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The implicit midpoint rule for nonexpansive mappings

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Abstract
The implicit midpoint rule (IMR) for nonexpansive mappings is established. The IMR generates a sequence by an implicit algorithm. Weak convergence of this algorithm is proved in a Hilbert space. Applications to the periodic solution of a nonlinear time-dependent evolution equation and to a Fredholm integral equation are included.

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1 Introduction

The implicit midpoint rule (IMR) is one of the powerful numerical methods for solving ordinary differential equations (in particular, the stiff equations) [1–6] and differential-algebra equations [7].

For the ordinary differential equation

$$y' = f(y), \qquad y(0) = y_0, \tag{1.1}$$

IMR generates a sequence $\{y_n\}$ by the recursion procedure

$$y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right), \quad n \geq 0, \tag{1.2}$$

where $h > 0$ is a stepsize. It is known that if $f : \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{y_n\}$ converges to the exact solution of (1.1) as $h \to 0$ uniformly over $t \in [0, \bar{t}]$ for any fixed $\bar{t} > 0$.

If we write the function f in the form $f(y) = y - g(y)$, then differential equation (1.1) becomes

$$y' = y - g(y), \qquad y(0) = y_0, \tag{1.3}$$

and the process (1.2) is rewritten as

$$y_{n+1} = y_n + h\left[\frac{y_n + y_{n+1}}{2} - g\left(\frac{y_n + y_{n+1}}{2}\right)\right], \quad n \geq 0. \tag{1.4}$$

The equilibrium problem associated with differential equation (1.3) is the fixed point problem

$$y = g(y). \tag{1.5}$$

This motivates us to transplant IMR (1.4) to the solving of the fixed point equation

$$x = Tx, \tag{1.6}$$

where T is, in general, a nonlinear operator in a Hilbert space. We below introduce our implicit midpoint rule (IMR) for the fixed point problem (1.6) in two iterative algorithms. The first algorithm generates a sequence $\{x_n\}$ in the following manner.

Algorithm I Initialize $x_0 \in H$ arbitrarily and iterate

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right], \quad n \geq 0, \tag{1.7}$$

where $t_n \in (0,1)$ for all n.

Our second IMR is an algorithm that generates a sequence $\{x_n\}$ as follows.

Algorithm II Initialize $x_0 \in H$ arbitrarily and iterate

$$x_{n+1} := (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.8}$$

where $t_n \in (0,1)$ for all n.

We observe that Algorithm I is equivalent to Algorithm II since it is easy to rewrite (1.7), by partially solving for x_{n+1}, as

$$x_{n+1} = (1 - s_n)x_n + s_n T\left(\frac{x_n + x_{n+1}}{2}\right), \tag{1.9}$$

where

$$s_n = \frac{2t_n}{2 + t_n}. \tag{1.10}$$

Consequently, we may concentrate on Algorithm II.

The purpose of this paper is to study the convergence of two IMR (1.7) and (1.8) in the case where the mapping T is a nonexpansive mapping in a general Hilbert space H, that is,

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in H. \tag{1.11}$$

The iterative methods for finding fixed points of nonexpansive mappings have received much attention due to the fact that in many practical problems, the governing operators are nonexpansive (*cf.* [8, 9]). Two iterative methods are basic and they are Mann's method [10, 11] and Halpern's method [12–16]. An implicit method is also proposed in [17].

2 Convergence analysis

Throughout this section we always assume that H is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ and that $T : H \to H$ is a nonexpansive mapping with a fixed point. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T. Namely, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. It is not hard to find that both IMR (1.7) and (1.8) are well defined. As a matter of fact, for each fixed $u \in H$ and $t \in (0,1)$, the mapping

$$x \mapsto T_u x := u - t\left[\frac{u+x}{2} - T\left(\frac{u+x}{2}\right)\right] \tag{2.1}$$

is a contraction with coefficient $\frac{1+t}{2} \in (0,1)$. That is,

$$\|T_u x - T_u y\| \le \frac{1+t}{2}\|x-y\|, \quad x,y \in H. \tag{2.2}$$

This is immediately clear due to the nonexpansivity of T.

It is also easily seen that the mapping

$$x \mapsto T^u x := (1-t)u + tT\left(\frac{u+x}{2}\right) \tag{2.3}$$

is a contraction with coefficient $t/2$.

2.1 Properties of Algorithm II

We first discuss the properties of Algorithm II.

Lemma 2.1 *Let* $\{x_n\}$ *be the sequence generated by Algorithm* II. *Then*

(i) $\|x_{n+1} - p\| \le \|x_n - p\|$ *for all* $n \ge 0$ *and* $p \in \mathrm{Fix}(T)$.

(ii) $\sum_{n=1}^{\infty} t_n\|x_n - x_{n+1}\|^2 < \infty$.

(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\|x_n - T(\frac{x_n+x_{n+1}}{2})\|^2 < \infty$.

Proof Let $p \in \mathrm{Fix}(T)$. We deduce that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|(1-t_n)(x_n-p) + t_n\left[T\left(\frac{x_n+x_{n+1}}{2}\right) - p\right]\right\|^2 \\
&= (1-t_n)\|x_n-p\|^2 + t_n\left\|T\left(\frac{x_n+x_{n+1}}{2}\right) - p\right\|^2 \\
&\quad - t_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2 \\
&\le (1-t_n)\|x_n-p\|^2 + t_n\left\|\frac{x_n+x_{n+1}}{2} - p\right\|^2 - t_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2 \\
&= (1-t_n)\|x_n-p\|^2 \\
&\quad + t_n\left(\frac{1}{2}\|x_n-p\|^2 + \frac{1}{2}\|x_{n+1}-p\|^2 - \frac{1}{4}\|x_n-x_{n+1}\|^2\right) \\
&\quad - t_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2.
\end{aligned}$$

It turns out that

$$\left(1-\frac{t_n}{2}\right)\|x_{n+1}-p\|^2 \le \left(1-\frac{t_n}{2}\right)\|x_n-p\|^2 - \frac{t_n}{4}\|x_n-x_{n+1}\|^2 - t_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2$$

and

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 - \frac{t_n}{2(2-t_n)}\|x_n-x_{n+1}\|^2 - \frac{2t_n(1-t_n)}{2-t_n}\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2. \tag{2.4}$$

It is then immediately evident that

$$\|x_{n+1}-p\| \le \|x_n-p\|, \quad n\ge 0. \tag{2.5}$$

Moreover, since $t_n \in (0,1)$, (2.4) also implies that

$$\sum_{n=1}^{\infty} t_n\|x_n-x_{n+1}\|^2 < \infty \tag{2.6}$$

and

$$\sum_{n=1}^{\infty} t_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2 < \infty. \tag{2.7}$$

The proof of the lemma is complete. □

Lemma 2.2 *Let* $\{x_n\}$ *be the sequence generated by Algorithm* II. *Suppose that* $t_{n+1}^2 \le at_n$ *for all* $n\ge 0$ *and some* $a>0$. *Then*

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{2.8}$$

Proof By definition (1.8) of Algorithm II, we derive that

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &= t_{n+1}\left\|x_{n+1} - T\left(\frac{x_{n+1}+x_{n+2}}{2}\right)\right\| \\
&\le t_{n+1}\left\|x_{n+1} - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\| \\
&\quad + t_{n+1}\left\|T\left(\frac{x_n+x_{n+1}}{2}\right) - T\left(\frac{x_{n+1}+x_{n+2}}{2}\right)\right\| \\
&\le t_{n+1}(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\| \\
&\quad + t_{n+1}\left\|\frac{x_n+x_{n+1}}{2} - \frac{x_{n+1}+x_{n+2}}{2}\right\|
\end{aligned}$$

$$\leq t_{n+1}(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\| + t_{n+1}\left(\frac{1}{2}\|x_{n+1}-x_n\| + \frac{1}{2}\|x_{n+2}-x_{n+1}\|\right).$$

Hence

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq \frac{t_{n+1}}{2-t_{n+1}}\|x_{n+1}-x_n\| + \frac{2t_{n+1}(1-t_n)}{2-t_{n+1}}\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\| \\ &\leq t_{n+1}\|x_{n+1}-x_n\| + 2t_{n+1}(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|.\end{aligned}$$

Using the assumption that $t_{n+1}^2 \leq at_n$, we further derive that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\|^2 &\leq 2t_{n+1}^2\|x_{n+1}-x_n\|^2 + 4t_{n+1}^2(1-t_n)^2\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2 \\ &\leq 2at_n\|x_{n+1}-x_n\|^2 + 4at_n(1-t_n)\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|^2.\end{aligned}$$

Now (2.6) and (2.7) imply that

$$\sum_{n=1}^{\infty}\|x_{n+2}-x_{n+1}\|^2 < \infty.$$

This in turn implies (2.8). □

2.2 Convergence of Algorithms I and II

As Algorithm I is a variant of Algorithm II, we focus on the convergence of Algorithm II. To this end, we need two conditions for the sequence of parameters $\{t_n\}$ as follows:

(C1) $t_{n+1}^2 \leq at_n$ for all $n \geq 0$ and some $a > 0$,

(C2) $\liminf_{n\to\infty} t_n > 0$.

These two conditions are not restrictive. As a matter of fact, it is not hard to find that, for each $p > 0$, the sequence

$$t_n = 1 - \frac{1}{(n+2)^p}, \quad n \geq 0,$$

satisfies (C1) and (C2).

Lemma 2.3 *Assume* (C1) *and* (C2). *Then the sequence* $\{x_n\}$ *generated by Algorithm* II *satisfies the property*

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{2.9}$$

Proof From (1.8) it follows that

$$\|x_{n+1}-x_n\| = t_n\left\|x_n - T\left(\frac{x_n+x_{n+1}}{2}\right)\right\|. \tag{2.10}$$

Now condition (C2) implies that $t_n \geq \bar{t} > 0$ for all large enough n. Hence from Lemma 2.2, we immediately get

$$\lim_{n\to\infty}\left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| = 0. \tag{2.11}$$

Conclusion (2.9) now follows from the following inference:

$$\begin{aligned}\|x_n - Tx_n\| &\leq \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \left\|Tx_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\ &\leq \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \frac{1}{2}\|x_n - x_{n+1}\| \to 0.\end{aligned}$$ □

To prove the convergence of Algorithm II, we need the following so-called demiclosedness principle for nonexpansive mappings.

Lemma 2.4 ([18]) *Let C be a nonempty closed convex subset of a Hilbert space H, and let $V : C \to H$ be a nonexpansive mapping with a fixed point. Assume that $\{x_n\}$ is a sequence in C such that $x_n \to x$ weakly and $(I - V)x_n \to 0$ strongly. Then $(I - T)x = 0$ (i.e., $Tx = x$).*

We use the notation $\omega_w(x_n)$ to denote the set of all weak cluster points of the sequence $\{x_n\}$.

The following result is easily proved (see [19]).

Lemma 2.5 *Let K be a nonempty closed convex subset of a Hilbert space H, and let $\{x_n\}$ be a bounded sequence in H. Assume that*

(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in K$,
(ii) $\omega_w(x_n) \subset K$.

Then $\{x_n\}$ weakly converges to a point in K.

We are now in a position to state and prove the main convergence result of this paper.

Theorem 2.6 *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Assume that $\{x_n\}$ is generated by IMR* (1.8) *where the sequence $\{t_n\}$ of parameters satisfies conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof By Lemmas 2.3 and 2.4, we have $\omega_w(x_n) \subset \mathrm{Fix}(T)$. Furthermore, by Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \mathrm{Fix}(T)$. Consequently, we can apply Lemma 2.5 with $K = \mathrm{Fix}(T)$ to assert the weak convergence of $\{x_n\}$ to a point in $\mathrm{Fix}(T)$. □

We then have the following convergence result for IMR (1.7).

Theorem 2.7 *Let H be a Hilbert space and $T : H \to H$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Assume that $\{x_n\}$ is generated by IMR* (1.7) *where the sequence $\{t_n\}$ of parameters satisfies conditions* (C1) *and* (C2). *Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof Since $\{x_n\}$ is also generated by algorithm (1.9), it suffices to verify that the sequence $\{s_n\}$ defined in (1.10) satisfies conditions (C1) and (C2). As $0 \leq t_n \leq 1$ and satisfies (C2), it is evident that $\{s_n\}$ satisfies (C2) as well. To see that $\{s_n\}$ also fulfils (C1), we argue as follows, using the fact that $\{t_n\}$ satisfies (C1):

$$s_{n+1}^2 = \frac{4t_{n+1}^2}{(2+t_{n+1})^2} \le t_{n+1}^2 \le at_n = as_n(2+t_n) \le 3as_n. \qquad \square$$

3 Applications

3.1 Periodic solution of a nonlinear evolution equation

Consider the time-dependent nonlinear evolution equation in a (possibly complex) Hilbert space H,

$$\frac{du}{dt} + A(t)u = f(t,u), \quad t > 0, \tag{3.1}$$

where $A(t)$ is a family of closed linear operators in H and $f : \mathbb{R} \times H \to H$.

Browder [20] proved the following existence of periodic solutions of equation (3.1).

Theorem 3.1 ([20]) *Suppose that $A(t)$ and $f(t,u)$ are periodic in t of period $\xi > 0$ and satisfy the following assumptions*:

(i) *For each t and each pair $u, v \in H$,*

$$\operatorname{Re}\langle f(t,u) - f(t,v), u - v\rangle \le 0.$$

(ii) *For each t and each $u \in D(A(t))$, $\operatorname{Re}\langle A(t)u, u\rangle \ge 0$.*

(iii) *There exists a mild solution u of equation* (3.1) *on $\mathbb{R}^+$ for each initial value $v \in H$. Recall that u is a mild solution of* (3.1) *with the initial value $u(0) = v$ if, for each $t > 0$,*

$$u(t) = U(t,0)v + \int_0^t U(t,s)f\bigl(s,u(s)\bigr)\,ds,$$

where $\{U(t,s)\}_{t\ge s\ge 0}$ is the evolution system for the homogeneous linear system

$$\frac{du}{dt} + A(t)u = 0 \quad (t > s). \tag{3.2}$$

(iv) *There exists some $R > 0$ such that*

$$\operatorname{Re}\langle f(t,u), u\rangle < 0$$

for $\|u\| = R$ and all $t \in [0,\xi]$.

Then there exists an element v of H with $\|v\| < R$ such that the mild solution of equation (3.1) *with the initial condition $u(0) = v$ is periodic of period ξ.*

We next apply our IMR for nonexpansive mappings to provide an iterative method for finding a periodic solution of (3.1).

As a matter of fact, define a mapping $T : H \to H$ by assigning to each $v \in H$ the value $u(\xi)$, where u is the solution of (3.1) satisfying the initial condition $u(0) = v$. Namely, we define T by

$$Tv = u(\xi), \quad \text{where } u \text{ solves (3.1) with } u(0) = v.$$

We then find that T is nonexpansive. Moreover, assumption (iv) forces T to map the closed ball $B := \{v \in H : \|v\| \le R\}$ into itself. Consequently, T has a fixed point which we denote by v, and the corresponding solution u of (3.1) with the initial condition $u(0) = v$ is a desired periodic solution of (3.1) with period ξ. In other words, to find a periodic solution u of (3.1) is equivalent to finding a fixed point of T. Our IMR is thus applicable to (3.1). It turns out that the sequence $\{v_n\}$ defined by the IMR

$$v_{n+1} = (1 - t_n)v_n + t_n T\left(\frac{v_n + v_{n+1}}{2}\right) \tag{3.3}$$

converges weakly to a fixed point v of T, and the mild solution of (3.1) with the initial value $u(0) = \xi$ is a periodic solution of (3.1). Note that the iteration method (3.3) is essentially to find a mild solution of (3.1) with the initial value of $(v_n + v_{n+1})/2$.

3.2 Fredholm integral equation

Consider a Fredholm integral equation of the form

$$x(t) = g(t) + \int_0^1 F\bigl(t, s, x(s)\bigr)\, ds, \quad t \in [0,1], \tag{3.4}$$

where g is a continuous function on $[0,1]$ and $F : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The existence of solutions has been investigated in the literature (see [21] and the references therein). In particular, if F satisfies the Lipschitz continuity condition

$$\bigl|F(t,s,x) - F(t,s,y)\bigr| \le |x - y|, \quad t, s \in [0,1], x, y \in \mathbb{R}, \tag{3.5}$$

then equation (3.6) has at least one solution in $L^2[0,1]$ ([21, Theorem 3.3]). Define a mapping $T : L^2[0,1] \to L^2[0,1]$ by

$$(Tx)(t) = g(t) + \int_0^1 F\bigl(t, s, x(s)\bigr)\, ds, \quad t \in [0,1]. \tag{3.6}$$

It is easily seen that T is nonexpansive. As a matter of fact, we have, for $x, y \in L^2[0,1]$,

$$\begin{aligned}
\|Tx - Ty\|^2 &= \int_0^1 \bigl|Tx(t) - Ty(t)\bigr|^2\, dt \\
&= \int_0^1 \left| \int_0^1 \bigl(F\bigl(t,s,x(s)\bigr) - F\bigl(t,s,y(s)\bigr)\bigr)\, ds \right|^2 dt \\
&\le \int_0^1 \left| \int_0^1 \bigl|x(s) - y(s)\bigr|\, ds \right|^2 dt \\
&\le \int_0^1 \bigl|x(s) - y(s)\bigr|^2\, ds = \|x - y\|^2.
\end{aligned}$$

This means that to find the solution of integral equation (3.6) is reduced to finding a fixed point of the nonexpansive mapping T in the Hilbert space $L^2[0,1]$. Hence our IMR is again applicable. Initiating with any function $x_0 \in L^2[0,1]$, we define a sequence of functions $\{x_n\}$

in $L^2[0,1]$ by

$$x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right). \tag{3.7}$$

Then the sequence $\{x_n\}$ converges weakly in $L^2[0,1]$ to the solution of integral equation (3.6).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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