Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the topology and *wt*-distance on metric type spaces

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Abstract

Recently, Khamsi and Hussain (Nonlinear Anal. 73:3123-3129, 2010) discussed a natural topology defined on any metric type space and noted that this topology enjoys most of the metric topology like properties. In this paper, we define topologically complete type metrizable space and prove that being of metrizability type is preserved under a countable Cartesian product and establish the fact that any G_δ set in a complete metric type space is a topologically metrizable type space. Next, we introduce the concept of *wt*-distance on a metric type space and prove some fixed point theorems in a partially ordered metric type space with some weak contractions induced by the *wt*-distance.

MSC: 47H09; 47H10

Keywords: metric type space; topologically complet; Alexandroff theorem; *wt*-distance; fixed point theorem

1 Preliminaries

The concept of metric type or b-metric space was introduced and studied by Bakhtin [1] and Czerwik [2]. Since then several papers have dealt with fixed point theory for single-valued and multivalued operators in b-metric and cone b-metric spaces (see [3–12] and references therein). Khmasi and Hussain [13] and Hussain and Shah [14] discussed KKM mappings and related results in metric and cone metric type spaces.

Definition 1.1 Let X be a set. Let $D : X \times X \to [0,\infty)$ be a function which satisfies

(1) $D(x,y) = 0$ if and only if $x = y$;

(2) $D(x,y) = D(y,x)$, for any $x,y \in X$;

(3) $D(x,y) \leq K(D(x,z) + D(z,y))$, for any points $x,y,z \in X$, for some constant $K \geq 1$.

The pair (X,D) is called a metric type space.

Definition 1.2 Let (X,D) be a metric type space.

(1) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n\to\infty} D(x_n,x) = 0$.

(2) The sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} D(x_n,x_m) = 0$.

(X,D) is complete if and only if any Cauchy sequence in X is convergent.

Example 1.3 Let X be the set of Lebesgue measurable functions on $[0,1]$ such that

$$\int_0^1 |f(x)|^2\,dx < \infty.$$

Define $D : X \times X \to [0,\infty)$ by

$$D(f,g) = \int_0^1 \left|f(x) - g(x)\right|^2 dx.$$

Then D satisfies the following properties:

(1) $D(f,g) = 0$ if and only if $f = g$;
(2) $D(f,g) = D(g,f)$, for any $f, g \in X$;
(3) $D(f,g) \leq 2(D(f,h) + D(h,g))$, for any functions $f, g, h \in X$.

Example 1.4 Let $(\mathbb{R}, |\cdot|)$ be metric space. Define

(1) $D_1(x,y) = |x - y|^2$ for any $x, y \in X$;
(2) $D_2(x,y) = |x - y|^2 + |\frac{1}{x} - \frac{1}{y}|^2$, for any $x, y \in X$.

Then $(\mathbb{R}, D_i)$, $i = 1, 2$ are metric type spaces with $K = 2$.

Definition 1.5 Let (X,D) be a metric type space. A subset $A \subset X$ is said to be open if and only if for any $a \in A$, there exists $\varepsilon > 0$ such that the open ball $B_o(a,\varepsilon) \subset A$. The family of all open subsets of X will be denoted by τ.

Theorem 1.6 ([13]) *τ defines a topology on (X,D).*

Theorem 1.7 ([13]) *Let (X,D) be a metric type space and τ be the topology defined above. Then for any nonempty subset $A \subset X$ we have*

(1) *A is closed if and only if for any sequence $\{x_n\}$ in A which converges to x, we have $x \in A$;*
(2) *if we define $\overline{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \overline{A}$ and for any $\varepsilon > 0$, we have*

$$B_o(a,\varepsilon) \cap A \neq \emptyset.$$

Theorem 1.8 ([13]) *Let (X,D) be a metric type space and τ be the topology defined above. Let $\emptyset \neq A \subset X$. The following properties are equivalent*:

(1) *A is compact*;
(2) *For any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k\to\infty} x_{n_k} \in A$.*

Definition 1.9 The subset A is called sequentially compact if and only if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, and $\lim_{n_k\to\infty} x_{n_k} \in A$. Also A is called totally bounded if for any $\varepsilon > 0$ there exits $x_1, x_2, \dots, x_n \in A$ such that

$$A \subset B_o(x_1,\varepsilon) \cup \cdots \cup B_o(x_n,\varepsilon).$$

Theorem 1.10 ([13]) *Let (X,D) be a metric type space and τ be the topology defined above. Let $\emptyset \neq A \subset X$. The following properties are equivalent*:

(1) *A is compact if and only if A is sequentially compact.*
(2) *If A is compact, then A is totally bounded.*

Corollary 1.11 *Every closed subset of a complete metric type space is complete.*

Theorem 1.12 ([15]) *Let (X,D) be a metric type space and suppose that $\{x_n\}$ and $\{y_n\}$ converge to $x,y \in X$, respectively. Then we have*

(1)

$$\frac{1}{K^2}D(x,y) \leq \liminf_n D(x_n,y_n) \leq \limsup_n D(x_n,y_n) \leq K^2 D(x,y).$$

In particular, if $x = y$, then $\lim_n D(x_n,y_n) = 0$.

(2) *Moreover, for each $z \in X$, we have*

$$\frac{1}{K}D(x,z) \leq \liminf_n D(x_n,z) \leq \limsup_n D(x_n,z) \leq KD(x,y).$$

2 Topologically complete metrizable type spaces

Lemma 2.1 *Let (X,D) be a metric type space and let $\lambda \in (0,1)$ then there exists a metric type E on X such that $E(x,y) \leq \lambda$, for each $x,y \in X$, and E and D induce the same topology on X.*

Proof We define $E(x,y) = \min\{\lambda, D(x,y)\}$. We claim that E is metric type on X. The properties (1) and (2) are immediate from the definition. For the triangle inequality, suppose that $x,y,z \in X$. Then $E(x,z) \leq \lambda$ and so $E(x,z) \leq E(x,y) + E(y,z)$ when either $E(x,y) = \lambda$ or $E(y,z) = \lambda$. The only remaining case is when $E(x,y) = D(x,y) < \lambda$ and $E(y,z) = D(y,z) < \lambda$. But $D(x,z) \leq K(D(x,y) + D(y,z))$ and $E(x,z) \leq D(x,z)$ and so $E(x,z) \leq K(E(x,y) + E(y,z))$. Thus E is a metric type on X. It only remains to show that the topology induced by E is the same as that induced by D. But we have $E(x_n,x) \longrightarrow 0$ if and only if $\min\{\lambda, D(x_n,x)\} \longrightarrow 0$ if and only if $D(x_n,x) \longrightarrow 0$, and we are done. □

The metric type E in the above lemma is said to be bounded by λ.

Definition 2.2 Let (X,D) be a metric type space, $x \in X$ and $\emptyset \neq A \subseteq X$. We define

$$\Delta(x,A) = \inf\{D(x,y) : y \in A\}.$$

Definition 2.3 A topological space is called a (topologically complete) metrizable type space if there exists a (topologically complete) metric type D inducing the given topology on it.

Example 2.4 Let $X = (0,1]$. The metric type space (X,D_1) is not complete because the Cauchy sequence $\{1/n\}$ in this space is not convergent. Now, if we consider (X,D_2). It is straightforward to show that (X,D_2) is complete. Since x_n tend to x with respect to the metric type D_1 if and only if $|x_n - x|^2 \longrightarrow 0$ if and only if x_n tend to x with respect to the metric type D_2, then D_1 and D_2 are equivalent. Hence the metric type space (X,D_1) is topologically complete metrizable type.

Lemma 2.5 *Metrizability type is preserved under countable Cartesian product.*

Proof Without loss of generality we may assume that the index set is $\mathbb{N}$. Let $\{(X_n,D_n) : n \in \mathbb{N}\}$ be a collection of metrizable type spaces. Let τ_n be the topology induced by D_n on X_n for

$n \in \mathbb{N}$ and let (X, τ) be the Cartesian product of $\{(X_n, \tau_n) : n \in \mathbb{N}\}$ with product topology. We have to prove that there is a metric type D on X which induces the topology τ. By the above lemma, we may suppose that D_n is bounded by 2^{-n} for all $n \in \mathbb{N}$, otherwise we replace D_n by another metric type which induces the same topology and which is bounded by 2^{-n}. Points of $X = \prod_{n\in N} X_n$ are denoted as sequences $x = \{x_n\}$ with $x_n \in X_n$ for $n \in \mathbb{N}$. Define $D(x,y) = \sum_{n=1}^{\infty} D_n(x_n, y_n)$, for each $x, y \in X$. First note that D is well defined since $\sum_{n=1}^{i} 2^{-n}$ is convergent. Also D is a metric type on X because each D_n is of a metric type. Let $\mathcal{U}$ be the topology induced by the metric type D. We claim that $\mathcal{U}$ coincides with τ. If $G \in \mathcal{U}$ and $x = \{x_n\} \in G$, then there exists $r > 0$ such that $B(x, r) \subset G$. Now choose $N \in \mathbb{N}$ such that $\sum_{n=1}^{i} 2^{-n} < \frac{r}{2}$. For each $n = 1, 2, \dots, N_0$, let $V_n = B(x_n, \frac{r}{2N})$, where the ball is with respect to the metric type D_n. Let $V_n = X_n$ for $n > N_0$. Put $V = \prod_{n\in N} V_n$, then $x \in V$ and V is an open set in the product topology τ on X. Furthermore $V \subset B(x, r)$, since for each $y \in V$

$$\begin{aligned} D(x,y) &= \sum_{n=1}^{\infty} D_n(x_n, y_n) \\ &= \sum_{n=1}^{N_0} D_n(x_n, y_n) + \sum_{n=N_0+1}^{\infty} D_n(x_n, y_n) \\ &\leq N\left(\frac{r}{2N}\right) + \sum_{n=N_0+1}^{\infty} 2^{-n} \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Hence $V \subset B(x, r) \subset G$. Therefore G is open in the product topology. Conversely suppose G is open in the product topology and let $x = \{x_n\} \in G$. Choose a standard basic open set V such that $x \in V$ and $V \subset G$. Let $V = \prod_{n\in N} V_n$, where each V_n is open in X_n and $V_n = X_n$ for all $n > N_0$. For $n = 1, 2, \dots, N_0$, let $r_n = \Delta_n(x_n, X_n - V_n)$, if $X_n \neq V_n$, and $r_n = 2^{-n}$, otherwise, let $r = \min\{r_1, r_2, \dots, r_{N_0}\}$. We claim that $B(x, r) \subset V$. If $y = \{y_n\} \in B(x, r)$, then $D(x,y) = \sum_{n=1}^{\infty} D_n(x_n, y_n) < r$ and so $D_n(x_n, y_n) < r \leq r_n$ for each $n = 1, 2, \dots, N_0$. Then $y_n \in V_n$, for $n = 1, 2, \dots, N_0$. Also for $n > N_0$, $y_n \in V_n = X_n$. Hence $y \in V$ and so $B(x, r, t) \subset V \subset G$. Therefore G is open with respect to the metric type topology and $\tau \subset \mathcal{U}$. Hence τ and $\mathcal{U}$ coincide. □

Theorem 2.6 *An open subspace of a complete metrizable type space is a complete metrizable type space.*

Proof Let (X, D) be a complete metric type space and G an open subspace of X. If the restriction of D to G is not complete we can replace D on G by another metric type as follows. Define $f : G \longrightarrow R^+$ by $f(x) = \frac{1}{\Delta(x, X-G)}$ (f is undefined if $X - G$ is empty, but then there is nothing to prove). For $x, y \in G$ define

$$E(x,y) = D(x,y) + \left|f(x) - f(y)\right|.$$

It is clear that D is of metric type on G.

We show that E and D are of the type of equivalent metrics on G. We do this by showing that for arbitrary sequence $\{x_n\}$ converges to $x \in X$, $D(x_n, x_m) \longrightarrow 0$ if and only if $E(x_n, x_m) \longrightarrow 0$. Since $E(x,y) \geq D(x,y)$ for all $x, y \in G$, $D(x_n, x_m) \longrightarrow 0$ whenever $E(x_n, x_m) \longrightarrow 0$. To prove the converse, let $D(x_n, x_m) \longrightarrow 0$, and using Theorem 1.12, we have

$$\begin{aligned}\limsup_n \Delta(x_n, X-G) &= \limsup_n \bigl(\inf\{D(x_n, y) : y \in X-G\}\bigr)\\ &\leq \limsup_n D(x_n, y)\\ &= KD(x,y).\end{aligned}$$

Therefore

$$\limsup_n \Delta(x_n, X-G) \leq K\Delta(x, X-G). \tag{1}$$

On the other hand, there exists a $y_0 \in X-G$, the positive sequence $\{a_n\}$ converges to zero, and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have

$$\frac{1}{K^2}\bigl(\Delta(x_n, X-G) + a_n\bigr) \geq D(x_n, y_0).$$

Then

$$\begin{aligned}\liminf_n \frac{1}{K^2}\Delta(x_n, X-G) &\geq \liminf_n D(x_n, y_0)\\ &\geq \frac{1}{K}D(x, y_0)\\ &\geq \frac{1}{K}\Delta(x, X-G),\end{aligned}$$

and

$$\liminf_n \Delta(x_n, X-G) \geq K\Delta(x, X-G). \tag{2}$$

By (1) and (2), we have

$$\lim_n \Delta(x_n, X-G) = K\Delta(x, X-G).$$

This implies $|f(x_n) - f(x_m)| \longrightarrow 0$. Hence $E(x_n, x_m) \longrightarrow 0$. Therefore E and D are equivalent. Next we show that E is a complete metric type. Suppose that $\{x_n\}$ is a Cauchy sequence in G with respect to E. Since for each $m, n \in \mathbb{N}$, $E(x_m, x_n) \geq D(x_m, x_n)$, therefore $\{x_n\}$ is also a Cauchy sequence with respect to D. By completeness of (X, D), $\{x_n\}$ converges to point p in X. We claim that $p \in G$. Assume otherwise, then for each $n \in \mathbb{N}$, if $p \in X-G$ and $D(x_n, p) \geq \Delta(x_n, X-G)$. Therefore

$$\frac{1}{\Delta(x_n, X-G)} \geq \frac{1}{D(x_n, p)}.$$

That is

$$f(x_n) \geq \frac{1}{D(x_n, p)}.$$

Therefore as $n \longrightarrow \infty$, we get $f(x_n) \longrightarrow \infty$. On the other hand, $|f(x_n) - f(x_m)| \leq E(x_m, x_n)$, for every $m, n \in \mathbb{N}$, that is, $\{f(x_n)\}$ is a bounded sequence. This contradiction shows that $p \in G$. Hence $\{x_n\}$ converges to p with respect to E and (G, E) is a complete metrizable type space. □

Theorem 2.7 (Alexandroff) *A G_δ set in a complete metric type space is a topologically complete metrizable type space.*

Proof Let (X, D) be a complete metric type space and G be a G_δ set in X, that is, $G = \bigcap_{n=1}^{\infty} G_n$, where each G_n is open in X. By the above theorem, there exists a complete metric type D_n on G_n and we may assume that D_n is bounded by 2^{-n}. Let $\mathcal{H}$ be the Cartesian product $\prod_{n=1}^{\infty} G_n$ with the product topology. Then $\mathcal{H}$ is a complete metrizable type space. Now, for each $n \in \mathbb{N}$ let $f_n : G \longrightarrow G_n$ be the inclusion map. So the evaluation map $e : G \longrightarrow \mathcal{H}$ is an embedding. Image of e is the diagonal $\mathfrak{D}G$ which is a closed subset of $\mathcal{H}$ and by Corollary 1.11, $\mathfrak{D}G$ is complete. Thus $\mathfrak{D}G$ is a complete metrizable type space and so is G which is homeomorphic to it. □

3 *wt*-Distance

Kada *et al.* [16] introduced in 1996, the concept of w-distance on a metric space and proved some fixed point theorems. In this section, we introduce the definition of a wt-distance and we state a lemma which we will use in the main sections of this work.

Definition 3.1 Let (X, D) be a metric type space with constant $K \geq 1$. Then a function $P : X \times X \longrightarrow [0, \infty)$ is called a wt-distance on X if the following are satisfied:

(a) $P(x, z) \leq K(P(x, y) + P(y, z))$ for any $x, y, z \in X$;
(b) for any $x \in X$, $P(x, \cdot) : X \longrightarrow [0, \infty)$ is K-lower semi-continuous;
(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P(z, x) \leq \delta$ and $P(z, y) \leq \delta$ imply $D(x, y) \leq \varepsilon$.

Let us recall that a real-valued function f defined on a metric type space X is said to be lower K-semi-continuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} Kf(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$ [17].

Let us give some examples of wt-distance.

Example 3.2 Let (X, D) be a metric type space. Then the metric D is a wt-distance on X.

Proof (a) and (b) are obvious. To show (c), for any $\varepsilon > 0$, put $\delta = \frac{\varepsilon}{2K}$. Then we see that $P(x, z) \leq \delta$ and $P(z, y) \leq \delta$ imply $D(x, y) \leq \varepsilon$. □

Example 3.3 Let $X = \mathbb{R}$ and $D_1(x, y) = (x - y)^2$. Then the function $P : X \times X \longrightarrow [0, \infty)$ defined by $P(x, y) = |x|^2 + |y|^2$ for every $x, y \in X$ is a wt-distance on X.

Proof (a) and (b) are obvious. To show (c), for any $\varepsilon > 0$, put $\delta = \frac{\varepsilon}{2}$. Then we have

$$D_1(x, y) = (x - y)^2 \leq |x|^2 + |y|^2 \leq P(z, x) + P(z, y) \leq \delta + \delta = \varepsilon.$$ □

Example 3.4 Let $X = \mathbb{R}$ and $D_1(x, y) = (x - y)^2$. Then the function $P : X \times X \longrightarrow [0, \infty)$ defined by $P(x, y) = |y|^2$ for every $x, y \in X$ is a wt-distance on X.

Proof (a) and (b) are obvious. To show (c), for any $\varepsilon > 0$, put $\delta = \frac{\varepsilon}{2}$. Then we have

$$D_1(x,y) = (x-y)^2 \leq |x|^2 + |y|^2 = P(z,x) + P(z,y) \leq \delta + \delta = \varepsilon. \qquad \square$$

Lemma 3.5 *Let (X,D) be a metric type space with constant $K \geq 1$ and P be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and let $x,y,z \in X$. Then the following hold*:

(1) *If $P(x_n,y) \leq \alpha_n$ and $P(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $P(x,y) = 0$ and $P(x,z) = 0$, then $y = z$;*
(2) *if $P(x_n,y_n) \leq \alpha_n$ and $P(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $D(y_n,z) \to 0$;*
(3) *if $P(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *if $P(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof The proof is similar to [16]. □

4 Fixed point theorems

We introduce first the following concept.

Definition 4.1 Suppose $(X,\leq)$ is a partially ordered set and $f : X \to X$ be a self mapping on X. We say f is inverse increasing if for $x,y \in X$,

$$f(x) \leq f(y) \quad \text{implies} \quad x \leq y. \tag{3}$$

Our first main result is a fixed point theorem for graphic contractions on a partially ordered metric space endowed with a *wt*-distance.

Theorem 4.2 *Let $(X,\leq)$ be a partially ordered set and let $D : X \times X \to [0,\infty)$ be a metric type on X such that (X,D) is a complete metric type space with constant $K \geq 1$. Suppose that P is a wt-distance in (X,D). Let $A : X \to X$ be a non-decreasing mapping and there exists $r \in [0,1)$ such that*

$$P\bigl(Ax,A^2x\bigr) \leq rP(x,Ax), \quad \text{for all } x \leq Ax, \tag{4}$$

and $Kr < 1$. Suppose also that:

(i) *for every $x \in X$ with $x \leq Ax$*

$$\inf\bigl\{P(x,y) + P(x,Ax)\bigr\} > 0, \quad \text{for every } y \in X \text{ with } y \neq Ay; \tag{5}$$

(ii) *there exists $x_0 \in X$ such that $x_0 \leq Ax_0$.*

Then A has a fixed point in X.

Proof If $Ax_0 = x_0$, then the proof is finished. Suppose that

$$Ax_0 \neq x_0.$$

Since $x_0 \leq Ax_0$ and A is non-decreasing, we obtain

$$x_0 \leq Ax_0 \leq A^2x_0 \leq \cdots \leq A^{n+1}x_0 \leq \cdots.$$

Hence, for each $n \in \mathbb{N}$ we have

$$P(A^n x_0, A^{n+1} x_0) \leq r^n P(x_0, Ax_0). \tag{6}$$

Then, for $n \in \mathbb{N}$ with $m > n$, we successively have

$$\begin{aligned} P(A^n x_0, A^m x_0) &\leq KP(A^n x_0, A^{n+1} x_0) + K^2 P(A^{n+1} x_0, A^{n+2} x_0) + \cdots \\ &\quad + K^{m-n-2}\big[P(A^{m-2} x_0, A^{m-1} x_0) + P(A^{m-1} x_0, A^m x_0)\big] \\ &\leq r^n KP(x_0, Ax_0) + \cdots + r^{m-1} K^{m-n-2} P(x_0, Ax_0) \\ &\leq Kr^n (1 + Kr + K^2 r^2 + \cdots) P(x_0, Ax_0) \\ &\leq \frac{Kr^n}{1 - Kr} P(x_0, Ax_0). \end{aligned}$$

By Lemma 3.5(3), we conclude that $\{A^n x_0\}$ is Cauchy sequence in (X, D). Since (X, D) is a complete metric type space, there exists $z \in X$ such that

$$\lim_{n \to \infty} A^n x_0 = z.$$

Let $n \in \mathbb{N}$ be an arbitrary but fixed. Then since $\{A^n x_0\}$ converges to z in (X, D) and $P(A^n x_0, \cdot)$ is K-lower semi-continuous, we have

$$P(A^n x_0, z) \leq \liminf_{m \to \infty} KP(A^n x_0, A^m x_0) \leq \frac{K^2 r^n}{1 - Kr} P(x_0, Ax_0).$$

Assume that $z \neq Az$. Since $A^n x_0 \leq A^{n+1} x_0$, by (5), we have

$$\begin{aligned} 0 &< \inf\big\{P(A^n x_0, z) + P(A^n x_0, A^{n+1} x_0)\big\} \\ &\leq \inf\left\{\frac{K^2 r^n}{1 - Kr} P(x_0, Ax_0) + r^n P(x_0, Ax_0)\right\} \\ &= 0. \end{aligned}$$

This is a contradiction. Therefore, we have $z = Az$. □

Another result of this type is the following.

Theorem 4.3 *Let $(X, \leq)$ be a partially ordered set, let $D : X \times X \to [0, \infty)$ be of a metric type on X such that (X, D) is a complete metric type space with constant $K \geq 1$. Suppose that P is a wt-distance in (X, D). Let $A : X \to X$ be a non-decreasing mapping and there exists $r \in [0, 1)$ such that*

$$P(Ax, A^2 x) \leq rP(x, Ax), \quad \textit{for all } x \leq Ax \tag{7}$$

and $Kr < 1$. Assume that one of the following assertions holds:

(i) *for every $x \in X$ with $x \leq Ax$*

$$\inf\big\{P(x, y) + P(x, Ax)\big\} > 0, \quad \textit{for every } y \in X \textit{ with } y \neq Ay; \tag{8}$$

(ii) *if both* $\{x_n\}$ *and* $\{Ax_n\}$ *converge to* y, *then* $y = Ay$;
(iii) A *is continuous*.
If there exists $x_0 \in X$ *with* $x_0 \preceq Ax_0$, *then* A *has a fixed point in* X.

Proof The case (i), was proved in Theorem 4.2.
Let us prove first that (ii) $\Longrightarrow$ (i). Assume that there exists $y \in X$ with $y \neq Ay$ such that

$$\inf\bigl\{P(x,y) + P(x,Ax) : x \preceq Ax\bigr\} = 0.$$

Then there exists $\{z_n\} \in X$ such that $z_n \preceq Az_n$ and

$$\lim_{n\to\infty}\bigl\{P(z_n,y) + P(z_n,Az_n)\bigr\} = 0.$$

Then $P(z_n,y) \longrightarrow 0$ and $P(z_n,Az_n) \longrightarrow 0$. By Lemma 3.5, we have that $Az_n \longrightarrow y$. We also have

$$\begin{aligned} P\bigl(z_n,A^2z_n\bigr) &\le P(z_n,Az_n) + P\bigl(Az_n,A^2z_n\bigr) \\ &\le (1+r)P(z_n,Az_n) \longrightarrow 0. \end{aligned}$$

Again by Lemma 3.5, we get $A^2z_n \longrightarrow y$. Put $x_n = Az_n$. Then both $\{x_n\}$ and $\{Ax_n\}$ converges to y. Thus, by (ii) we have $y = Ay$. Thus (ii) $\Longrightarrow$ (i) holds.
Now, we show that (iii) $\Longrightarrow$ (ii). Let A be continuous. Further assume that $\{x_n\}$ and $\{Ax_n\}$ converges to y. Then we have

$$Ay = A\Bigl(\lim_{n\to\infty} x_n\Bigr) = \lim_{n\to\infty} x_n = y.$$ □

5 Common fixed point theorem for commuting mappings

The following theorem was given by Jungck [18] and it represents a generalization of the Banach contraction principle in complete metric spaces.

Theorem 5.1 *Let* f *be a continuous self mapping on a complete metric space* (X,d) *and let* $g : X \longrightarrow X$ *be another mapping, such that the following conditions are satisfied*:
(a) $g(X) \subseteq f(X)$;
(b) g *commutes with* f;
(c) $d(g(x),g(y)) \le kd(f(x),f(y))$, *for all* $x,y \in X$ *and for some* $0 \le k < 1$.
Then f *and* g *have a unique common fixed point*.

The next example shows that if the mapping $f : X \to X$ is continuous with respect to a metric type D on X and $g : X \to X$ satisfies the condition

$$P\bigl(g(x),g(y)\bigr) \le rP\bigl(f(x),f(y)\bigr), \quad \text{for all } x,y \in X \text{ and some } r \in [0,1),$$

then, in general, g may be not continuous in (X,D).

Example 5.2 Let $X := (\mathbb{R}, |\cdot|)$ be a normed linear space. Consider Example 3.4 with *wt*-distance defined by

$$P(x,y) = |y|^2 \quad \text{for every } x,y \in \mathbb{R}.$$

Consider the functions f and g defined by $f(x) = 4$ and

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then

$$P\bigl(g(x), g(y)\bigr) = \bigl|g(y)\bigr|^2 \leq 1 \leq \left(\frac{1}{3}\right) P\bigl(f(x), f(y)\bigr) = \frac{|f(y)|^2}{3} = \frac{16}{3}.$$

Definition 5.3 Let $(X, \preceq)$ be a partially ordered set and $g, h : X \to X$. By definition, we say that g is h-non-decreasing if for $x, y \in X$,

$$h(x) \preceq h(y) \quad \text{implies} \quad g(x) \preceq g(y). \tag{9}$$

Our next result is a generalization of the above mentioned result of Jungck [18], for the case of a weak contraction with respect to a wt-distance.

Theorem 5.4 *Let $(X, \preceq)$ be a partially ordered set, let $D : X \times X \to [0, \infty)$ be a metric type on X such that (X, D) is a complete metric type space with constant $K \geq 1$. Suppose that P is a wt-distance on X. Let $f, g : X \longrightarrow X$ be mappings that satisfy the following conditions*:
(a) $g(X) \subseteq f(X)$;
(b) *g is f-non-decreasing and f is inverse increasing*;
(c) *g commutes with f and f, g are continuous in (X, D)*;
(d) *$P(g(x), g(y)) \leq rP(f(x), f(y))$ for all $x, y \in X$ with $x \preceq y$ and some $0 < r < 1$ such that $rK < 1$.*
(e) *there exists $x_0 \in X$ such that*:
  (i) *$f(x_0) \preceq g(x_0)$ and*
  (ii) *$f(x_0) \preceq f(g(x_0))$.*

Then f and g have a common fixed point $u \in X$. Moreover, if $g(v) = g^2(v)$ for all $v \in X$, then $P(u, u) = 0$.

Proof We claim that for every $f(x) \preceq g(x)$

$$\inf\bigl\{P\bigl(f(x), g(x)\bigr) + P\bigl(f(x), z\bigr) + P\bigl(g(x), z\bigr) + P\bigl(g(x), g\bigl(g(x)\bigr)\bigr)\bigr\} > 0$$

for every $z \in X$ with $g(z) \neq g(g(z))$. For the moment suppose the claim is true. Let $x_0 \in X$ with $f(x_0) \preceq g(x_0)$. By (a) we can find $x_1 \in X$ such that $f(x_1) = g(x_0)$. By induction, we can define a sequence $\{x_n\}_n \in X$ such that

$$f(x_n) = g(x_{n-1}). \tag{10}$$

Since $f(x_0) \preceq g(x_0)$ and $f(x_1) = g(x_0)$, we have

$$f(x_0) \preceq f(x_1). \tag{11}$$

Then from (b),

$$g(x_0) \preceq g(x_1),$$

that means, by (10), that $f(x_1) \preceq f(x_2)$. Again by (b) we get

$$g(x_1) \preceq g(x_2),$$

that is, $f(x_2) \preceq f(x_3)$. By this procedure, we obtain

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq g(x_3) \preceq \cdots \preceq g(x_n) \preceq g(x_{n+1}) \preceq \cdots. \tag{12}$$

Hence from (10) and (12) we have $f(x_{n-1}) \preceq f(x_n)$ and by (3) we have $x_{n-1} \preceq x_n$. By induction we get

$$\begin{aligned} P\big(f(x_n),f(x_{n+1})\big) &= P\big(g(x_{n-1}),g(x_n)\big) \\ &\leq rP\big(f(x_{n-1}),f(x_n)\big) \\ &\leq \cdots \leq r^n P\big(f(x_0),f(x_1)\big) \end{aligned}$$

for $n = 1,2,\ldots$. This implies that, for $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} &P\big(f(x_n),f(x_m)\big) \\ &\quad\leq K^{m-n-2}\big[P\big(f\big(x_{m-1},f(x_m)\big) + p\big(f(x_{m-2}),f(x_{m-1})\big)\big] + \cdots + KP\big(f(x_n),f(x_{n+1})\big) \\ &\quad< Kr^n\big(1 + Kr + Kr^2 + \cdots\big)P\big(f(x_0),f(x_1)\big) \\ &\quad\leq \frac{Kr^n}{1-Kr}P\big(f(x_0),f(x_1)\big). \end{aligned}$$

Thus, by Lemma 3.5, we find that $\{f(x_n)\}$ is a Cauchy sequence in (X,D). Since (X,D) is complete, there exists $y \in X$ such that $\lim_{n\to\infty} f(x_n) = y$. As a result the sequence $g(x_{n-1}) = f(x_n)$ tends to y as $n \to +\infty$ and hence $\{g(f(x_n))\}_n$ converges to $g(y)$ as $n \to +\infty$. However, $g(f(x_n)) = f(g(x_n))$, by the commutativity of f and g, implies that $f(g(x_n))$ converges to $f(y)$ as $n \to +\infty$. Because limit is unique, we get $f(y) = g(y)$ and, thus, $f(f(y)) = f(g(y))$. On the other hand, by K-lower semi-continuity of $P(x,\cdot)$ we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} P\big(f(x_n),y\big) &\leq \liminf_{m\to\infty} P\big(f(x_n),f(x_m)\big) \leq \frac{K^2 r^n}{1-Kr}P\big(f(x_0),f(x_1)\big), \\ P\big(g(x_n),y\big) &\leq \liminf_{m\to\infty} P\big(f(x_{n+1}),f(x_m)\big) \leq \frac{K^2 r^n}{1-Kr}P\big(f(x_0),f(x_1)\big). \end{aligned}$$

Notice that, by (11), (10), and (9), we obtain $f(x_0) \preceq f(f(x_1))$ and thus, by (9), we get $g(x_0) \preceq g(f(x_1))$. Then

$$f(x_1) \preceq g\big(f(x_1)\big) = f\big(g(x_1)\big) = f\big(f(x_2)\big).$$

By (9) we get $g(x_1) \preceq g(f(x_2))$ and thus $f(x_2) \preceq f(g(x_2))$. Continuing this process we get

$$f(x_n) \preceq f\big(g(x_n)\big), \quad n = 0,1,2,3,\ldots,$$

and by (3) we get

$$x_n \preceq g(x_n), \quad n = 0, 1, 2, 3, \ldots.$$

Using now the condition (d), we have

$$\begin{aligned} P\big(g(x_n), g\big(g(x_n)\big)\big) &\leq rP\big(f(x_n), f\big(g(x_n)\big)\big) \\ &= rP\big(g(x_{n-1}), g\big(g(x_{n-1})\big)\big) \\ &\leq r^2 P\big(f(x_{n-1}), f\big(g(x_{n-1})\big)\big) \\ &= r^2 P\big(g(x_{n-2}), g\big(g(x_{n-2})\big)\big) \\ &\leq \cdots \leq r^n P\big(f(x_1), g\big(f(x_1)\big)\big). \end{aligned}$$

We will show that $g(y) = g(g(y))$. Suppose, by contradiction, that $g(y) \neq g(g(y))$. Then we have

$$\begin{aligned} 0 &< \inf\big\{P\big(f(x), g(x)\big) + P\big(f(x), y\big) + P\big(g(x), y\big) + P\big(g(x), g\big(g(x)\big)\big) : x \in X\big\} \\ &\leq \inf\big\{P\big(f(x_n), g(x_n)\big) + P\big(f(x_n), y\big) + P\big(g(x_n), y\big) + P\big(g(x_n), g\big(g(x_n)\big)\big) : n \in \mathbb{N}\big\} \\ &= \inf\big\{P\big(f(x_n), f(x_{n+1})\big) + P\big(f(x_n), y\big) + P\big(g(x_n), y\big) + P\big(g(x_n), g\big(g(x_n)\big)\big) : n \in \mathbb{N}\big\} \\ &\leq \inf_n \bigg\{ r^n P\big(f(x_0), f(x_1)\big) + \frac{K^2 r^n}{1 - Kr} P\big(f(x_0), f(x_1)\big) + \frac{K^2 r^{n+1}}{1 - Kr} P\big(f(x_0), f(x_1)\big) \\ &\quad + r^n P\big(f(x_1), g\big(f(x_1)\big)\big) : n \in \mathbb{N} \bigg\} = 0. \end{aligned}$$

This is a contradiction. Therefore $g(y) = g(g(y))$. Thus, $g(y) = g(g(y)) = f(g(y))$. Hence $u := g(y)$ is a common fixed point of f and g.

Furthermore, since $g(v) = g(g(v))$ for all $v \in X$, we have

$$\begin{aligned} P\big(g(y), g(y)\big) &= P\big(g\big(g(y)\big), g\big(g(y)\big)\big) \\ &\leq rP\big(f\big(g(y)\big), f\big(g(y)\big)\big) \\ &= rP\big(g(y), g(y)\big), \end{aligned}$$

which implies that $P(g(y), g(y)) = 0$.

Now it remains to prove the initial claim. Assume that there exists $y \in X$ with $g(y) \neq g(g(y))$ and

$$\inf\big\{P\big(f(x), g(x)\big) + P\big(f(x), y\big) + p\big(g(x), y\big) + Pp\big(g(x), g\big(g(x)\big)\big) : x \in X\big\} = 0.$$

Then there exists $\{x_n\}$ such that

$$\lim_{n \to \infty} \big\{P\big(f(x_n), g(x_n)\big) + P\big(f(x_n), y\big) + P\big(g(x_n), y\big) + P\big(g(x_n), g\big(g(x_n)\big)\big)\big\} = 0.$$

Since $P(f(x_n), g(x_n)) \longrightarrow 0$ and $P(f(x_n), y) \longrightarrow 0$, by Lemma 3.5, we have

$$\lim_{n \to \infty} g(x_n) = y. \tag{13}$$

Also, since $P(g(x_n),y) \longrightarrow 0$ and $P(g(x_n),g(g(x_n))) \longrightarrow 0$, by Lemma 3.5, we have

$$\lim_{n\to\infty} g\bigl(g(x_n)\bigr) = y. \tag{14}$$

By (13), (14), and the continuity of g we have

$$g(y) = g\Bigl(\lim_n g(x_n)\Bigr) = \lim_n g\bigl(g(x_n)\bigr) = y.$$

Therefore, $g(y) = g(g(y))$, which is a contradiction. Hence, if $g(y) \neq g(g(y))$, then

$$\inf\bigl\{P\bigl(f(x),g(x)\bigr) + P\bigl(f(x),y\bigr) + P\bigl(g(x),y\bigr) + P\bigl(g(x),g\bigl(g(x)\bigr)\bigr) : x \in X\bigr\} > 0. \qquad \square$$

Example 5.5 Let $X := (\mathbb{R}, |\cdot|)$ be a normed linear space. Consider Example 3.4 with *wt*-distance defined by

$$P(x,y) = |y|^2 \quad \text{for every } x,y \in \mathbb{R}.$$

Consider the functions f and g defined by $f(x) = 3x$ and $g(x) = \sqrt{2}x$. Then

$$P\bigl(g(x),g(y)\bigr) = \bigl|g(y)\bigr|^2 = 2y^2 \leq \left(\frac{1}{3}\right)P\bigl(f(x),f(y)\bigr) = \frac{|f(y)|^2}{3} = \frac{9y^2}{3}.$$

Put $K = 2$. Then all conditions of Theorem 5.4 hold and $u = 0$ is the common fixed point of f and g and $P(0,0) = |0|^2 = 0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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Acknowledgements

The authors would like to thank the referees for giving useful suggestions and comments for the improvement of this paper. This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the first author acknowledges with thanks DSR, KAU for financial support.

Received: 17 January 2014 Accepted: 21 March 2014 Published: 03 Apr 2014

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10.1186/1687-1812-2014-88
Cite this article as: Hussain et al.: On the topology and *wt*-distance on metric type spaces. *Fixed Point Theory and Applications* 2014, 2014:88