there exist sequences $\{w_n^1\}, \{w_n^2\} \in H$, and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$ such that

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq (1 + \frac{1}{n})\mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \|w_n^2 - w_{n+1}^2\| \leq (1 + \frac{1}{n})\mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n}\langle u_n - x_n, u - u_n\rangle \geq 0, \quad \forall u \in C, \\ \Phi(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n}\langle v_n - x_n, v - v_n\rangle \geq 0, \quad \forall v \in C, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n)P_C(I - \eta B)v_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $D, T : C \to CB(H)$ are $\mathcal{H}$-Lipschitz continuous with constants μ_1, μ_2, respectively, $\Phi_1, \Phi_2 : H \times C \times C \to \mathbb{R}$ are equilibrium-like functions satisfying (H1)-(H3), $A : C \to H$ is an α-inverse strongly monotone mapping and $B : C \to H$ is a β-inverse strongly monotone mapping.

He proved under some control conditions on $\{\delta_n\}$, $\{\alpha_n\}$, $\{s_n\}$, and $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{\mathcal{F}} x_1$, where $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, $G_1, G_2 : C \to C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ and $P_{\mathcal{F}} x_1$ is a solution of the following system of variational inequalities:

$$\begin{cases} \langle Ax^*, x - x^*\rangle \geq 0, \\ \langle Bx^*, x - x^*\rangle \geq 0. \end{cases}$$

By modifying the generalized equilibrium problem (1.3), we introduced *the modified generalized equilibrium problem* (*MGEP*) as follows:

$$(MGEP) \qquad \begin{cases} \text{Find } u \in C \text{ and } w \in T(I - \lambda A)u, \quad \forall \lambda > 0, \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) + \langle v - u, Au\rangle \geq 0, \quad \forall v \in C, \end{cases} \tag{1.6}$$

where $A : C \to C$ is a mapping. The set of such solutions of ($MGEP$) is denoted by $(MGEP)_s(\Phi, \varphi, A)$. If $A = 0$, (1.6) reduces to (1.3).

In this paper, motivated by Theorem 1.2, Algorithm 1.3 and (1.6), we modify the generalized equilibrium problem introduced by Ceng *et al.* [16] and introduce the K-mapping generated by a finite family of strictly pseudo-contractive mappings and finite real numbers modifying the results of Kangtunyakarn and Suantai [13]. Then we prove the strong convergence theorem for finding a common element of the set of fixed points of a finite family of strictly pseudo-contractive mappings and a finite family of the set of solutions of the modified generalized equilibrium problem. Moreover, using our main result, we obtain the additional results related to the generalized equilibrium problem.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by the notations '$\rightharpoonup$' and '$\to$', respectively.

Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemmas are needed to prove the main theorem.

Lemma 2.1 ([18]) *Let H be a real Hilbert space. Then the following identities hold*:

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$, $\forall x, y \in H$;

(ii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in H$.

Lemma 2.2 ([19]) *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 0, 1, 2, \dots, n$ such that $\sum_{i=0}^{n} \alpha_i = 1$ the following equality holds*:

$$\left\| \sum_{i=0}^{n} \alpha_i x_i \right\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \le i,j \le n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3 ([18]) *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Leftrightarrow \quad \langle u - z, v - u\rangle \ge 0, \quad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Lemma 2.4 (Demiclosedness principle [20]) *Assume that T is a nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y it follows that $(I - T)x = y$. Here, I is the identity mapping of H.*

Lemma 2.5 ([21]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contractive mapping, then S*

satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.6 ([22]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Definition 2.1 A multivalued mapping $T : C \to CB(H)$ is said to be $\mathcal{H}$-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\mathcal{H}\big(T(u), T(v)\big) \leq \mu\|u - v\|, \quad \forall u, v \in C,$$

where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.7 (Nadler's theorem [17]) *Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$. If $U, V \in CB(H)$, then for every $\epsilon > 0$ and $u \in U$, there exists $v \in V$ such that*

$$\|u - v\| \leq (1+\epsilon)\mathcal{H}(U, V).$$

Theorem 2.8 ([16]) *Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with constant μ, and $\Phi : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function satisfying* (H1)-(H3). *Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarily and define a mapping $T_r : C \to C$ as follows*:

$$T_r(x) = \left\{u \in C : \Phi(w_x, u, v) + \varphi(v) - \varphi(u) + \frac{1}{r}\langle u - x, v - u\rangle \geq 0, \forall v \in C\right\}.$$

Then we have the following:

(a) *T_r is single-valued*;

(b) *T_r is firmly nonexpansive* (*that is*, *for any $u, v \in C$, $\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v\rangle$*) *if*

$$\Phi\big(w_1, T_r(x_1), T_r(x_2)\big) + \Phi\big(w_2, T_r(x_2), T_r(x_1)\big) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $w_i \in T(x_i)$, $i = 1, 2$;

(c) *$F(T_r) = (GEP)_s(\Phi, \varphi)$*;

(d) *$(GEP)_s(\Phi, \varphi)$ is closed and convex.*

Definition 2.2 ([13]) Let C be a nonempty closed convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers with $0 \le \lambda_i \le 1$ for every $i = 1, 2, \ldots, N$. Define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
&U_1 = \lambda_1 T_1 + (1 - \lambda_1)I, \\
&U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\
&U_3 = \lambda_3 T_3 U_2 + (1 - \lambda_3)U_2, \\
&\vdots \\
&U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\
&K = U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}.
\end{aligned} \tag{2.1}$$

Such a mapping K is called the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

The following lemmas are needed to prove our main result.

Lemma 2.9 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mapping of C into itself with $\kappa_i \le \gamma_1$, for all $i = 1, 2, \ldots, N$, and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers with $0 < \lambda_i < \gamma_2$, for all $i = 1, 2, \ldots, N$ and $\gamma_1 + \gamma_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then the following properties hold*:

(i) $F(K) = \bigcap_{i=1}^N F(T_i)$;

(ii) *K is a nonexpansive mapping.*

Proof To prove (i), it is easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(K)$.

Next, we claim that $F(K) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this, let $x \in F(K)$ and $y \in \bigcap_{i=1}^N F(T_i)$.

By the definition of K-mapping, we get

$$\begin{aligned}
&\|x - y\| \\
&\quad = \|Kx - y\|^2 \\
&\quad = \big\|\lambda_N T_N U_{N-1}x + (1 - \lambda_N)U_{N-1}x - y\big\|^2 \\
&\quad = \big\|\lambda_N (T_N U_{N-1}x - y) + (1 - \lambda_N)(U_{N-1}x - y)\big\|^2 \\
&\quad = \lambda_N^2 \|T_N U_{N-1}x - y\|^2 + (1 - \lambda_N)^2 \|U_{N-1}x - y\|^2 \\
&\qquad + 2\lambda_N(1 - \lambda_N)\langle T_N U_{N-1}x - y, U_{N-1}x - y\rangle \\
&\quad = \lambda_N^2 \big(\|U_{N-1}x - y\|^2 + \kappa_N \|T_N U_{N-1}x - U_{N-1}x\|^2\big) + (1 - \lambda_N)^2 \|U_{N-1}x - y\|^2 \\
&\qquad + 2\lambda_N(1 - \lambda_N)\left(\|U_{N-1}x - y\|^2 - \frac{1 - \kappa_N}{2}\|T_N U_{N-1}x - U_{N-1}x\|^2\right) \\
&\quad = \big(\lambda_N^2 + (1 - \lambda_N)^2 + 2\lambda_N(1 - \lambda_N)\big)\|U_{N-1}x - y\|^2 \\
&\qquad + \big(\lambda_N^2 \kappa_N - \lambda_N(1 - \lambda_N)(1 - \kappa_N)\big)\|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\quad = (\lambda_N + 1 - \lambda_N)^2 \|U_{N-1}x - y\|^2 \\
&\qquad + \lambda_N\big(\lambda_N \kappa_N - (1 - \lambda_N)(1 - \kappa_N)\big)\|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\quad = \|U_{N-1}x - y\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \lambda_N\big(\lambda_N\kappa_N - (1-\kappa_N) + \lambda_N(1-\kappa_N)\big)\|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&= \|U_{N-1}x - y\|^2 + \lambda_N(\kappa_N + \lambda_N - 1)\|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\leq \|U_{N-1}x - y\|^2 + \lambda_N(\gamma_1 + \gamma_2 - 1)\|T_N U_{N-1}x - U_{N-1}x\|^2 \\
&\leq \|U_{N-1}x - y\|^2 \\
&\quad \vdots \\
&= \|U_2x - y\|^2 \\
&= \big\|\lambda_2(T_2U_1x - y) + (1-\lambda_2)(U_1x - y)\big\|^2 \\
&= \lambda_2^2\|T_2U_1x - y\|^2 + (1-\lambda_2)^2\|U_1x - y\|^2 \\
&\quad + 2\lambda_2(1-\lambda_2)\langle T_2U_1x - y, U_1x - y\rangle \\
&= \lambda_2^2\big(\|U_1x - y\|^2 + \kappa_2\|T_2U_1x - U_1x\|^2\big) + (1-\lambda_2)^2\|U_1x - y\|^2 \\
&\quad + 2\lambda_2(1-\lambda_2)\left(\|U_1x - y\|^2 - \frac{1-\kappa_2}{2}\|T_2U_1x - U_1x\|^2\right) \\
&= \big(\lambda_2^2 + (1-\lambda_2)^2 + 2\lambda_2(1-\lambda_2)\big)\|U_1x - y\|^2 \\
&\quad + \big(\lambda_2^2\kappa_2 - \lambda_2(1-\lambda_2)(1-\kappa_2)\big)\|T_2U_1x - U_1x\|^2 \\
&= (\lambda_2 + 1 - \lambda_2)^2\|U_1x - y\|^2 \\
&\quad + \lambda_2\big(\lambda_2\kappa_2 - (1-\lambda_2)(1-\kappa_2)\big)\|T_2U_1x - U_1x\|^2 \\
&= \|U_1x - y\|^2 + \lambda_2(\kappa_2 + \lambda_2 - 1)\|T_2U_1x - U_1x\|^2 \\
&\leq \|U_1x - y\|^2 + \lambda_2\big((\gamma_1 + \gamma_2) - 1\big)\|T_2U_1x - U_1x\|^2 \\
&\leq \|U_1x - y\|^2 \\
&= \big\|\lambda_1(T_1x - y) + (1-\lambda_1)(x - y)\big\|^2 \\
&= \lambda_1^2\|T_1x - y\|^2 + (1-\lambda_1)^2\|x - y\|^2 \\
&\quad + 2\lambda_1(1-\lambda_1)\langle T_1x - y, x - y\rangle \\
&= \lambda_1^2\big(\|x - y\|^2 + \kappa_1\|T_1x - x\|^2\big) + (1-\lambda_1)^2\|x - y\|^2 \\
&\quad + 2\lambda_1(1-\lambda_1)\left(\|x - y\|^2 - \frac{1-\kappa_1}{2}\|T_1x - x\|^2\right) \\
&= \big(\lambda_1^2 + (1-\lambda_1)^2 + 2\lambda_1(1-\lambda_1)\big)\|x - y\|^2 \\
&\quad + \big(\lambda_1^2\kappa_1 - \lambda_1(1-\lambda_1)(1-\kappa_1)\big)\|T_1x - x\|^2 \\
&= (\lambda_1 + 1 - \lambda_1)^2\|x - y\|^2 \\
&\quad + \lambda_1\big(\lambda_1\kappa_1 - (1-\lambda_1)(1-\kappa_1)\big)\|T_1x - x\|^2 \\
&= \|x - y\|^2 + \lambda_1(\kappa_1 + \lambda_1 - 1)\|T_1x - x\|^2 \\
&\leq \|x - y\|^2 + \lambda_1\big((\gamma_1 + \gamma_2) - 1\big)\|T_1x - x\|^2. \qquad (2.2)
\end{aligned}
$$

From (2.2), it yields

$$\lambda_1\bigl(1-(\gamma_1+\gamma_2)\bigr)\|T_1x-x\|^2 \le 0.$$

This implies that

$$\|T_1x-x\| = 0.$$

Therefore $x = T_1x$, that is,

$$x \in F(T_1). \tag{2.3}$$

By the definition of U_1 and (2.3), we have

$$U_1x = \lambda_1T_1x + (1-\lambda_1)x = x,$$

that is,

$$x \in F(U_1). \tag{2.4}$$

Again by (2.2) and (2.4), we obtain

$$\begin{aligned}\|x-y\|^2 &\le \|U_1x-y\|^2 + \lambda_2\bigl((\gamma_1+\gamma_2)-1\bigr)\|T_2U_1x-U_1x\|^2 \\ &= \|x-y\|^2 + \lambda_2\bigl((\gamma_1+\gamma_2)-1\bigr)\|T_2x-x\|^2,\end{aligned}$$

which implies that $x = T_2x$, that is,

$$x \in F(T_2). \tag{2.5}$$

By the definition of U_2, (2.4), and (2.5), we get

$$U_2x = \lambda_2T_2U_1x + (1-\lambda_2)U_1x = x,$$

from which it follows that

$$x \in F(U_2).$$

Using the same argument, we can conclude that

$$x \in F(T_i) \quad \text{and} \quad x \in F(U_i), \quad \forall i = 1,2,\dots,N-1.$$

Next, we show that $x \in F(T_N)$. Since

$$\begin{aligned}0 &= Kx - x \\ &= \lambda_NT_NU_{N-1}x + (1-\lambda_N)U_{N-1}x - x \\ &= \lambda_N(T_Nx - x)\end{aligned}$$

and $\lambda_N \in (0,1]$, we obtain

$$x \in F(T_N),$$

from which it follows that

$$x \in \bigcap_{i=1}^{N} F(T_i). \tag{2.6}$$

Therefore

$$F(K) \subseteq \bigcap_{i=1}^{N} F(T_i). \tag{2.7}$$

Hence

$$F(K) = \bigcap_{i=1}^{N} F(T_i). \tag{2.8}$$

To prove (ii), we claim that K is a nonexpansive mapping.

Let $x, y \in C$. Then we obtain

$$\begin{aligned}
&\|Kx - Ky\|^2 \\
&\quad = \bigl\| \bigl(\lambda_N T_N U_{N-1}x + (1-\lambda_N)U_{N-1}x\bigr) - \bigl(\lambda_N T_N U_{N-1}y + (1-\lambda_N)U_{N-1}y\bigr) \bigr\|^2 \\
&\quad = \bigl\| \bigl(U_{N-1}x - \lambda_N(U_{N-1}x - T_N U_{N-1}x)\bigr) - \bigl(U_{N-1}y - \lambda_N(U_{N-1}y - T_N U_{N-1}y)\bigr) \bigr\|^2 \\
&\quad = \bigl\| (U_{N-1}x - U_{N-1}y) - \lambda_N\bigl((I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr) \bigr\|^2 \\
&\quad = \|U_{N-1}x - U_{N-1}y\|^2 + \lambda_N^2 \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\qquad - 2\lambda_N \bigl\langle U_{N-1}x - U_{N-1}y, (I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y \bigr\rangle \\
&\quad \le \|U_{N-1}x - U_{N-1}y\|^2 + \lambda_N^2 \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\qquad - 2\lambda_N \left(\frac{1-\kappa_N}{2}\right) \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\quad = \|U_{N-1}x - U_{N-1}y\|^2 \\
&\qquad + \lambda_N\bigl(\lambda_N - (1-\kappa_N)\bigr) \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\quad \le \|U_{N-1}x - U_{N-1}y\|^2 \\
&\qquad + \lambda_N(\gamma_1 + \gamma_2 - 1) \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\quad = \|U_{N-1}x - U_{N-1}y\|^2 \\
&\qquad - \lambda_N\bigl(1 - (\gamma_1 + \gamma_2)\bigr) \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\quad = \bigl\| \bigl(\lambda_{N-1} T_{N-1} U_{N-2}x + (1-\lambda_{N-1})U_{N-2}x\bigr) - \bigl(\lambda_{N-1} T_{N-1} U_{N-2}y + (1-\lambda_{N-1})U_{N-2}y\bigr) \bigr\|^2 \\
&\qquad - \lambda_N\bigl(1 - (\gamma_1 + \gamma_2)\bigr) \bigl\|(I-T_N)U_{N-1}x - (I-T_N)U_{N-1}y\bigr\|^2 \\
&\quad = \bigl\| \bigl(U_{N-2}x - \lambda_{N-1}(I-T_{N-1})U_{N-2}x\bigr) - \bigl(U_{N-2}y - \lambda_{N-1}(I-T_{N-1})U_{N-2}y\bigr) \bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&= \bigl\|(U_{N-2}x-U_{N-2}y)-\lambda_{N-1}\bigl((I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr)\bigr\|^2 \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&= \|U_{N-2}x-U_{N-2}y\|^2+\lambda_{N-1}^2\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - 2\lambda_{N-1}\bigl\langle U_{N-2}x-U_{N-2}y,(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\rangle \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&\le \|U_{N-2}x-U_{N-2}y\|^2+\lambda_{N-1}^2\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - 2\lambda_{N-1}\left(\frac{1-\kappa_{N-1}}{2}\right)\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&= \|U_{N-2}x-U_{N-2}y\|^2 \\
&\quad + \lambda_{N-1}\bigl(\lambda_{N-1}-(1-\kappa_{N-1})\bigr)\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&\le \|U_{N-2}x-U_{N-2}y\|^2 \\
&\quad + \lambda_{N-1}(\gamma_1+\gamma_2-1)\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&= \|U_{N-2}x-U_{N-2}y\|^2 \\
&\quad - \lambda_{N-1}\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_{N-1})U_{N-2}x-(I-T_{N-1})U_{N-2}y\bigr\|^2 \\
&\quad - \lambda_N\bigl(1-(\gamma_1+\gamma_2)\bigr)\bigl\|(I-T_N)U_{N-1}x-(I-T_N)U_{N-1}y\bigr\|^2 \\
&= \|U_{N-2}x-U_{N-2}y\|^2 \\
&\quad - \bigl(1-(\gamma_1+\gamma_2)\bigr)\sum_{i=N-1}^{N}\lambda_i\bigl\|(I-T_i)U_{i-1}x-(I-T_i)U_{i-1}y\bigr\|^2 \\
&\quad \vdots \\
&\le \|x-y\|^2-\bigl(1-(\gamma_1+\gamma_2)\bigr)\sum_{i=1}^{N}\lambda_i\bigl\|(I-T_i)U_{i-1}x-(I-T_i)U_{i-1}y\bigr\|^2,
\end{aligned}
$$

which implies that

$$
\|Kx-Ky\|^2 \le \|x-y\|^2-\bigl(1-(\gamma_1+\gamma_2)\bigr)\sum_{i=1}^{N}\lambda_i\bigl\|(I-T_i)U_{i-1}x-(I-T_i)U_{i-1}y\bigr\|^2. \tag{2.9}
$$

From (2.9) and $\gamma_1+\gamma_2<1$, we obtain

$$
\|Kx-Ky\| \le \|x-y\|, \quad \forall x,y\in C,
$$

that is, K is a nonexpansive mapping. □

Lemma 2.10 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself with $\kappa_i \leq \gamma_1$ and $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. For every $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, let $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $\lambda_1^n, \lambda_2^n, \ldots, \lambda_N^n$ be real numbers with $0 < \lambda_i, \lambda_i^n < \gamma_2$ and $\gamma_1 + \gamma_2 < 1$ such that $\lambda_i^n \to \lambda_i$ as $n \to \infty$ and $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$. For every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_1, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_1^n, \lambda_2^n, \ldots, \lambda_N^n$, respectively. Then, for every bounded sequence $\{x_n\}$ in C, the following properties hold*:

(i) $\lim_{n\to\infty} \|K_n x_n - K x_n\| = 0$;

(ii) $\sum_{n=1}^{\infty} \|K_n x_{n-1} - K_{n-1} x_{n-1}\| < \infty$.

Proof Let $\{x_n\}$ be a bounded sequence in C and let U_k and $U_{n,k}$ be generated by $T_1, T_1, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_1, \ldots, T_N$ and $\lambda_1^n, \lambda_2^n, \ldots, \lambda_N^n$, respectively.

First, we shall prove that (i) holds. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \|U_{n,1}x_n - U_1 x_n\| &= \|\lambda_1^n T_1 x_n + (1-\lambda_1^n)x_n - (\lambda_1 T_1 x_n + (1-\lambda_1)x_n)\| \\ &= \|\lambda_1^n T_1 x_n - \lambda_1^n x_n - \lambda_1 T_1 x_n + \lambda_1 x_n\| \\ &= \|(\lambda_1^n - \lambda_1)T_1 x_n - (\lambda_1^n - \lambda_1)x_n\| \\ &= |\lambda_1^n - \lambda_1| \|T_1 x_n - x_n\|. \end{aligned} \tag{2.10}$$

For $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned} &\|U_{n,k}x_n - U_k x_n\| \\ &\quad= \|\lambda_k^n T_k U_{n,k-1}x_n + (1-\lambda_k^n)U_{n,k-1}x_n - (\lambda_k T_k U_{k-1}x_n + (1-\lambda_k)U_{k-1}x_n)\| \\ &\quad= \|\lambda_k^n T_k U_{n,k-1}x_n - \lambda_k T_k U_{k-1}x_n + (1-\lambda_k^n)U_{n,k-1}x_n - (1-\lambda_k)U_{k-1}x_n\| \\ &\quad= \|\lambda_k^n T_k U_{n,k-1}x_n - \lambda_k^n T_k U_{k-1}x_n + \lambda_k^n T_k U_{k-1}x_n - \lambda_k T_k U_{k-1}x_n \\ &\qquad + (1-\lambda_k^n)U_{n,k-1}x_n - (1-\lambda_k^n)U_{k-1}x_n + (1-\lambda_k^n)U_{k-1}x_n \\ &\qquad - (1-\lambda_k)U_{k-1}x_n\| \\ &\quad= \|\lambda_k^n(T_k U_{n,k-1}x_n - T_k U_{k-1}x_n) + (\lambda_k^n - \lambda_k)T_k U_{k-1}x_n \\ &\qquad + (1-\lambda_k^n)(U_{n,k-1}x_n - U_{k-1}x_n) + (1-\lambda_k^n - (1-\lambda_k))U_{k-1}x_n\| \\ &\quad\leq \lambda_k^n \|T_k U_{n,k-1}x_n - T_k U_{k-1}x_n\| + |\lambda_k^n - \lambda_k| \|T_k U_{k-1}x_n\| \\ &\qquad + (1-\lambda_k^n)\|U_{n,k-1}x_n - U_{k-1}x_n\| + |\lambda_k - \lambda_k^n| \|U_{k-1}x_n\| \\ &\quad\leq \lambda_k^n \frac{1+\kappa_k}{1-\kappa_k}\|U_{n,k-1}x_n - U_{k-1}x_n\| + |\lambda_k^n - \lambda_k| \|T_k U_{k-1}x_n\| \\ &\qquad + (1-\lambda_k^n)\|U_{n,k-1}x_n - U_{k-1}x_n\| + |\lambda_k - \lambda_k^n| \|U_{k-1}x_n\| \\ &\quad\leq \frac{1+\kappa_k}{1-\kappa_k}\|U_{n,k-1}x_n - U_{k-1}x_n\| + \frac{1-\kappa_k}{1-\kappa_k}\|U_{n,k-1}x_n - U_{k-1}x_n\| \\ &\qquad + |\lambda_k^n - \lambda_k| (\|T_k U_{k-1}x_n\| + \|U_{k-1}x_n\|) \\ &\quad= \frac{2}{1-\kappa_k}\|U_{n,k-1}x_n - U_{k-1}x_n\| \\ &\qquad + |\lambda_k^n - \lambda_k| (\|T_k U_{k-1}x_n\| + \|U_{k-1}x_n\|). \end{aligned} \tag{2.11}$$

By (2.10) and (2.11), we get

$$\begin{aligned}
&\|K_n x_n - K x_n\| \\
&\quad = \|U_{n,N} x_n - U_N x_n\| \\
&\quad \le \frac{2}{1-\kappa_N} \|U_{n,N-1} x_n - U_{N-1} x_n\| \\
&\qquad + \left|\lambda_N^n - \lambda_N\right| \left(\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|\right) \\
&\quad \le \frac{2}{1-\kappa_N} \left(\frac{2}{1-\kappa_{N-1}} \|U_{n,N-2} x_n - U_{N-2} x_n\| \right.\\
&\qquad \left. + \left|\lambda_{N-1}^n - \lambda_{N-1}\right| \left(\|T_{N-1} U_{N-2} x_n\| + \|U_{N-2} x_n\|\right)\right) \\
&\qquad + \left|\lambda_N^n - \lambda_N\right| \left(\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|\right) \\
&\quad = \left(\frac{2}{1-\kappa_N}\right)\left(\frac{2}{1-\kappa_{N-1}}\right) \|U_{n,N-2} x_n - U_{N-2} x_n\| \\
&\qquad + \frac{2}{1-\kappa_N} \left|\lambda_{N-1}^n - \lambda_{N-1}\right| \left(\|T_{N-1} U_{N-2} x_n\| + \|U_{N-2} x_n\|\right) \\
&\qquad + \left|\lambda_N^n - \lambda_N\right| \left(\|T_N U_{N-1} x_n\| + \|U_{N-1} x_n\|\right) \\
&\quad = \prod_{j=N-1}^{N} \left(\frac{2}{1-\kappa_j}\right) \|U_{n,N-2} x_n - U_{N-2} x_n\| \\
&\qquad + \sum_{j=N-1}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_j^n - \lambda_j\right| \left(\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|\right) \\
&\quad \vdots \\
&\quad \le \prod_{j=2}^{N} \left(\frac{2}{1-\kappa_j}\right) \|U_{n,1} x_n - U_1 x_n\| \\
&\qquad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_j^n - \lambda_j\right| \left(\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|\right) \\
&\quad = \prod_{j=2}^{N} \left(\frac{2}{1-\kappa_j}\right) \left|\lambda_1^n - \lambda_1\right| \|T_1 x_n - x_n\| \\
&\qquad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_j^n - \lambda_j\right| \left(\|T_j U_{j-1} x_n\| + \|U_{j-1} x_n\|\right).
\end{aligned} \tag{2.12}$$

By (2.12) and the fact that $\lambda_i^n \to \lambda_i$ as $n \to \infty$ for all $i = 1, 2, \dots, N$, we deduce that $\lim_{n\to\infty} \|K_n x_n - K x_n\| = 0$.

Next, we will claim that (ii) holds. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\|U_{n,1} x_{n-1} - U_{n-1,1} x_{n-1}\| \\
&\quad = \left\|\lambda_1^n T_1 x_{n-1} + \left(1-\lambda_1^n\right) x_{n-1} - \left(\lambda_1^{n-1} T_1 x_{n-1} + \left(1-\lambda_1^{n-1}\right) x_{n-1}\right)\right\| \\
&\quad = \left\|\lambda_1^n T_1 x_{n-1} - \lambda_1^n x_{n-1} - \lambda_1^{n-1} T_1 x_{n-1} + \lambda_1^{n-1} x_{n-1}\right\|
\end{aligned}$$

$$\begin{aligned}
&= \left\| \left(\lambda_1^n - \lambda_1^{n-1}\right) T_1 x_{n-1} - \left(\lambda_1^n - \lambda_1^{n-1}\right) x_{n-1} \right\| \\
&= \left|\lambda_1^n - \lambda_1^{n-1}\right| \|T_1 x_{n-1} - x_{n-1}\|. \qquad (2.13)
\end{aligned}$$

For $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned}
&\|U_{n,k} x_{n-1} - U_{n-1,k} x_{n-1}\| \\
&\quad = \left\| \lambda_k^n T_k U_{n,k-1} x_{n-1} + \left(1 - \lambda_k^n\right) U_{n,k-1} x_{n-1} - \left(\lambda_k^{n-1} T_k U_{n-1,k-1} x_{n-1} \right.\right. \\
&\qquad \left.\left. + \left(1 - \lambda_k^{n-1}\right) U_{n-1,k-1} x_{n-1}\right) \right\| \\
&\quad = \left\| \lambda_k^n T_k U_{n,k-1} x_{n-1} - \lambda_k^{n-1} T_k U_{n-1,k-1} x_{n-1} + \left(1 - \lambda_k^n\right) U_{n,k-1} x_{n-1} \right. \\
&\qquad \left. - \left(1 - \lambda_k^{n-1}\right) U_{n-1,k-1} x_{n-1} \right\| \\
&\quad = \left\| \lambda_k^n T_k U_{n,k-1} x_{n-1} - \lambda_k^n T_k U_{n-1,k-1} x_{n-1} + \lambda_k^n T_k U_{n-1,k-1} x_{n-1} \right. \\
&\qquad - \lambda_k^{n-1} T_k U_{n-1,k-1} x_{n-1} + \left(1 - \lambda_k^n\right) U_{n,k-1} x_{n-1} - \left(1 - \lambda_k^n\right) U_{n-1,k-1} x_{n-1} \\
&\qquad \left. + \left(1 - \lambda_k^n\right) U_{n-1,k-1} x_{n-1} - \left(1 - \lambda_k^{n-1}\right) U_{n-1,k-1} x_{n-1} \right\| \\
&\quad = \left\| \lambda_k^n (T_k U_{n,k-1} x_{n-1} - T_k U_{n-1,k-1} x_{n-1}) + \left(\lambda_k^n - \lambda_k^{n-1}\right) T_k U_{n-1,k-1} x_{n-1} \right. \\
&\qquad + \left(1 - \lambda_k^n\right)(U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}) \\
&\qquad \left. + \left(1 - \lambda_k^n - \left(1 - \lambda_k^{n-1}\right)\right) U_{n-1,k-1} x_{n-1} \right\| \\
&\quad \le \lambda_k^n \|T_k U_{n,k-1} x_{n-1} - T_k U_{n-1,k-1} x_{n-1}\| + \left|\lambda_k^n - \lambda_k^{n-1}\right| \|T_k U_{n-1,k-1} x_{n-1}\| \\
&\qquad + \left(1 - \lambda_k^n\right) \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| + \left|\lambda_k^n - \lambda_k^{n-1}\right| \|U_{n-1,k-1} x_{n-1}\| \\
&\quad \le \lambda_k^n \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| + \left|\lambda_k^n - \lambda_k^{n-1}\right| \|T_k U_{n-1,k-1} x_{n-1}\| \\
&\qquad + \left(1 - \lambda_k^n\right) \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| + \left|\lambda_k^n - \lambda_k^{n-1}\right| \|U_{n-1,k-1} x_{n-1}\| \\
&\quad \le \frac{1 + \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| \\
&\qquad + \frac{1 - \kappa_k}{1 - \kappa_k} \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| \\
&\qquad + \left|\lambda_k^n - \lambda_k^{n-1}\right| \left(\|T_k U_{n-1,k-1} x_{n-1}\| + \|U_{n-1,k-1} x_{n-1}\|\right) \\
&\quad = \frac{2}{1 - \kappa_k} \|U_{n,k-1} x_{n-1} - U_{n-1,k-1} x_{n-1}\| \\
&\qquad + \left|\lambda_k^n - \lambda_k^{n-1}\right| \left(\|T_k U_{n-1,k-1} x_{n-1}\| + \|U_{n-1,k-1} x_{n-1}\|\right). \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14), we obtain

$$\begin{aligned}
&\|K_n x_{n-1} - K_{n-1} x_{n-1}\| \\
&\quad = \|U_{n,N} x_{n-1} - U_{n-1,N} x_{n-1}\| \\
&\quad \le \frac{2}{1 - \kappa_N} \|U_{n,N-1} x_{n-1} - U_{n-1,N-1} x_{n-1}\| \\
&\qquad + \left|\lambda_N^n - \lambda_N^{n-1}\right| \left(\|T_N U_{n-1,N-1} x_{n-1}\| + \|U_{n-1,N-1} x_{n-1}\|\right) \\
&\quad \le \frac{2}{1 - \kappa_N} \left(\frac{2}{1 - \kappa_{N-1}} \|U_{n,N-2} x_{n-1} - U_{n-1,N-2} x_{n-1}\| \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left|\lambda_{N-1}^{n} - \lambda_{N-1}^{n-1}\right| \left(\|T_{N-1}U_{n-1,N-2}x_{n-1}\| + \|U_{n-1,N-2}x_{n-1}\| \right) \Bigg) \\
&\quad + \left|\lambda_{N}^{n} - \lambda_{N}^{n-1}\right| \left(\|T_{N}U_{n-1,N-1}x_{n-1}\| + \|U_{n-1,N-1}x_{n-1}\| \right) \\
&= \left(\frac{2}{1-\kappa_N}\right)\left(\frac{2}{1-\kappa_{N-1}}\right) \|U_{n,N-2}x_{n-1} - U_{n-1,N-2}x_{n-1}\| \\
&\quad + \frac{2}{1-\kappa_N} \left|\lambda_{N-1}^{n} - \lambda_{N-1}^{n-1}\right| \left(\|T_{N-1}U_{n-1,N-2}x_{n-1}\| + \|U_{n-1,N-2}x_{n-1}\| \right) \\
&\quad + \left|\lambda_{N}^{n} - \lambda_{N}^{n-1}\right| \left(\|T_{N}U_{n-1,N-1}x_{n-1}\| + \|U_{n-1,N-1}x_{n-1}\| \right) \\
&= \prod_{j=N-1}^{N} \left(\frac{2}{1-\kappa_j}\right) \|U_{n,N-2}x_{n-1} - U_{n-1,N-2}x_{n-1}\| \\
&\quad + \sum_{j=N-1}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_{j}^{n} - \lambda_{j}^{n-1}\right| \left(\|T_{j}U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\| \right) \\
&\quad \vdots \\
&\leq \prod_{j=2}^{N} \left(\frac{2}{1-\kappa_j}\right) \|U_{n,1}x_{n-1} - U_{n-1,1}x_{n-1}\| \\
&\quad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_{j}^{n} - \lambda_{j}^{n-1}\right| \left(\|T_{j}U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\| \right) \\
&= \prod_{j=2}^{N} \left(\frac{2}{1-\kappa_j}\right) \left|\lambda_{1}^{n} - \lambda_{1}^{n-1}\right| \|T_1 x_{n-1} - x_{n-1}\| \\
&\quad + \sum_{j=2}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_{j}^{n} - \lambda_{j}^{n-1}\right| \left(\|T_{j}U_{n-1,j-1}x_{n-1}\| + \|U_{n-1,j-1}x_{n-1}\| \right) \\
&\leq \prod_{j=2}^{N} \left(\frac{2}{1-\kappa_j}\right) \left|\lambda_{1}^{n} - \lambda_{1}^{n-1}\right| M + 2\sum_{j=2}^{N} \left(\frac{2}{1-\kappa_{j+1}}\right)^{N-j} \left|\lambda_{j}^{n} - \lambda_{j}^{n-1}\right| M, \qquad (2.15)
\end{aligned}
$$

where $M = \max_{n\in\mathbb{N}}\{\|T_1x_{n-1} - x_{n-1}\|, \|T_jU_{n-1,j-1}x_{n-1}\|, \|U_{n-1,j-1}x_{n-1}\|\}$, for all $j = 2,3,\ldots,N$. Hence, by (2.15) and $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$ for all $i = 1,2,\ldots,N$, we have $\sum_{n=1}^{\infty} \|K_nx_{n-1} - K_{n-1}x_{n-1}\| < \infty$. □

In 2010, Kangtunyakarn and Suantai [23] introduced the S-mapping generated by the finite family of κ_i-strictly pseudo-contractions in Hilbert space as in the following definition.

Definition 2.3 ([23]) Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractions of C into itself. For each $j = 1,2,\ldots,N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mappings $S : C \to C$ as follows:

$$
\begin{aligned}
&U_0 = I, \\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I,
\end{aligned}
$$

$$U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I,$$
$$\vdots$$
$$U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I,$$
$$S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.$$

This mapping is called S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Furthermore, they obtained the following important lemma.

Lemma 2.11 ([23]) *Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractions of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_2^j \in (\kappa, 1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N \in [\kappa, 1)$, $\alpha_2^j \in [\kappa, 1)$ for all $j = 1,2,\ldots,N$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.*

By putting $\alpha_1^j = \lambda_j$ and $\alpha_2^j = 0$, for all $j = 1,2,\ldots,N$, we see that the S-mapping reduces to the K-mapping as defined in Definition 2.2. Moreover, from Lemma 2.11, we have the following result.

Lemma 2.12 *Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractions of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$ and let $\lambda_j \in (\kappa, 1) \subset [0,1]$, for all $j = 1,2,\ldots,N-1$ and $\lambda_N \in (\kappa, 1]$. Let K be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^N F(T_i)$ and K is a nonexpansive mapping.*

Remark 2.13 For the result of Lemma 2.9 in our work, we obtain some improvement as follows:

(i) We relax the conditions of κ_i and λ_i in Lemma 2.12 in sense that κ_i is not depended on λ_i, for all $i = 1,2,\ldots,N$.
(ii) We do not assume the condition $\kappa = \max\{\kappa_i : i = 1,2,\ldots,N\}$.

Example 2.14 Let $\mathbb{R}$ be the set of real numbers and let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_i x = -(i+1)x, \quad \text{for all } x \in \mathbb{R},$$

and $\lambda_i = \frac{i+5}{i+6}$, for all $i = 1,2,\ldots,5$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_5$ and $\lambda_1, \lambda_2, \ldots, \lambda_5$. Then $F(K) = \bigcap_{i=1}^5 F(T_i) = \{0\}$.

Solution. It is easy to see that T_i is κ_i-strictly pseudo-contractive mapping with $\kappa_i = \frac{i}{i+2}$. We obtain $\kappa = \max\{\kappa_i : i = 1,2,\ldots,5\} = \frac{5}{7}$ and $\lambda_i \in (\frac{5}{7}, 1]$, for all $i = 1,2,\ldots,5$. By the definition of a K-mapping, we have

$$U_1 x = \left(\frac{6}{7}\right)(-2x) + \left(1 - \frac{6}{7}\right)x,$$
$$U_2 x = \left(\frac{7}{8}\right)(-3U_1 x) + \left(1 - \frac{7}{8}\right)U_1 x,$$

$$U_3x = \left(\frac{8}{9}\right)(-4U_2x) + \left(1 - \frac{8}{9}\right)U_2x,$$
$$U_4x = \left(\frac{9}{10}\right)(-5U_3x) + \left(1 - \frac{9}{10}\right)U_3x,$$
$$Kx = U_5x = \left(\frac{10}{11}\right)(-6U_4x) + \left(1 - \frac{10}{11}\right)U_4x. \tag{2.16}$$

Observe that $\bigcap_{i=1}^{5} F(T_i) = \{0\}$. Then, by Lemma 2.12, we obtain

$$F(K) = \bigcap_{i=1}^{5} F(T_i) = \{0\}.$$

Next, we give an example for Lemma 2.9.

Example 2.15 Let $\mathbb{R}$ be the set of real numbers and let $T_i : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_ix = -(i+1)x, \quad \text{for all } x \in \mathbb{R},$$

and $\lambda_i = \frac{i}{5i+1}$, for all $i = 1, 2, \ldots, 5$. Let K be the K-mapping generated by $T_1, T_2, \ldots, T_5$ and $\lambda_1, \lambda_2, \ldots, \lambda_5$. Choose $\gamma_1 = \frac{11}{14}$ and $\gamma_2 = \frac{11}{52}$, from which it follows that $\gamma_1 + \gamma_2 = \frac{11}{14} + \frac{11}{52} = \frac{726}{728} = \frac{363}{364} < 1$. Then, by Lemma 2.9, we obtain $F(K) = \bigcap_{i=1}^{5} F(T_i) = \{0\}$.

3 Strong convergence theorem

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i, $\Phi_i : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying* (H1)-(H3). *Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and $A : C \to C$ be an α-inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^{N}$ be a finite family of κ_i-strictly pseudo-contractive mappings and $\kappa_i \le \gamma_1$ with $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} (MGEP)_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1^n, \lambda_2^n, \ldots, \lambda_N^n$ where $0 < \phi \le \lambda_i^n \le \psi < \gamma_2$, for all $i = 1, 2, \ldots, N$ and $\gamma_1 + \gamma_2 < 1$. For every $i = 1, 2, \ldots, N$, let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $w_1^i \in S_i(I - r_1^iA)x_1$, there exist sequences $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C$ such that*

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \le (1 + \frac{1}{n})\mathcal{H}(S_i(I - r_n^iA)x_n, S_i(I - r_{n+1}^iA)x_{n+1}), \\ \quad w_n^i \in S_i(I - r_n^iA)x_n \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i}\langle u_n^i - x_n, y - u_n^i\rangle + \langle Ax_n, y - u_n^i\rangle \\ \quad \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n(\sum_{i=1}^{N} a_n^i u_n^i) + \delta_n K_n x_n, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $f : C \to C$ be a contraction mapping with a constant ξ and $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ with $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \ge 1$. Suppose the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \le \beta_n, \delta_n \le \upsilon < 1$;

(iii) $0 \le \eta \le a_n^i \le \sigma < 1$, *for all* $i = 1, 2, \ldots, N - 1$ *and* $0 < \eta \le a_n^N \le \sigma \le 1$ *with* $\sum_{n=1}^{N} a_n^i = 1$;

(iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, *for all* $n \in \mathbb{N}$ *and* $i = 1, 2, \dots, N$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty} |\lambda_i^{n+1} - \lambda_i^n| < \infty$, *for all* $i = 1, 2, \dots, N$;

(vi) *for each* $i = 1, 2, \dots, N$, *there exists* $\rho_i > 0$ *such that*

$$\begin{aligned}
&\Phi_i\big(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\big) + \Phi_i\big(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\big) \\
&\quad \leq -\rho_i \big\| T_{r_1^i}(x_1) - T_{r_2^i}(x_2) \big\|^2,
\end{aligned} \tag{3.2}$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ *and* $w_j^i \in S_i(x_j)$, *for* $j = 1, 2$, *where* $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ *and* $\{u_n^i\}$ *converges strongly to* $q = P_{\mathcal{F}} f(q)$, *for every* $i = 1, 2, \dots, N$.

Proof The proof shall be divided into seven steps.

Step 1. We will prove that $I - r_n^i A$ is nonexpansive, for all $i = 1, 2, \dots, N$.

From (3.1), we have

$$\Phi_i\big(w_n^i, u_n^i, y\big) + \varphi(y) - \varphi\big(u_n^i\big) + \frac{1}{r_n^i}\big\langle u_n^i - \big(I - r_n^i A\big)x_n, y - u_n^i\big\rangle \geq 0, \tag{3.3}$$

for every $y \in C$. From (3.3) and Theorem 2.8, we obtain

$$u_n^i = T_{r_n^i}\big(I - r_n^i A\big)x_n, \quad \forall i = 1, 2, \dots, N.$$

Put $r^i \in \Theta_i$ for all $i = 1, 2, \dots, N$. From (3.2), we have

$$\begin{aligned}
&\Phi_i\big(w_1^i, T_{r^i}(x_1), T_{r^i}(x_2)\big) + \Phi_i\big(w_2^i, T_{r^i}(x_2), T_{r^i}(x_1)\big) \\
&\quad \leq -\rho_i \big\| T_{r^i}(x_1) - T_{r^i}(x_2) \big\|^2 \leq 0,
\end{aligned} \tag{3.4}$$

for all $(x_1, x_2) \in C \times C$ and $w_j^i \in S_i(x_j)$, $j = 1, 2$.

From (3.4), we find the implication that Theorem 2.8 holds.

It obvious to see that $I - r_n^i A$ is a nonexpansive mapping, for every $i = 1, 2, \dots, N$.

Indeed, A is α-inverse strongly monotone with $r_n^i \in (0, 2\alpha)$, we get

$$\begin{aligned}
&\big\| \big(I - r_n^i A\big)x - \big(I - r_n^i A\big)y \big\|^2 \\
&\quad = \big\| x - y - r_n^i(Ax - Ay) \big\|^2 \\
&\quad = \|x - y\|^2 - 2r_n^i \langle x - y, Ax - Ay\rangle + \big(r_n^i\big)^2 \|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^2 - 2\alpha r_n^i \|Ax - Ay\|^2 + \big(r_n^i\big)^2 \|Ax - Ay\|^2 \\
&\quad = \|x - y\|^2 + r_n^i\big(r_n^i - 2\alpha\big) \|Ax - Ay\|^2 \\
&\quad \leq \|x - y\|^2.
\end{aligned}$$

Thus $I - r_n^i A$ is a nonexpansive mapping, for all $i = 1, 2, \dots, N$.

Step 2. We will show that $\{x_n\}$ is bounded.

Let $z \in \mathcal{F}$. By nonexpansiveness of K_n, we have

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&\quad \le \alpha_n \big\|f(x_n) - z\big\| + \beta_n \left\| \sum_{i=1}^{N} a_n^i \big(u_n^i - z\big) \right\| + \delta_n \|K_n x_n - z\| \\
&\quad \le \alpha_n \big\|f(x_n) - f(z) + f(z) - z\big\| + \beta_n \sum_{i=1}^{N} a_n^i \big\|u_n^i - z\big\| + \delta_n \|x_n - z\| \\
&\quad \le \alpha_n \big(\big\|f(x_n) - f(z)\big\| + \big\|f(z) - z\big\|\big) + \beta_n \sum_{i=1}^{N} a_n^i \big\| T_{r_n^i} \big(I - r_n^i A\big) x_n - z\big\| \\
&\qquad + \delta_n \|x_n - z\| \\
&\quad \le \alpha_n \big(\xi \|x_n - z\| + \big\|f(z) - z\big\|\big) + \beta_n \sum_{i=1}^{N} a_n^i \|x_n - z\| + \delta_n \|x_n - z\| \\
&\quad = \big(1 - \alpha_n(1 - \xi)\big) \|x_n - z\| + \alpha_n \big\|f(z) - z\big\| \\
&\quad \le \max\left\{ \|x_1 - z\|, \frac{\|f(z) - z\|}{1 - \xi} \right\}.
\end{aligned}$$

By induction, we have $\|x_n - z\| \le \max\{\|x_1 - z\|, \frac{\|f(z)-z\|}{1-\xi}\}$, $\forall n \in \mathbb{N}$. It follows that $\{x_n\}$ is bounded and so is $\{u_n^i\}$, $\forall i = 1, 2, \ldots, N$.

Step 3. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

By the definition of x_n, we obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \left\| \alpha_n f(x_n) + \beta_n \left(\sum_{i=1}^{N} a_n^i u_n^i \right) + \delta_n K_n x_n \right. \\
&\qquad \left. - \left(\alpha_{n-1} f(x_{n-1}) + \beta_{n-1} \left(\sum_{i=1}^{N} a_{n-1}^i u_{n-1}^i \right) + \delta_{n-1} K_{n-1} x_{n-1} \right) \right\| \\
&\quad \le \alpha_n \big\|f(x_n) - f(x_{n-1})\big\| + |\alpha_n - \alpha_{n-1}| \big\|f(x_{n-1})\big\| \\
&\qquad + \beta_n \left\| \sum_{i=1}^{N} a_n^i u_n^i - \sum_{i=1}^{N} a_{n-1}^i u_{n-1}^i \right\| + |\beta_n - \beta_{n-1}| \left\| \sum_{i=1}^{N} a_{n-1}^i u_{n-1}^i \right\| \\
&\qquad + \delta_n \|K_n x_n - K_n x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| \\
&\qquad + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\quad \le \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \big\|f(x_{n-1})\big\| \\
&\qquad + \beta_n \left\| \sum_{i=1}^{N} a_n^i u_n^i - \sum_{i=1}^{N} a_n^i u_{n-1}^i + \sum_{i=1}^{N} a_n^i u_{n-1}^i - \sum_{i=1}^{N} a_{n-1}^i u_{n-1}^i \right\| \\
&\qquad + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \big\|u_{n-1}^i\big\| + \delta_n \|x_n - x_{n-1}\| \\
&\qquad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \left\| \sum_{i=1}^{N} a_n^i (u_n^i - u_{n-1}^i) \right\| \\
&\quad + \beta_n \left\| \sum_{i=1}^{N} (a_n^i - a_{n-1}^i) u_{n-1}^i \right\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \|u_{n-1}^i\| \\
&\quad + \delta_n \|x_n - x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\le \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| + \beta_n \sum_{i=1}^{N} a_n^i \|u_n^i - u_{n-1}^i\| \\
&\quad + \beta_n \sum_{i=1}^{N} |a_n^i - a_{n-1}^i| \|u_{n-1}^i\| + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \|u_{n-1}^i\| \\
&\quad + \delta_n \|x_n - x_{n-1}\| + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\|. \qquad (3.5)
\end{aligned}$$

From $u_n^i = T_{r_n^i}(I - r_n^i A)x_n$, for all $i = 1, 2, \ldots, N$, we have

$$\Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, y - u_n^i \rangle \ge 0, \quad \forall y \in C$$

and

$$\begin{aligned}
&\Phi_i(w_{n+1}^i, u_{n+1}^i, y) + \varphi(y) - \varphi(u_{n+1}^i) \\
&\quad + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, y - u_{n+1}^i \rangle \ge 0, \quad \forall y \in C.
\end{aligned}$$

In particular, we obtain

$$\Phi_i(w_n^i, u_n^i, u_{n+1}^i) + \varphi(u_{n+1}^i) - \varphi(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, u_{n+1}^i - u_n^i \rangle \ge 0 \qquad (3.6)$$

and

$$\begin{aligned}
&\Phi_i(w_{n+1}^i, u_{n+1}^i, u_n^i) + \varphi(u_n^i) - \varphi(u_{n+1}^i) \\
&\quad + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, u_n^i - u_{n+1}^i \rangle \ge 0. \qquad (3.7)
\end{aligned}$$

Summing up (3.6) with (3.7) and applying (3.4), we get

$$\begin{aligned}
&\frac{1}{r_n^i} \langle u_n^i - (I - r_n^i A)x_n, u_{n+1}^i - u_n^i \rangle \\
&\quad + \frac{1}{r_{n+1}^i} \langle u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}, u_n^i - u_{n+1}^i \rangle \ge 0,
\end{aligned}$$

which implies that

$$\left\langle u_{n+1}^i - u_n^i, \frac{u_n^i - (I - r_n^i A)x_n}{r_n^i} - \frac{u_{n+1}^i - (I - r_{n+1}^i A)x_{n+1}}{r_{n+1}^i} \right\rangle \ge 0.$$

It follows that

$$\left\langle u_{n+1}^i - u_n^i, u_n^i - u_{n+1}^i + u_{n+1}^i - \left(I - r_n^i A\right)x_n - \frac{r_n^i}{r_{n+1}^i}\left(u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right)\right\rangle \geq 0. \tag{3.8}$$

From (3.8), we obtain

$$\begin{aligned}
&\left\|u_{n+1}^i - u_n^i\right\|^2 \\
&\quad \leq \left\langle u_{n+1}^i - u_n^i, u_{n+1}^i - \left(I - r_n^i A\right)x_n - \frac{r_n^i}{r_{n+1}^i}\left(u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right)\right\rangle \\
&\quad = \left\langle u_{n+1}^i - u_n^i, \left(I - r_{n+1}^i A\right)x_{n+1} - \left(I - r_n^i A\right)x_n \right. \\
&\qquad \left. + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right)\left(u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right)\right\rangle \\
&\quad \leq \left\|u_{n+1}^i - u_n^i\right\| \left\| \left(I - r_{n+1}^i A\right)x_{n+1} - \left(I - r_n^i A\right)x_n \right. \\
&\qquad \left. + \left(1 - \frac{r_n^i}{r_{n+1}^i}\right)\left(u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right)\right\| \\
&\quad \leq \left\|u_{n+1}^i - u_n^i\right\| \left[\left\| \left(I - r_{n+1}^i A\right)x_{n+1} - \left(I - r_{n+1}^i A\right)x_n + \left(I - r_{n+1}^i A\right)x_n \right.\right. \\
&\qquad \left.\left. - \left(I - r_n^i A\right)x_n \right\| + \left|1 - \frac{r_n^i}{r_{n+1}^i}\right| \left\|u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right\| \right] \\
&\quad \leq \left\|u_{n+1}^i - u_n^i\right\| \left[\left\| \left(I - r_{n+1}^i A\right)x_{n+1} - \left(I - r_{n+1}^i A\right)x_n \right\| \right. \\
&\qquad + \left\| \left(I - r_{n+1}^i A\right)x_n - \left(I - r_n^i A\right)x_n \right\| \\
&\qquad \left. + \frac{1}{r_{n+1}^i}\left|r_{n+1}^i - r_n^i\right| \left\|u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right\| \right] \\
&\quad \leq \left\|u_{n+1}^i - u_n^i\right\| \left[\|x_{n+1} - x_n\| + \left|r_{n+1}^i - r_n^i\right| \|Ax_n\| \right. \\
&\qquad \left. + \frac{1}{\epsilon}\left|r_{n+1}^i - r_n^i\right| \left\|u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right\| \right],
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\left\|u_{n+1}^i - u_n^i\right\| \leq{}& \|x_{n+1} - x_n\| + \left|r_{n+1}^i - r_n^i\right| \|Ax_n\| \\
&+ \frac{1}{\epsilon}\left|r_{n+1}^i - r_n^i\right| \left\|u_{n+1}^i - \left(I - r_{n+1}^i A\right)x_{n+1}\right\|.
\end{aligned} \tag{3.9}$$

From (3.9), we have

$$\begin{aligned}
\left\|u_n^i - u_{n-1}^i\right\| \leq{}& \|x_n - x_{n-1}\| + \left|r_n^i - r_{n-1}^i\right| \|Ax_{n-1}\| \\
&+ \frac{1}{\epsilon}\left|r_n^i - r_{n-1}^i\right| \left\|u_n^i - \left(I - r_n^i A\right)x_n\right\|.
\end{aligned} \tag{3.10}$$

From (3.5) and (3.10), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \leq \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + \beta_n \sum_{i=1}^{N} a_n^i \Big[\|x_n - x_{n-1}\| \\
&\qquad + |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{1}{\epsilon} |r_n^i - r_{n-1}^i| \| u_n^i - (I - r_n^i A)x_n \| \Big] \\
&\qquad + \beta_n \sum_{i=1}^{N} |a_n^i - a_{n-1}^i| \| u_{n-1}^i \| + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \| u_{n-1}^i \| + \delta_n \|x_n - x_{n-1}\| \\
&\qquad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\quad = \alpha_n \xi \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| + \beta_n \sum_{i=1}^{N} a_n^i \|x_n - x_{n-1}\| \\
&\qquad + \beta_n \sum_{i=1}^{N} a_n^i |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{\beta_n}{\epsilon} \sum_{i=1}^{N} a_n^i |r_n^i - r_{n-1}^i| \| u_n^i - (I - r_n^i A)x_n \| \\
&\qquad + \beta_n \sum_{i=1}^{N} |a_n^i - a_{n-1}^i| \| u_{n-1}^i \| + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \| u_{n-1}^i \| + \delta_n \|x_n - x_{n-1}\| \\
&\qquad + \delta_n \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\| \\
&\quad \leq \big(1 - \alpha_n (1 - \xi)\big) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \| f(x_{n-1}) \| \\
&\qquad + \sum_{i=1}^{N} |r_n^i - r_{n-1}^i| \|Ax_{n-1}\| + \frac{1}{\epsilon} \sum_{i=1}^{N} |r_n^i - r_{n-1}^i| \| u_n^i - (I - r_n^i A)x_n \| \\
&\qquad + \sum_{i=1}^{N} |a_n^i - a_{n-1}^i| \| u_{n-1}^i \| + |\beta_n - \beta_{n-1}| \sum_{i=1}^{N} a_{n-1}^i \| u_{n-1}^i \| \\
&\qquad + \|K_n x_{n-1} - K_{n-1} x_{n-1}\| + |\delta_n - \delta_{n-1}| \|K_{n-1} x_{n-1}\|.
\end{aligned}
$$

Applying the conditions (i), (v), Lemma 2.6, and Lemma 2.10(ii), we obtain

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}
$$

Step 4. We will show that $\lim_{n\to\infty} \|u_n^i - x_n\| = \lim_{n\to\infty} \|K_n x_n - x_n\| = 0$, $\forall i = 1, 2, \ldots, N$. Since $T_{r_n^i}$ is a firmly nonexpansive mapping, for every $i = 1, 2, \ldots, N$, we obtain

$$
\begin{aligned}
&\big\| T_{r_n^i} (I - r_n^i A)x_n - z \big\|^2 \\
&\quad = \big\| T_{r_n^i} (I - r_n^i A)x_n - T_{r_n^i} (I - r_n^i A)z \big\|^2 \\
&\quad \leq \big\langle (I - r_n^i A)x_n - (I - r_n^i A)z, u_n^i - z \big\rangle \\
&\quad = \frac{1}{2} \big(\big\| (I - r_n^i A)x_n - (I - r_n^i A)z \big\|^2 + \big\| u_n^i - z \big\|^2 \\
&\qquad - \big\| (I - r_n^i A)x_n - (I - r_n^i A)z - (u_n^i - z) \big\|^2 \big)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{2}\left(\|x_n - z\|^2 + \left\|u_n^i - z\right\|^2 - \left\|\left(x_n - u_n^i\right) - r_n^i(Ax_n - Az)\right\|^2\right) \\
&= \frac{1}{2}\left(\|x_n - z\|^2 + \left\|u_n^i - z\right\|^2 - \left\|x_n - u_n^i\right\|^2 - \left(r_n^i\right)^2\|Ax_n - Az\|^2 \right. \\
&\quad \left. + 2r_n^i\left\langle x_n - u_n^i, Ax_n - Az\right\rangle\right) \\
&\leq \frac{1}{2}\left(\|x_n - z\|^2 + \left\|u_n^i - z\right\|^2 - \left\|x_n - u_n^i\right\|^2 + 2r_n^i\left\|x_n - u_n^i\right\|\|Ax_n - Az\|\right),
\end{aligned}$$

which implies that

$$\left\|u_n^i - z\right\|^2 \leq \|x_n - z\|^2 - \left\|x_n - u_n^i\right\|^2 + 2r_n^i\left\|x_n - u_n^i\right\|\|Ax_n - Az\|. \tag{3.12}$$

From the nonexpansiveness of $T_{r_n^i}$ and $u_n^i = T_{r_n^i}(I - r_n^i A)x_n$, for every $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\left\|u_n^i - z\right\|^2 &= \left\|T_{r_n^i}\left(I - r_n^i A\right)x_n - T_{r_n^i}\left(I - r_n^i A\right)z\right\|^2 \\
&\leq \left\|(x_n - z) - r_n^i(Ax_n - Az)\right\|^2 \\
&= \|x_n - z\|^2 - 2r_n^i\langle x_n - z, Ax_n - Az\rangle + \left(r_n^i\right)^2\|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 - 2\alpha r_n^i\|Ax_n - Az\|^2 + \left(r_n^i\right)^2\|Ax_n - Az\|^2 \\
&= \|x_n - z\|^2 - r_n^i\left(2\alpha - r_n^i\right)\|Ax_n - Az\|^2.
\end{aligned} \tag{3.13}$$

From the definition of x_n and (3.13), we get

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad \leq \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\sum_{i=1}^N a_n^i\left\|u_n^i - z\right\|^2 + \delta_n\|x_n - z\|^2 \\
&\quad \leq \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\sum_{i=1}^N a_n^i\left(\|x_n - z\|^2 - r_n^i\left(2\alpha - r_n^i\right)\|Ax_n - Az\|^2\right) + \delta_n\|x_n - z\|^2 \\
&\quad = \alpha_n\left\|f(x_n) - z\right\|^2 + \beta_n\sum_{i=1}^N a_n^i\|x_n - z\|^2 - \beta_n\sum_{i=1}^N a_n^i r_n^i\left(2\alpha - r_n^i\right)\|Ax_n - Az\|^2 \\
&\qquad + \delta_n\|x_n - z\|^2 \\
&\quad \leq \|x_n - z\|^2 + \alpha_n\left\|f(x_n) - z\right\|^2 - \beta_n\sum_{i=1}^N a_n^i r_n^i\left(2\alpha - r_n^i\right)\|Ax_n - Az\|^2,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\beta_n\sum_{i=1}^N a_n^i r_n^i\left(2\alpha - r_n^i\right)\|Ax_n - Az\|^2 \\
&\quad \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\left\|f(x_n) - z\right\|^2 \\
&\quad \leq \left(\|x_n - z\| + \|x_{n+1} - z\|\right)\|x_{n+1} - x_n\| + \alpha_n\left\|f(x_n) - z\right\|^2.
\end{aligned} \tag{3.14}$$

From (3.11), (3.14), and the conditions (i), (ii), (iii), and (iv), we obtain

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.15}$$

From the definition of x_n and (3.12), we have

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le \alpha_n\big\|f(x_n)-z\big\|^2+\beta_n\sum_{i=1}^N a_n^i\big\|u_n^i-z\big\|^2+\delta_n\|x_n-z\|^2\\
&\quad\le \alpha_n\big\|f(x_n)-z\big\|^2+\beta_n\sum_{i=1}^N a_n^i\big(\|x_n-z\|^2-\big\|x_n-u_n^i\big\|^2\\
&\qquad+2r_n^i\big\|x_n-u_n^i\big\|\|Ax_n-Az\|\big)+\delta_n\|x_n-z\|^2\\
&\quad\le \alpha_n\big\|f(x_n)-z\big\|^2+\beta_n\sum_{i=1}^N a_n^i\|x_n-z\|^2-\beta_n\sum_{i=1}^N a_n^i\big\|x_n-u_n^i\big\|^2\\
&\qquad+2\beta_n\sum_{i=1}^N a_n^i r_n^i\big\|x_n-u_n^i\big\|\|Ax_n-Az\|+\delta_n\|x_n-z\|^2\\
&\quad\le \|x_n-z\|^2+\alpha_n\big\|f(x_n)-z\big\|^2-\beta_n\sum_{i=1}^N a_n^i\big\|x_n-u_n^i\big\|^2\\
&\qquad+2\beta_n\sum_{i=1}^N a_n^i r_n^i\big\|x_n-u_n^i\big\|\|Ax_n-Az\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\beta_n\sum_{i=1}^N a_n^i\big\|x_n-u_n^i\big\|^2\\
&\quad\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\big\|f(x_n)-z\big\|^2\\
&\qquad+2\beta_n\sum_{i=1}^N a_n^i r_n^i\big\|x_n-u_n^i\big\|\|Ax_n-Az\|\\
&\quad\le \big(\|x_n-z\|+\|x_{n+1}-z\|\big)\|x_{n+1}-x_n\|+\alpha_n\big\|f(x_n)-z\big\|^2\\
&\qquad+2\beta_n\sum_{i=1}^N a_n^i r_n^i\big\|x_n-u_n^i\big\|\|Ax_n-Az\|.
\end{aligned}\tag{3.16}$$

From (3.11), (3.15), (3.16), and the conditions (i), (ii), (iii), we get

$$\lim_{n\to\infty}\big\|x_n-u_n^i\big\|=0,\quad \text{for all } i=1,2,\ldots,N. \tag{3.17}$$

By the definition of x_n, we obtain

$$\begin{aligned}
x_{n+1}-x_n&=\alpha_n f(x_n)+\beta_n\left(\sum_{i=1}^N a_n^i u_n^i\right)+\delta_n K_n x_n-x_n\\
&=\alpha_n\big(f(x_n)-x_n\big)+\beta_n\sum_{i=1}^N a_n^i\big(u_n^i-x_n\big)+\delta_n(K_n x_n-x_n).
\end{aligned}$$

From (3.11), (3.17), and the conditions (i) and (ii), we get

$$\lim_{n\to\infty} \|K_n x_n - x_n\| = 0. \tag{3.18}$$

Step 5. We show that $\{x_n\}$, $\{w_n^i\}$ and $\{r_n^i\}$ are Cauchy sequences, for every $i = 1, 2, \dots, N$.
Let $a \in (0, 1)$, by (3.11), there exists $N \in \mathbb{N}$ such that

$$\|x_{n+1} - x_n\| < a^n, \quad \forall n \geq N. \tag{3.19}$$

Thus, for any $n \geq N \in \mathbb{N}$ and $p \in \mathbb{N}$, we have

$$\|x_{n+p} - x_n\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k < \sum_{k=n}^{\infty} a^k = \frac{a^n}{1-a}. \tag{3.20}$$

Since $a \in (0, 1)$, we get $\lim_{n\to\infty} a^n = 0$. From (3.20), taking $n \to \infty$, we obtain $\{x_n\}$ is a Cauchy sequence in a Hilbert space H. Let $\lim_{n\to\infty} x_n = x^*$. Since $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous on H with coefficients μ_i, for every $i = 1, 2, \dots, N$, and (3.1), we have

$$\begin{aligned}
&\|w_n^i - w_{n+1}^i\| \\
&\quad\leq \left(1 + \frac{1}{n}\right)\mathcal{H}\big(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}\big) \\
&\quad\leq \left(1 + \frac{1}{n}\right)\mu_i \big\|(I - r_n^i A)x_n - (I - r_{n+1}^i A)x_{n+1}\big\| \\
&\quad\leq \left(1 + \frac{1}{n}\right)\mu_i \big(\big\|(I - r_n^i A)x_n - (I - r_n^i A)x_{n+1}\big\| \\
&\qquad + \big\|(I - r_n^i A)x_{n+1} - (I - r_{n+1}^i A)x_{n+1}\big\|\big) \\
&\quad\leq \left(1 + \frac{1}{n}\right)\mu_i \big(\|x_n - x_{n+1}\| + \big|r_{n+1}^i - r_n^i\big|\|Ax_{n+1}\|\big) \\
&\quad\leq \left(1 + \frac{1}{n}\right)\mu_i \big(\|x_n - x_{n+1}\| + \big|r_{n+1}^i - r_n^i\big|M\big),
\end{aligned} \tag{3.21}$$

where $M = \max_{n\in\mathbb{N}}\{\|Ax_n\|\}$. From (3.11), (3.21), and the condition (v), we obtain

$$\lim_{n\to\infty} \|w_n^i - w_{n+1}^i\| = 0, \quad \text{for every } i = 1, 2, \dots, N.$$

By continuing the same argument as (3.19) and (3.20), we have $\{w_n^i\}$ is a Cauchy sequence in a Hilbert space H, for all $i = 1, 2, \dots, N$. Let $\lim_{n\to\infty} w_n^i = w_i^*$, for every $i = 1, 2, \dots, N$. Using the same method as above and the condition (v), we see that $\{r_n^i\}$ is a Cauchy sequence, for all $i = 1, 2, \dots, N$. Put $\lim_{n\to\infty} r_n^i = r_i^*$, for every $i = 1, 2, \dots, N$.

Next, we will prove that $w_i^* \in S_i(I - r_i^* A)x^*$, for all $i = 1, 2, \dots, N$.

Since $w_n^i \in S_i(I - r_n^i A)x_n$, we obtain

$$\begin{aligned}
&d\big(w_n^i, S_i(I - r_i^* A)x^*\big) \\
&\quad\leq \max\Big\{d\big(w_n^i, S_i(I - r_i^* A)x^*\big), \sup_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d\big(S_i(I - r_n^i A)x_n, \tilde{w}_i\big)\Big\}
\end{aligned}$$

$$\leq \max\Big\{ \sup_{\hat{w}_i \in S_i(I - r_n^i A)x_n} d\big(\hat{w}_i, S_i\big(I - r_i^* A\big)x^*\big), \sup_{\tilde{w}_i \in S_i(I - r_i^* A)x^*} d\big(S_i\big(I - r_n^i A\big)x_n, \tilde{w}_i\big) \Big\}$$

$$= \mathcal{H}\big(S_i\big(I - r_n^i A\big)x_n, S_i\big(I - r_i^* A\big)x^*\big), \quad \text{for every } i = 1, 2, \ldots, N. \tag{3.22}$$

Since

$$\begin{aligned} d\big(w_i^*, S_i\big(I - r_i^* A\big)x^*\big) &\leq \big\|w_i^* - w_n^i\big\| + d\big(w_n^i, S_i\big(I - r_i^* A\big)x^*\big) \\ &\leq \big\|w_i^* - w_n^i\big\| + \mathcal{H}\big(S_i\big(I - r_n^i A\big)x_n, S_i\big(I - r_i^* A\big)x^*\big) \\ &\leq \big\|w_i^* - w_n^i\big\| + \mu_i \big\|\big(I - r_n^i A\big)x_n - \big(I - r_i^* A\big)x^*\big\| \\ &= \big\|w_i^* - w_n^i\big\| + \mu_i \big\|\big(x_n - x^*\big) - \big(r_n^i A x_n - r_i^* A x^*\big)\big\|, \end{aligned}$$

taking $n \to \infty$, we have

$$d\big(w_i^*, S_i\big(I - r_i^* A\big)x^*\big) = 0,$$

which implies that

$$w_i^* \in S_i\big(I - r_i^* A\big)x^*, \quad \text{for all } i = 1, 2, \ldots, N. \tag{3.23}$$

Step 6. We will show that $\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle \leq 0$, where $q = P_{\mathcal{F}} f(q)$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \big\langle f(q) - q, x_n - q\big\rangle = \lim_{k\to\infty} \big\langle f(q) - q, x_{n_k} - q\big\rangle.$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$.

For every $i = 1, 2, \ldots, N$, $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2 < 1$, for all $i = 1, 2, \ldots, N$, without loss of generality, we may assume that

$$\lambda_i^{n_k} \to \lambda_i \in (0, 1) \quad \text{as } k \to \infty, \text{for every } i = 1, 2, \ldots, N.$$

Let K be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. By Lemma 2.9, we see that K is nonexpansive and $F(K) = \bigcap_{i=1}^N F(T_i)$.

From Lemma 2.10(i), we obtain

$$\lim_{k\to\infty} \|K_{n_k} x_{n_k} - K x_{n_k}\| = 0. \tag{3.24}$$

Since

$$\|x_{n_k} - K x_{n_k}\| \leq \|x_{n_k} - K_{n_k} x_{n_k}\| + \|K_{n_k} x_{n_k} - K x_{n_k}\|,$$

by (3.18) and (3.24), we have

$$\lim_{k\to\infty} \|x_{n_k} - K x_{n_k}\| = 0. \tag{3.25}$$

Since $x_{n_k} \rightharpoonup \tilde{x}$ as $n \to \infty$, by (3.25) and Lemma 2.4, we have

$$\tilde{x} \in F(K) = \bigcap_{i=1}^{N} F(T_i). \tag{3.26}$$

Next, we show that $x^* \in \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi, A)$.

Since $x_{n_k} \to x^*$ as $k \to \infty$ and (3.17), we have

$$u_{n_k}^i \to x^* \quad \text{as } k \to \infty, \text{for all } i = 1, 2, \ldots, N. \tag{3.27}$$

From (3.1), we obtain

$$\Phi_i\big(w_{n_k}^i, u_{n_k}^i, y\big) + \varphi(y) - \varphi\big(u_{n_k}^i\big) + \frac{1}{r_{n_k}^i}\big\langle u_{n_k}^i - x_{n_k}, y - u_{n_k}^i\big\rangle + \big\langle Ax_{n_k}, y - u_{n_k}^i\big\rangle \geq 0,$$

for every $y \in C$ and $i = 1, 2, \ldots, N$. From (3.17), (3.27), the condition (H1), and the lower semicontinuity of φ, we get

$$\Phi_i\big(w_i^*, x^*, y\big) + \varphi(y) - \varphi\big(x^*\big) + \big\langle Ax^*, y - x^*\big\rangle \geq 0,$$

for every $y \in C$ and $i = 1, 2, \ldots, N$, from which it follows by (3.23) that

$$x^* \in (GEP)_s(\Phi_i, \varphi, A), \quad \text{for every } i = 1, 2, \ldots, N.$$

It implies that

$$x^* \in \bigcap_{i=1}^{N} (GEP)_s(\Phi_i, \varphi, A). \tag{3.28}$$

Since $x_{n_k} \rightharpoonup \tilde{x}$ and $x_{n_k} \to x^*$ as $n \to \infty$, then $\tilde{x} = x^*$. From (3.26) and (3.28), we have

$$x^* \in \mathcal{F}. \tag{3.29}$$

Indeed, since $x_{n_k} \to x^*$ as $k \to \infty$, by (3.29) and Lemma 2.3, we obtain

$$\limsup_{n\to\infty} \big\langle f(q) - q, x_n - q\big\rangle = \lim_{k\to\infty} \big\langle f(q) - q, x_{n_k} - q\big\rangle = \big\langle f(q) - q, x^* - q\big\rangle \leq 0. \tag{3.30}$$

Step 7. Finally, we will prove that $\{x_n\}$ and $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}} f(q)$, for every $i = 1, 2, \ldots, N$.

By Lemma 2.1(ii), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \left\| \alpha_n\big(f(x_n) - q\big) + \beta_n \sum_{i=1}^{N} a_n^i\big(u_n^i - q\big) + \delta_n(K_n x_n - q) \right\|^2 \\
&\quad \leq \left\| \beta_n \sum_{i=1}^{N} a_n^i\big(u_n^i - q\big) + \delta_n(K_n x_n - q) \right\|^2 + 2\alpha_n \big\langle f(x_n) - q, x_{n+1} - q\big\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\beta_n \left\| \sum_{i=1}^{N} a_n^i (u_n^i - q) \right\| + \delta_n \|K_n x_n - q\| \right)^2 \\
&\quad + 2\alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\
&\leq \left(\beta_n \sum_{i=1}^{N} a_n^i \|x_n - q\| + \delta_n \|x_n - q\| \right)^2 + 2\alpha_n \|f(x_n) - f(q)\| \|x_{n+1} - q\| \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\
&\leq \big((1-\alpha_n)\|x_n - q\|\big)^2 + 2\alpha_n \xi \|x_n - q\| \|x_{n+1} - q\| \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle \\
&\leq (1-\alpha_n)^2 \|x_n - q\|^2 + \alpha_n \xi \big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\big) \\
&\quad + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \frac{(1-\alpha_n)^2 + \alpha_n \xi}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \\
&\quad = \frac{1 - \alpha_n \xi - 2\alpha_n (1-\xi)}{1 - \alpha_n \xi} \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \\
&\quad = \left(1 - \frac{2\alpha_n (1-\xi)}{1 - \alpha_n \xi}\right) \|x_n - q\|^2 + \frac{\alpha_n^2}{1 - \alpha_n \xi} \|x_n - q\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n \xi} \langle f(q) - q, x_{n+1} - q \rangle \\
&\quad = \left(1 - \frac{2\alpha_n (1-\xi)}{1 - \alpha_n \xi}\right) \|x_n - q\|^2 + \frac{2\alpha_n (1-\xi)}{1 - \alpha_n \xi} \left(\frac{\alpha_n}{2(1-\xi)} \|x_n - q\|^2 \right. \\
&\qquad \left. + \frac{1}{1-\xi} \langle f(q) - q, x_{n+1} - q \rangle \right).
\end{aligned}$$

Applying the condition (i), (3.30), and Lemma 2.6, we have the sequence $\{x_n\}$ converges strongly to $q = P_{\mathcal{F}} f(q)$. From (3.17), we also obtain $\{u_n^i\}$ converges strongly to $q = P_{\mathcal{F}} f(q)$, for every $i = 1, 2, \ldots, N$. This completes the proof. □

The following corollaries are consequences which are applied by Theorem 3.1. Therefore, we omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, N$, $S_i : C \to CB(H)$ be $\mathcal{H}$-Lipschitz continuous with coefficients μ_i, $\Phi_i : H \times C \times C \to \mathbb{R}$ be equilibrium-like function satisfying* (H1)-(H3). *Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and $A : C \to C$ be an α-inverse strongly monotone mapping. Let $T : C \to C$ be κ-strictly pseudo-contractive mapping with $\kappa \leq \gamma_1$ and $\mathcal{F} := F(T) \cap \bigcap_{i=1}^{N} (MGEP)_s(\Phi_i, \varphi, A) \neq \emptyset$. For every $n \in \mathbb{N}$, let $\{\lambda_n\}$ be a sequence of real numbers* *where* $0 < \lambda_n < \gamma_2$ *and* $\gamma_1 + \gamma_2 < 1$. *For every* $i = 1, 2, \ldots, N$, *let* $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and* $w_1^i \in S_i(I - r_1^i A)x_1$, *there exist sequences* $\{w_n^i\} \in H$ *and* $\{x_n\}, \{u_n^i\} \subseteq C$ *such that*

$$\begin{cases} \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i(I - r_n^i A)x_n, S_i(I - r_{n+1}^i A)x_{n+1}), \\ \quad w_n^i \in S_i(I - r_n^i A)x_n \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i}\langle u_n^i - x_n, y - u_n^i\rangle + \langle Ax_n, y - u_n^i\rangle \\ \quad \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n(\sum_{i=1}^N a_n^i u_n^i) + \delta_n(\lambda_n T + (1 - \lambda_n)I)x_n, \quad \forall n \geq 1, \end{cases} \tag{3.31}$$

where $f : C \to C$ *be a contraction mapping with a constant* ξ *and* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n, \delta_n \leq \upsilon < 1$;

(iii) $0 \leq \eta \leq a_n^i \leq \sigma < 1$, *for all* $i = 1, 2, \ldots, N - 1$ *and* $0 < \eta \leq a_n^N \leq \sigma \leq 1$ *with* $\sum_{n=1}^N a_n^i = 1$;

(iv) $0 < \epsilon \leq r_n^i \leq \omega < 2\alpha$, *for all* $n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, N$;

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1}^i - r_n^i| < \infty$, $\sum_{n=1}^{\infty} |a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, *for all* $i = 1, 2, \ldots, N$;

(vi) *for each* $i = 1, 2, \ldots, N$, *there exists* $\rho_i > 0$ *such that*

$$\begin{aligned} &\Phi_i\big(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\big) + \Phi_i\big(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\big) \\ &\quad \leq -\rho_i \big\| T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\big\|^2, \end{aligned} \tag{3.32}$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ *and* $w_j^i \in S_i(x_j)$, *for* $j = 1, 2$, *where* $\Theta_i = \{r_n^i : n \geq 1\}$.

Then $\{x_n\}$ *and* $\{u_n^i\}$ *converges strongly to* $q = P_{\mathcal{F}} f(q)$, *for every* $i = 1, 2, \ldots, N$.

Corollary 3.3 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *For every* $i = 1, 2, \ldots, N$, $S_i : C \to CB(H)$ *be* $\mathcal{H}$-*Lipschitz continuous with coefficients* μ_i, $\Phi_i : H \times C \times C \to \mathbb{R}$ *be equilibrium-like function satisfying* (H1)-(H3). *Let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Let* $\{T_i\}_{i=1}^N$ *be a finite family of* κ_i-*strictly pseudo-contractive mappings and* $\kappa_i \leq \gamma_1$ *with* $\mathcal{F} := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N (GEP)_s(\Phi_i, \varphi) \neq \emptyset$. *For every* $n \in \mathbb{N}$, *let* K_n *be the* K-*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_1^n, \lambda_2^n, \ldots, \lambda_N^n$ *where* $0 < \phi \leq \lambda_i^n \leq \psi < \gamma_2 < 1$, *for all* $i = 1, 2, \ldots, N$ *and* $\gamma_1 + \gamma_2 < 1$. *For every* $i = 1, 2, \ldots, N$, *let* $\{x_n\}$ *be the sequence generated by* $x_1 \in C$ *and* $w_1^i \in S_i(x_1)$, *there exist sequences* $\{w_n^i\} \in H$ *and* $\{x_n\}, \{u_n^i\} \subseteq C$ *such that*

$$\begin{cases} w_n^i \in S_i(x_n), \quad \|w_n^i - w_{n+1}^i\| \leq (1 + \frac{1}{n})\mathcal{H}(S_i(x_n), S_i(x_{n+1})), \\ \Phi_i(w_n^i, u_n^i, y) + \varphi(y) - \varphi(u_n^i) + \frac{1}{r_n^i}\langle u_n^i - x_n, y - u_n^i\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n(\sum_{i=1}^N a_n^i u_n^i) + \delta_n K_n x_n, \quad \forall n \geq 1, \end{cases} \tag{3.33}$$

where $f : C \to C$ *is a contraction mapping with a constant* ξ *and* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\} \subseteq (0, 1)$ *with* $\alpha_n + \beta_n + \delta_n = 1$, $\forall n \geq 1$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(ii) $0 < \tau \le \beta_n, \delta_n \le \upsilon < 1$;
(iii) $0 \le \eta \le a_n^i \le \sigma < 1$, *for all* $i = 1, 2, \ldots, N-1$ *and* $0 < \eta \le a_n^N \le \sigma \le 1$ *with* $\sum_{n=1}^{N} a_n^i = 1$;
(iv) $0 < \epsilon \le r_n^i \le \omega < 1$, *for all* $n \in \mathbb{N}$ *and* $i = 1, 2, \ldots, N$;
(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$, $\sum_{n=1}^{\infty}|r_{n+1}^i - r_n^i| < \infty$, $\sum_{n=1}^{\infty}|a_{n+1}^i - a_n^i| < \infty$, $\sum_{n=1}^{\infty}|\lambda_i^{n+1}-\lambda_i^n| < \infty$, *for all* $i = 1, 2, \ldots, N$;
(vi) *for each* $i = 1, 2, \ldots, N$, *there exists* $\rho_i > 0$ *such that*

$$\begin{aligned}&\Phi_i\big(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2)\big) + \Phi_i\big(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1)\big)\\ &\quad\le -\rho_i\big\|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\big\|^2,\end{aligned}\tag{3.34}$$

for all $(r_1^i, r_2^i) \in \Theta_i \times \Theta_i$, $(x_1, x_2) \in C \times C$ *and* $w_j^i \in S_i(x_j)$, *for* $j = 1, 2$, *where* $\Theta_i = \{r_n^i : n \ge 1\}$.

Then $\{x_n\}$ *and* $\{u_n^i\}$ *converges strongly to* $q = P_{\mathcal{F}}f(q)$, *for every* $i = 1, 2, \ldots, N$.

Remark 3.4 From Corollary 3.3, put $N = 1$, then the iterative scheme (3.33) reduces to

$$\begin{cases} w_n^1 \in S_1(x_n), \quad \|w_n^1 - w_{n+1}^1\| \le (1+\frac{1}{n})\mathcal{H}(S_1(x_n), S_1(x_{n+1})),\\ \Phi_1(w_n^1, u_n^1, y) + \varphi(y) - \varphi(u_n^1) + \frac{1}{r_n^1}\langle u_n^1 - x_n, y - u_n^1\rangle \ge 0, \quad \forall y \in C,\\ x_{n+1} = \alpha_n f(x_n) + \beta_n u_n^1 + \delta_n(\lambda_1^n T_1 + (1-\lambda_1^n)I)x_n, \quad \forall n \ge 1,\end{cases}$$

which is a modification of iterative scheme (1.4) in the results of Ceng *et al.* [16]. By assuming the initial condition $x_1 \in C$, $w_1^1 \in S_1(x_1)$ and the following conditions hold:
(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(ii) $0 < \tau \le \beta_n, \delta_n \le \upsilon < 1$;
(iii) $0 < \epsilon \le r_n^1 \le \omega < 1$, for all $n \in \mathbb{N}$;
(iv) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$, $\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$, $\sum_{n=1}^{\infty}|r_{n+1}^1 - r_n^1| < \infty$, $\sum_{n=1}^{\infty}|\lambda_1^{n+1}-\lambda_1^n| < \infty$;
(v) there exists $\rho_1 > 0$ such that

$$\begin{aligned}&\Phi_1\big(w_1^1, T_{r_1^1}(x_1), T_{r_2^1}(x_2)\big) + \Phi_1\big(w_2^1, T_{r_2^1}(x_2), T_{r_1^1}(x_1)\big)\\ &\quad\le -\rho_1\big\|T_{r_1^1}(x_1) - T_{r_2^1}(x_2)\big\|^2,\end{aligned}$$

for all $(r_1^1, r_2^1) \in \Theta_1 \times \Theta_1$, $(x_1, x_2) \in C \times C$ and $w_j^1 \in S_1(x_j)$, for $j = 1, 2$, where $\Theta_1 = \{r_n^1 : n \ge 1\}$.

Then $\{x_n\}$ and $\{u_n^1\}$ converge strongly to $q = P_{\mathcal{F}}f(q)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly to this research article. Both authors read and approved the final manuscript.

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