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On new fixed point results for (α,ψ,ξ)-contractive multi-valued mappings on α-complete metric spaces and their consequences

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Abstract
The purpose of this paper is to establish new fixed point results for multi-valued mappings satisfying an (α,ψ,ξ)-contractive condition, and via Bianchini-Grandolfi gauge functions, on α-complete metric spaces. Our results unify, generalize, and complement various results from the literature. We also give examples which support our main result while previous results in the literature are not applicable. Some of the fixed point results in metric spaces endowed with an arbitrary binary relation and endowed with a graph are given here to illustrate the usability of the obtained results.
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1 Introduction and preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers, and real numbers, respectively.

We recollect some essential notations, required definitions, and primary results coherent with the literature. For a nonempty set X, we denote by $N(X)$ the class of all nonempty subsets of X. Let (X,d) be a metric space, we denote by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X. For $A,B \in CL(X)$, let the functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{\infty\}$ be defined by

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise} \end{cases}$$

for every $A,B \in CL(X)$, where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from a to $B \subseteq X$. Such a functional is called the generalized Pompeiu-Hausdorff metric induced by d.

In this paper, we denote by Ψ the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is a nondecreasing function;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as Bianchini-Grandolfi gauge functions in some sources (see *e.g.* [1–3]).

Remark 1.1 For each $\psi \in \Psi$, we see that the following assertions hold:

1. $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$.
2. $\psi(t) < t$ for each $t > 0$;
3. $\psi(0) = 0$;

Example 1.2 The function $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = kt$, where $k \in [0,1)$, is a Bianchini-Grandolfi gauge function.

Example 1.3 The function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \begin{cases} \frac{1}{4}t, & 0 \le t < 1, \\ \frac{1}{2}t, & t = 1, \\ \frac{3}{4}t, & t > 1, \end{cases}$$

is a Bianchini-Grandolfi gauge function.

In [4], Samet *et al.* introduced the concepts of an α-admissible mapping and an α-ψ-contractive mapping as follows.

Definition 1.4 ([4]) Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be a mapping. We say that T is α*-admissible* if the following condition holds:

$$x, y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \ge 1.$$

Definition 1.5 ([4]) Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi\bigl(d(x,y)\bigr)$$

for all $x, y \in X$.

One also proved some fixed point theorems for such mappings on complete metric spaces and showed that these results can be utilized to derive fixed point theorems in partially ordered metric spaces.

Afterwards, Asl *et al.* [5] introduced the concept of an α_*-admissible mapping which is a multi-valued version of the α-admissible mapping provided in [4].

Definition 1.6 ([5]) Let X be a nonempty set, $T : X \to N(X)$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. We say that T is α_*-admissible if the following condition holds:

$$x, y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha_*(Tx, Ty) \ge 1,$$

where $\alpha_*(Tx, Ty) := \inf\{\alpha(a,b) | a \in Tx, b \in Ty\}$.

They extended the α-ψ-contractive condition of Samet *et al.* [4] from a single-valued version to a multi-valued version as follows.

Definition 1.7 ([5]) Let (X,d) be a metric space, $T: X \to CL(X)$ be a multi-valued mapping and $\alpha: X \times X \to [0,\infty)$ be a given mapping. We say that T is an α-ψ-contractive multi-valued mapping if there exists $\psi \in \Psi$ such that

$$\alpha(x,y)H(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr)$$

for all $x,y \in X$.

Asl *et al.* [5] also established a fixed point result for multi-valued mappings on complete metric spaces satisfying an α-ψ-contractive condition.

Recently, Ali *et al.* [6] introduced the notion of (α,ψ,ξ)-contractive multi-valued mappings, where $\xi \in \Xi$ and Ξ is the family of functions $\xi: [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ξ_1) ξ is continuous;
(ξ_2) ξ is nondecreasing on $[0,\infty)$;
(ξ_3) $\xi(t) = 0$ if and only if $t = 0$;
(ξ_4) ξ is subadditive.

Remark 1.8 From (ξ_2) and (ξ_3), we have $\xi(t) > 0$, for all $t \in (0,\infty)$.

Example 1.9 Let $\xi: [0,\infty) \to [0,\infty)$ be a mapping which is defined by

$$\xi(t) = \int_0^t \phi(s)\,ds$$

for each $t \in [0,\infty)$, where $\phi: [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$ and satisfies the following conditions:

- for each $\epsilon > 0$, we have $\int_0^\epsilon \phi(t)\,dt > 0$;
- for each $a, b > 0$, we have

$$\int_0^{a+b} \phi(t)\,dt \leq \int_0^a \phi(t)\,dt + \int_0^b \phi(t)\,dt.$$

Then $\xi \in \Xi$.

Lemma 1.10 *Let (X,d) be a metric space. If $\xi \in \Xi$, then $(X, \xi \circ d)$ is a metric space.*

Lemma 1.11 ([6]) *Let (X,d) be a metric space, $\xi \in \Xi$, and $B \in CL(X)$. If there exists $x \in X$ such that $\xi(d(x,B)) > 0$, then there exists $y \in B$ such that*

$$\xi\bigl(d(x,y)\bigr) < q\xi\bigl(d(x,B)\bigr),$$

where $q > 1$.

Definition 1.12 ([6]) Let (X,d) be a metric space. A multi-valued mapping $T: X \to CL(X)$ is called an (α,ψ,ξ)-contractive mapping if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$, and

$\alpha : X \times X \to [0,\infty)$ such that

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \xi\bigl(H(Tx,Ty)\bigr) \leq \psi\bigl(\xi\bigl(M(x,y)\bigr)\bigr), \tag{1.1}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

In the case when $\psi \in \Psi$ is strictly increasing, the (α,ψ,ξ)-contractive mapping is called a strictly (α,ψ,ξ)-contractive mapping.

Ali *et al.* [6] also prove fixed point results for (α,ψ,ξ)-contractive multi-valued mapping on complete metric spaces.

Question 1 Is it possible to prove fixed point results for (α,ψ,ξ)-contractive multi-valued mapping T under some weaker condition for T?

Question 2 Is it possible to prove fixed point results for (α,ψ,ξ)-contractive multi-valued mapping in some space which is more general than complete metric spaces?

Question 3 Is it possible to find some consequences or applications of the fixed point results?

On the other hand, Mohammadi *et al.* [7] extended the concept of an α_*-admissible mapping to α-admissible as follows.

Definition 1.13 ([7]) Let X be a nonempty set, $T : X \to N(X)$ and $\alpha : X \times X \to [0,\infty)$ be two given mappings. We say that T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, we have $\alpha(y,z) \geq 1$, for all $z \in Ty$.

Remark 1.14 It is clear that α_*-admissible mapping is also α-admissible, but the converse may not be true as shown in Example 15 of [8].

Recently, Hussain *et al.* [9] introduced the concept of α-completeness for metric space which is a weaker than the concept of completeness.

Definition 1.15 ([9]) Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. The metric space X is said to be *α-complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, converges in X.

Remark 1.16 If X is complete metric space, then X is also α-complete metric space. But the converse is not true.

Example 1.17 Let $X = (0,\infty)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$, for all $x, y \in X$. Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} e^{\frac{xy}{x+y}}, & x, y \in [2,5], \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$$

It is easy to see that (X,d) is not a complete metric space, but (X,d) is an α-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, then $x_n \in [2,5]$, for all $n \in \mathbb{N}$. Since $[2,5]$ is a closed subset of $\mathbb{R}$, we see that $([2,5],d)$ is a complete metric space and then there exists $x^* \in [2,5]$ such that $x_n \to x^*$ as $n \to \infty$.

In this paper, we establish new fixed point results for (α,ψ,ξ)-contractive multi-valued mappings on α-complete metric spaces by using the idea of α-admissible multi-valued mapping due to Mohammadi *et al.* [7]. These results are real generalization of main results of Ali *et al.* [6] and many results in literature. We furnish some interesting examples which support our main theorems while results of Ali *et al.* [6] are not applicable. We also obtain fixed point results in metric space endowed with an arbitrary binary relation and fixed point results in metric space endowed with graph.

2 Main results

First, we introduce the concept of α-continuity for multi-valued mappings in metric spaces.

Definition 2.1 Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ and $T : X \to CL(X)$ be two given mappings. We say T is an α-continuous multi-valued mapping on $(CL(X),H)$ if, for all sequences $\{x_n\}$ with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$, and $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $Tx_n \xrightarrow{H} Tx$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n,x) = 0 \quad \text{and} \quad \alpha(x_n,x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \quad \Longrightarrow \quad \lim_{n\to\infty} H(Tx_n,Tx) = 0.$$

Note that the continuity of T implies the α-continuity of T, for all mappings α. In general, the converse is not true (see in Example 2.2).

Example 2.2 Let $X = [0,\infty)$, $\lambda \in [10,20]$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$, for all $x,y \in X$. Define $T : X \to CL(X)$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} \{\lambda x^2\}, & x \in [0,1], \\ \{x\}, & x > 1 \end{cases}$$

and

$$\alpha(x,y) \begin{cases} \cosh(x^2+y^2), & x,y \in [0,1], \\ \tanh(x+y), & \text{otherwise}. \end{cases}$$

Clearly, T is not a continuous multi-valued mapping on $(CL(X),H)$. Indeed, for sequence $\{x_n\} = \{1+\frac{1}{n}\}$ in X, we see that $x_n = 1+\frac{1}{n} \xrightarrow{d} 1$, but $Tx_n = \{1+\frac{1}{n}\} \xrightarrow{H} \{1\} \neq \{\lambda\} = T1$.

Next, we show that T is an α-continue multi-valued mapping on $(CL(X),H)$. Let $\{x_n\}$ be a sequence in X such that $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Then we have $x,x_n \in [0,1]$, for all $n \in \mathbb{N}$. Therefore, $Tx_n = \{\lambda x_n^2\} \xrightarrow{H} \{\lambda x^2\} = Tx$. This shows that T is an α-continuous multi-valued mapping on $(CL(X),H)$.

Now we give first main result in this paper.

Theorem 2.3 *Let (X,d) be a metric space and $T : X \to CL(X)$ be a strictly (α,ψ,ξ)-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X,d) is an α-complete metric space*;

(S_2) *T is an α-admissible multi-valued mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
(S_4) *T is an α-continuous multi-valued mapping.*

Then T has a fixed point.

Proof Starting from x_0 and $x_1 \in Tx_0$ in (S_3), we have $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then we see that x_0 is a fixed point of T. Assume that $x_0 \neq x_1$. If $x_1 \in Tx_1$, we obtain that x_1 is a fixed point of T. Then we have nothing to prove. So we let $x_1 \notin Tx_1$. From (α, ψ, ξ)-contractive condition, we get

$$\begin{aligned}
&\xi\bigl(H(Tx_0, Tx_1)\bigr) \\
&\quad\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right)\right) \\
&\quad= \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\}\right)\right) \\
&\quad\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2}\right\}\right)\right) \\
&\quad= \psi\bigl(\xi\bigl(\max\bigl\{d(x_0, x_1), d(x_1, Tx_1)\bigr\}\bigr)\bigr). \qquad (2.1)
\end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we get

$$\begin{aligned}
0 &< \xi\bigl(d(x_1, Tx_1)\bigr) \\
&\leq \xi\bigl(H(Tx_0, Tx_1)\bigr) \\
&\leq \psi\bigl(\xi\bigl(\max\bigl\{d(x_0, x_1), d(x_1, Tx_1)\bigr\}\bigr)\bigr) \\
&= \psi\bigl(\xi\bigl(d(x_1, Tx_1)\bigr)\bigr), \qquad (2.2)
\end{aligned}$$

which is a contradiction. Therefore, $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$. From (2.1), we get

$$0 < \xi\bigl(d(x_1, Tx_1)\bigr) \leq \xi\bigl(H(Tx_0, Tx_1)\bigr) \leq \psi\bigl(\xi\bigl(d(x_0, x_1)\bigr)\bigr). \qquad (2.3)$$

For fixed $q > 1$ by using Lemma 1.11, there exists $x_2 \in Tx_1$ such that

$$0 < \xi\bigl(d(x_1, x_2)\bigr) < q\xi\bigl(d(x_1, Tx_1)\bigr). \qquad (2.4)$$

From (2.3) and (2.4), we have

$$0 < \xi\bigl(d(x_1, x_2)\bigr) < q\psi\bigl(\xi\bigl(d(x_0, x_1)\bigr)\bigr). \qquad (2.5)$$

Since ψ is strictly increasing function, we have

$$0 < \psi\bigl(\xi\bigl(d(x_1, x_2)\bigr)\bigr) < \psi\bigl(q\psi\bigl(\xi\bigl(d(x_0, x_1)\bigr)\bigr)\bigr). \qquad (2.6)$$

Put $q_1 = \frac{\psi(q\psi(\xi(d(x_0,x_1))))}{\psi(\xi(d(x_1,x_2)))}$ and then $q_1 > 1$.

If $x_1 = x_2$ or $x_2 \in Tx_2$, then we find that x_2 is a fixed point of T and thus we have nothing to prove. Therefore, we may assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Since $x_1 \in Tx_0$, $x_2 \in Tx_1$, $\alpha(x_0, x_1) \geq 1$, and T is an α-admissible multi-valued mapping, we have $\alpha(x_1, x_2) \geq 1$. Applying from (α, ψ, ξ)-contractive condition, we have

$$\begin{aligned}
&\xi\bigl(H(Tx_1, Tx_2)\bigr) \\
&\quad\leq \psi\left(\xi\left(\max\left\{d(x_1,x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right\}\right)\right) \\
&\quad= \psi\left(\xi\left(\max\left\{d(x_1,x_2), d(x_2, Tx_2), \frac{d(x_1, Tx_2)}{2}\right\}\right)\right) \\
&\quad\leq \psi\left(\xi\left(\max\left\{d(x_1,x_2), d(x_2, Tx_2), \frac{d(x_1,x_2) + d(x_2, Tx_2)}{2}\right\}\right)\right) \\
&\quad= \psi\bigl(\xi\bigl(\max\bigl\{d(x_1,x_2), d(x_2, Tx_2)\bigr\}\bigr)\bigr).
\end{aligned} \tag{2.7}$$

Suppose that $\max\{d(x_1,x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$. From (2.7), we get

$$\begin{aligned}
0 &< \xi\bigl(d(x_2, Tx_2)\bigr) \\
&\leq \xi\bigl(H(Tx_1, Tx_2)\bigr) \\
&\leq \psi\bigl(\xi\bigl(\max\bigl\{d(x_1,x_2), d(x_2, Tx_2)\bigr\}\bigr)\bigr) \\
&= \psi\bigl(\xi\bigl(d(x_2, Tx_2)\bigr)\bigr),
\end{aligned} \tag{2.8}$$

which is a contradiction. Therefore, we may let $\max\{d(x_1,x_2), d(x_2, Tx_2)\} = d(x_1,x_2)$. From (2.7), we have

$$0 < \xi\bigl(d(x_2, Tx_2)\bigr) \leq \xi\bigl(H(Tx_1, Tx_2)\bigr) \leq \psi\bigl(\xi\bigl(d(x_1,x_2)\bigr)\bigr). \tag{2.9}$$

By using Lemma 1.11 with q_1, there exists $x_3 \in Tx_2$ such that

$$0 < \xi\bigl(d(x_2,x_3)\bigr) < q_1\xi\bigl(d(x_2, Tx_2)\bigr). \tag{2.10}$$

From (2.9) and (2.10), we get

$$0 < \xi\bigl(d(x_2,x_3)\bigr) < q_1\psi\bigl(\xi\bigl(d(x_1,x_2)\bigr)\bigr) = \psi\bigl(q\psi\bigl(\xi\bigl(d(x_0,x_1)\bigr)\bigr)\bigr). \tag{2.11}$$

It follows from ψ being a strictly increasing function that

$$0 < \psi\bigl(\xi\bigl(d(x_2,x_3)\bigr)\bigr) < \psi^2\bigl(q\psi\bigl(\xi\bigl(d(x_0,x_1)\bigr)\bigr)\bigr). \tag{2.12}$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1} \in Tx_n$,

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.13}$$

and

$$0 < \xi\bigl(d(x_{n+1}, x_{n+2})\bigr) < \psi^n\bigl(q\psi\bigl(\xi\bigl(d(x_0,x_1)\bigr)\bigr)\bigr) \tag{2.14}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Let $m, n \in \mathbb{N}$ such that $m > n$. By the triangle inequality, we have

$$\xi\big(d(x_m, x_n)\big) \leq \sum_{i=n}^{m-1} \xi\big(d(x_i, x_{i+1})\big) < \sum_{i=n}^{m-1} \psi^{i-1}\big(q\psi\big(\xi\big(d(x_0, x_1)\big)\big)\big).$$

Since $\psi \in \Psi$, we have $\lim_{n,m\to\infty} \xi(d(x_m, x_n)) = 0$. Using ($\xi_1$), we get $\lim_{n,m\to\infty} d(x_m, x_n) = 0$. This implies that $\{x_n\}$ is a Cauchy sequence in (X, d). From (2.13) and the α-completeness of (X, d), there exists $x^* \in X$ such that $x_n \overset{d}{\to} x^*$ as $n \to \infty$.

By α-continuity of the multi-valued mapping T, we get

$$\lim_{n\to\infty} H(Tx_n, Tx) = 0. \tag{2.15}$$

Now we obtain

$$d\big(x^*, Tx^*\big) = \lim_{n\to\infty} d(x_{n+1}, Tx) \leq \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Therefore, $x^* \in Tx^*$ and hence T has a fixed point. This completes the proof. □

Corollary 2.4 *Let (X, d) be a metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X, d) is an α-complete metric space*;
(S_2') *T is an α_*-admissible multi-valued mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
(S_4) *T is an α-continuous multi-valued mapping.*

Then T has a fixed point.

Corollary 2.5 (Theorem 2.5 in [6]) *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Suppose that the following conditions hold*:

(A_1) *T is an α_*-admissible multi-valued mapping*;
(A_2) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
(A_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point.

Next, we give second main result in this work.

Theorem 2.6 *Let (X, d) be a metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X, d) is an α-complete metric space*;
(S_2) *T is an α-admissible multi-valued mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
(S_4') *if $\{x_n\}$ is a sequence in X with $x_n \overset{d}{\to} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, then we have $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof Following the proof of Theorem 2.3, we know that $\{x_n\}$ is a Cauchy sequence in X such that $x_n \xrightarrow{d} x^*$ as $n \to \infty$ and

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.16}$$

for all $n \in \mathbb{N}$. From condition (S'_4), we get

$$\alpha(x_n, x^*) \geq 1 \tag{2.17}$$

for all $n \in \mathbb{N}$. By using the (α, ψ, ξ)-contractive condition of T, we have

$$\begin{aligned} &\xi(H(Tx_n, Tx^*)) \\ &\quad \leq \psi\left(\xi\left(\max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right\}\right)\right) \end{aligned} \tag{2.18}$$

for all $n \in \mathbb{N}$. Suppose that $d(x^*, Tx^*) > 0$. Let $\epsilon := \frac{d(x^*,Tx^*)}{2}$. Since $x_n \xrightarrow{d} x^*$ as $n \to \infty$, we can find $N_1 \in \mathbb{N}$ such that

$$d(x^*, x_n) < \frac{d(x^*, Tx^*)}{2} \tag{2.19}$$

for all $n \geq N_1$. Furthermore, we obtain that

$$d(x^*, Tx_n) \leq d(x^*, x_{n+1}) < \frac{d(x^*, Tx^*)}{2} \tag{2.20}$$

for all $n \geq N_1$. Also, as $\{x_n\}$ is a Cauchy sequence, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) < \frac{d(x^*, Tx^*)}{2} \tag{2.21}$$

for all $n \geq N_2$. It follows from $d(x_n, Tx^*) \to d(x^*, Tx^*)$ as $n \to \infty$ that we can find $N_3 \in \mathbb{N}$ such that

$$d(x_n, Tx^*) < \frac{3d(x^*, Tx^*)}{2} \tag{2.22}$$

for all $n \geq N_3$. Using (2.19)-(2.22), we get

$$\max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right\} = d(x^*, Tx^*) \tag{2.23}$$

for all $n \geq N := \max\{N_1, N_2, N_3\}$. For $n \geq N$, by (2.18) and the triangle inequality, we have

$$\begin{aligned} \xi(d(x^*, Tx^*)) &\leq \xi(d(x^*, x_{n+1})) + \xi(H(Tx_n, Tx^*)) \\ &\leq \xi(d(x^*, x_{n+1})) + \psi\Bigg(\xi\Bigg(\max\Bigg\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \\ &\qquad \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\Bigg\}\Bigg)\Bigg) \\ &= \xi(d(x^*, x_{n+1})) + \psi(\xi(d(x^*, Tx^*))). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\xi\big(d\big(x^*, Tx^*\big)\big) \le \psi\big(\xi\big(d\big(x^*, Tx^*\big)\big)\big).$$

This implies that $\xi(d(x^*, Tx^*)) = 0$, which is a contradiction. Therefore, $d(x^*, Tx^*) = 0$, that is, $x^* \in Tx^*$. This completes the proof. □

Corollary 2.7 *Let (X, d) be a metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X, d) is an α-complete metric space*;
(S_2') *T is an α_*-admissible multi-valued mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$*;
(S_4') *if $\{x_n\}$ is a sequence in X with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, then we have $\alpha(x_n, x) \ge 1$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Corollary 2.8 (Theorem 2.6 in [6]) *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Suppose that the following conditions hold*:

(A_1) *T is an α_*-admissible multi-valued mapping*;
(A_2) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$*;
(A_3') *if $\{x_n\}$ is a sequence in X with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \ge 1$, for all $n \in \mathbb{N}$, then we have $\alpha(x_n, x) \ge 1$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Remark 2.9 Theorems 2.3 and 2.6 generalize many results in the following sense:

- The condition (1.1) is weaker than some kinds of the contractive conditions such as Banach's contractive condition [10], Kannan's contractive condition [11], Chatterjea's contractive condition [12], Nadler's contractive condition [13], *etc.*;
- the condition of being α-admissible of a multi-valued mapping T is weaker than the condition of being α_*-admissible of T;
- for the existence of fixed point, we merely require that α-continuity of T and α-completeness of X, whereas other result demands stronger than these conditions.

Consequently, Theorems 2.3 and 2.6 extend and improve the following results:

- Theorem 2.5 and Theorem 2.6 of Ali *et al.* [6];
- Theorem 2.1 and Theorem 2.2 of Samet *et al.* [4];
- Theorem 2.3 of Asl *et al.* [5];
- Theorem 2.1 and Theorem 2.2 of Amiri *et al.* [14];
- Theorem 2.1 of Salimi *et al.* [15];
- Theorem 3.1 and Theorem 3.4 of Mohammadi *et al.* [7].

Next, we give an example to show that our result is more general than the results of Ali *et al.* [6] and many known results in the literature.

Example 2.10 Let $X = (-10, 10)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Define $T : X \to CL(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} [-9, |x|], & x \in (-10, 0), \\ [0, \frac{x}{4}], & x \in [0, 2], \\ [\frac{x+10}{4}, 9], & x \in (2, 10) \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} e^{xy} \cosh(x + y), & x, y \in [0, 2], \\ \tanh(e^x - e^y), & \text{otherwise.} \end{cases}$$

Clearly, (X, d) is not complete metric space. Therefore, the results of Ali *et al.* [6] are not applicable here.

Next, we show that by Theorem 2.6 can be guaranteed the existence of a fixed point of T. Define functions $\psi, \xi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\xi(t) = \sqrt{t}$, for all $t \in [0, \infty)$. It is easy to see that $\psi \in \Psi$ and $\xi \in \Xi$.

Firstly, we will show that T is a strictly (α, ψ, ξ)-contractive mapping. For $x, y \in X$ and $\alpha(x, y) \geq 1$, we have $x, y \in [0, 2]$ and then

$$\begin{aligned} \xi\bigl(H(Tx, Ty)\bigr) &= \sqrt{\frac{|x - y|}{4}} \\ &= \frac{1}{2}\sqrt{|x - y|} \\ &\leq \frac{1}{2}\sqrt{M(x, y)} \\ &= \psi\bigl(\xi\bigl(M(x, y)\bigr)\bigr). \end{aligned}$$

It is to be observed that ψ is strictly increasing function. Therefore, T is a strictly (α, ψ, ξ)-contractive mapping.

Moreover, it is easy to see that T is an α-admissible multi-valued mapping and there exists $x_0 = 1 \in X$ and $x_1 = 1/4 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha(1, 1/4) \geq 1.$$

Also, T is an α-continuous mapping.

Finally, for each sequence $\{x_n\}$ in X with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$. Thus the condition (S_4') in Theorem 2.6 holds.

Therefore, by using Theorem 2.3 or 2.6, we get T has a fixed point in X. In this case, T has infinitely fixed points such as -2, -1, and 0.

3 Consequences

3.1 Fixed point results in metric spaces endowed with an arbitrary binary relation

It has been pointed out in some studies that some results in metric spaces endowed with an arbitrary binary relation can be concluded from the fixed point results related with α-admissible mappings on metric spaces. In this section, we give some fixed point results on metric spaces endowed with an arbitrary binary relation which can be regarded as consequences of the results presented in the previous section. The following notions and definitions are needed.

Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1},$$

i.e.,

$$x,y \in X, \quad x\mathcal{S}y \quad \iff \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

Definition 3.1 Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. A multi-valued mapping $T : X \to N(X)$ is said to be a *weakly comparative* if for each $x \in X$ and $y \in Tx$ with $x\mathcal{S}y$, we have $y\mathcal{S}z$, for all $z \in Ty$.

Definition 3.2 Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. The metric space X is said to be $\mathcal{S}$-*complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{S}x_{n+1}$, for all $n \in \mathbb{N}$, converges in X.

Definition 3.3 Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. We say that $T : X \to CL(X)$ is a $\mathcal{S}$-*continuous mapping* to $(CL(X),H)$ if for given $x \in X$ and sequence $\{x_n\}$ with

$$\lim_{n\to\infty} d(x_n,x) = 0 \quad \text{and} \quad x_n\mathcal{S}x_{n+1} \quad \text{for all } n \in \mathbb{N} \quad \Longrightarrow \quad \lim_{n\to\infty} H(Tx_n,Tx) = 0.$$

Definition 3.4 Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. A mapping $T : X \to CL(X)$ is called an $(\mathcal{S},\psi,\xi)$-contractive mapping if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that

$$x,y \in X, \quad x\mathcal{S}y \quad \Longrightarrow \quad \xi\bigl(H(Tx,Ty)\bigr) \le \psi\bigl(\xi\bigl(M(x,y)\bigr)\bigr), \tag{3.1}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

In the case when $\psi \in \Psi$ is strictly increasing, the $(\mathcal{S},\psi,\xi)$-contractive mapping is called a strictly $(\mathcal{S},\psi,\xi)$-contractive mapping.

Theorem 3.5 *Let (X,d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CL(X)$ be a strictly $(\mathcal{S},\psi,\xi)$-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X,d) is an $\mathcal{S}$-complete metric space*;
(S_2) *T is a weakly comparative mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0\mathcal{S}x_1$*;
(S_4) *T is a $\mathcal{S}$-continuous multi-valued mapping.*

Then T has a fixed point.

Proof This result can be obtain from Theorem 2.3 by define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in x\mathcal{S}y, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. $\square$

By using Theorem 2.6, we get the following result.

Theorem 3.6 *Let (X,d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CL(X)$ be a strictly $(\mathcal{S},\psi,\xi)$-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X,d) is an $\mathcal{S}$-complete metric space*;
(S_2) *T is a weakly comparative mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0\mathcal{S}x_1$*;
(S'_4) *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $x_n\mathcal{S}x_{n+1}$, for all $n \in \mathbb{N}$, then we have $x_n\mathcal{S}x$, for all $n \in \mathbb{N}$.*

Then T has a fixed point.

3.2 Fixed point results in metric spaces endowed with graph

In 2008, Jachymski [16] obtained a generalization of Banach's contraction principle for mappings on a metric space endowed with a graph. Afterwards, Dinevari and Frigon [17] extended some results of Jachymski [16] to multi-valued mappings. For more fixed point results on a metric space with a graph, one can refer to [18–20].

In this section, we give fixed point results on a metric space endowed with a graph. Before presenting our results, we give the following notions and definitions.

Throughout this section, let (X,d) be a metric space. A set $\{(x,x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, *i.e.*, $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 3.7 Let X be a nonempty set endowed with a graph G and $T : X \to N(X)$ be a multi-valued mapping, where X is a nonempty set X. We say that T *weakly preserves edges* if for each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$, for all $z \in Ty$.

Definition 3.8 Let (X,d) be a metric space endowed with a graph G. The metric space X is said to be $E(G)$*-complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n,x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, converges in X.

Definition 3.9 Let (X,d) be a metric space endowed with a graph G. We say that $T : X \to CL(X)$ is an $E(G)$*-continuous mapping* to $(CL(X),H)$ if for given $x \in X$ and sequence $\{x_n\}$ with

$$\lim_{n\to\infty} d(x_n,x) = 0 \quad \text{and} \quad (x_n,x_{n+1}) \in E(G) \quad \text{for all } n \in \mathbb{N} \quad \Longrightarrow \quad \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Definition 3.10 Let (X,d) be a metric space endowed with a graph G. A mapping $T : X \to CL(X)$ is called an $(E(G),\psi,\xi)$-contractive mapping if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that

$$x,y \in X, \quad (x,y) \in E(G) \quad \Longrightarrow \quad \xi\big(H(Tx,Ty)\big) \le \psi\big(\xi\big(M(x,y)\big)\big), \tag{3.2}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

In the case when $\psi \in \Psi$ is strictly increasing, the $(E(G), \psi, \xi)$-contractive mapping is called a strictly $(E(G), \psi, \xi)$-contractive mapping.

Theorem 3.11 *Let (X, d) be a metric space endowed with a graph G, and $T : X \to CL(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X, d) is an $E(G)$-complete metric space*;
(S_2) *T weakly preserves edges*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$*;
(S_4) *T is an $E(G)$-continuous multi-valued mapping.*

Then T has a fixed point.

Proof This result can be obtained from Theorem 2.3 by defining a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. □

By using Theorem 2.6, we get the following result.

Theorem 3.12 *Let (X, d) be a metric space endowed with a graph G and $T : X \to CL(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Suppose that the following conditions hold*:

(S_1) *(X, d) is an $E(G)$-complete metric space*;
(S_2) *T weakly preserves edges*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$*;
(S_4') *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Remark 3.13

1. If we assume G is such that $E(G) := X \times X$, then clearly G is connected and our Theorems 3.11 and 3.12 improve Nadler's contraction principle [13] and in the case of a single-valued mapping, we improve Banach's contraction principle [10], Kannan's contraction theorem [11], Chatterjea's contraction theorem [12], and Bianchini and Grandolfi's fixed point theorem.
2. Theorems 3.11 and 3.12 are partial some generalized fixed point results endowed with a graph of Jachymski [16] and Dinevari and Frigon [17].
3. Theorems 3.11 and 3.12 are generalizations of fixed point results of Theorem 2.5 and Theorem 2.6 of Ali *et al.* [6] in a graph version.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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