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Some convergence results for multivalued quasi-nonexpansive mappings in CAT(κ) spaces

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Abstract

In this paper, we prove some strong and Δ-convergence theorems for a finite family of multivalued quasi-nonexpansive mappings satisfying condition (E) in CAT(κ) spaces. Our results extend the corresponding results of Abkar and Eslamian (Nonlinear Anal. 75:1895-1903, 2012), Panyanak (Fixed Point Theory Appl. 2014:1, 2014), Shahzad and Zegeye (Nonlinear Anal. 71:838-844, 2009) and many others.

MSC: 47H10; 47J25

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1 Introduction

Fixed point theory for multivalued contractions and nonexpansive mappings using the Hausdorff metric was first studied by Markin [1] and Nadler [2]. Since then different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings. Sastry and Babu [3] defined Mann and Ishikawa iterates for a multivalued map T in a Hilbert space. Panyanak [4] and Song and Wang [5] generalized the results of Sastry and Babu [3] to uniformly convex Banach spaces. Later, Shahzad and Zegeye [6] defined two types of Ishikawa iteration processes and extended the results of [3–5]. The reader may consult [7] for more detail. Recently, Abkar and Eslamian [8] established strong and Δ-convergence theorems for the following iterative process for a finite family of multivalued quasi-nonexpansive mappings satisfying condition (E) in CAT(0) spaces:

$$\begin{cases} y_{n,1} = (1-\alpha_{n,1})x_n \oplus \alpha_{n,1}z_{n,1}, \\ y_{n,2} = (1-\alpha_{n,2})x_n \oplus \alpha_{n,2}z_{n,2}, \\ \dots \\ y_{n,m-1} = (1-\alpha_{n,m-1})x_n \oplus \alpha_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1-\alpha_{n,m})x_n \oplus \alpha_{n,m}z_{n,m}, \quad n \geq 1, \end{cases} \tag{1}$$

where $z_{n,1} \in T_1(x_n)$ and $z_{n,k} \in T_k(y_{n,k-1})$ for $k = 2,\dots,m$. It is easy to see that if $m = 2$ and $T_1 = T_2 = T$, then the sequence $\{x_n\}$ defined by (1) is the Ishikawa iteration:

$$\begin{cases} y_n = (1-\alpha_{n,1})x_n \oplus \alpha_{n,1}z_n, \\ x_{n+1} = (1-\alpha_{n,2})x_n \oplus \alpha_{n,2}z'_n, \quad n \geq 1, \end{cases}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$.

The purpose of the paper is to extend and improve the corresponding results of Abkar and Eslamian [8] to the general setting of CAT(κ) spaces, which are geodesic spaces of bounded curvature, where $\kappa \in \mathbb{R}$ is the curvature bound. For example, the n-dimensional hyperbolic space $\mathbb{H}^n$ is a CAT(−1) space and the n-dimensional unit sphere $\mathbb{S}^n$ is a CAT(1) space (see Section 2 for details). It is worth mentioning that any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$. Thus all results for CAT(κ) spaces with $\kappa > 0$ immediately apply to any CAT(0) space.

Let D be a subset of a metric space (X, d). Recall that an element $p \in D$ is called a *fixed point* of a single-valued mapping T if $p = Tp$ and of a multivalued mapping T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$. D is said to be *proximinal* if, for each $x \in X$, there exists an element $x^* \in D$ such that

$$d(x, D) = \inf\{d(x, y) : y \in D\} = d(x, x^*).$$

It is evident that every proximinal set is closed and every compact set is proximinal (see [9]).

Let 2^D be a family of nonempty subsets of D. We denote by $\mathcal{C}(D)$, $\mathcal{P}(D)$ and $\mathcal{K}(D)$ the families of nonempty closed subsets, nonempty proximinal subsets and nonempty compact subsets of D, respectively. The *Hausdorff metric* on $\mathcal{K}(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for all $A, B \in \mathcal{K}(D)$, where $d(x, B) = \inf\{d(x, z) : z \in B\}$.

Definition 1 A multivalued mapping $T : D \to 2^D$ is said to

(i) be *nonexpansive* if, for all $x, y \in D$,

$$H(Tx, Ty) \leq d(x, y);$$

(ii) be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq d(x, p), \quad \forall p \in F(T), x \in D;$$

(iii) satisfy *condition* (E_μ) provided that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y), \quad x, y \in D \text{ and } \mu \geq 1.$$

We say that T satisfies *condition* (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 1 There exist multivalued quasi-nonexpansive mappings satisfying condition (E). For example, define a mapping $T : [0, 5] \to [0, 5]$ by

$$Tx = \begin{cases} [0, \frac{x}{5}], & x \neq 5, \\ \{1\}, & x = 5. \end{cases}$$

Let $x, y \in [0, 5)$, then we get

$$H(Tx, Ty) = \left| \frac{x - y}{5} \right| \leq d(x, y).$$

If $x \in [0, 4]$ and $y = 5$, then

$$H(Tx, Ty) = 1 \leq 5 - x = d(x, y).$$

If $x \in (4, 5)$ and $y = 5$, we have

$$d(x, Tx) = \frac{4x}{5}, \qquad d(x, y) = 5 - x, \qquad H(Tx, Ty) = 1 \quad \text{and} \quad d(x, Ty) = x - 1.$$

Then it is easy to prove that T has the required properties.

In 1991, Xu [10] introduced the best approximation operator P_T to find fixed points of *-nonexpansive multivalued mappings. In 2013, Dehghan [11] obtained the demiclosed principle of such mappings and approximated their fixed points using P_T. Let $P_T : D \to 2^D$ be a multivalued mapping defined by

$$P_T(x) = \{u \in Tx : d(x, u) = d(x, Tx)\}.$$

By [12] we have the following lemma.

Lemma 1 [12] *Let D be a nonempty subset of a metric space (X, d) and $T : D \to \mathcal{P}(D)$ be a multivalued mapping. Then*

(i) $d(x, Tx) = d(x, P_T(x))$ *for all* $x \in D$;
(ii) $x \in F(T) \Leftrightarrow x \in F(P_T) \Leftrightarrow P_T(x) = \{x\}$;
(iii) $F(T) = F(P_T)$.

2 Preliminaries

The study of fixed points in CAT(κ) spaces was initiated by Kirk [13, 14]. A few recent new convergence results of classical iterations on CAT(κ) spaces have been obtained (see, *e.g.*, [15–19] and the references therein). For example, Panyanak [19] in 2014 proved the strong convergence of two types of Ishikawa iteration processes introduced in Shahzad and Zegeye [6] for some multivalued quasi-nonexpansive mappings in CAT(1) spaces.

Let (X, d) be a metric space and $x, y \in X$ with $l = d(x, y)$. For $x, y \in X$, a *geodesic path* joining x to y is an isometry $c : [0, l] \to X$ such that $c(0) = x$, $c(l) = y$. The image of a geodesic path is called a *geodesic segment*, and we shall denote a definite choice of this geodesic segment by $[x, y]$. A metric space X is a *geodesic space* (*r-geodesic space*) if every two points of X (every two points with distance smaller than r) are joined by a geodesic segment, and X is a *uniquely geodesic space* (*r-uniquely geodesic space*) if there is exactly one geodesic segment joining x and y for any $x, y \in X$ (for any $x, y \in X$ with $d(x, y) < r$). A subset D of X is said to be *convex* if D includes every geodesic segment joining any two of its points.

The n-dimensional sphere $\mathbb{S}^n$ is the set $\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \langle x|x\rangle = 1\}$, where $\langle\cdot|\cdot\rangle$ is the Euclidean scalar product. It is endowed with the following metric: $d_{\mathbb{S}^n}(x,y) = \arccos\langle x|y\rangle$, $x, y \in \mathbb{S}^n$.

Definition 2 Given $\kappa \in \mathbb{R}$, denote by M_κ^n the following metric spaces:
(i) if $\kappa = 0$, then M_0^n is the Euclidean space $\mathbb{R}^n$;
(ii) if $\kappa > 0$, then M_κ^n is obtained from the sphere $\mathbb{S}^n$ by multiplying the distance function by $1/\sqrt{\kappa}$;
(iii) if $\kappa < 0$, then M_κ^n is obtained from the hyperbolic n-space $\mathbb{H}^n$ by multiplying the distance function by $1/\sqrt{-\kappa}$.

A *geodesic triangle* $\triangle(x,y,z)$ in a geodesic space (X,d) consists of three points x, y, z of X and three geodesic segments joining each pair of vertices. A *comparison triangle* of a geodesic triangle $\triangle(x,y,z)$ is the triangle $\overline{\triangle}(\bar{x},\bar{y},\bar{z})$ in M_κ^2 such that

$$d(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y}), \qquad d(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z}), \qquad d(z,x) = d_{M_\kappa^2}(\bar{z},\bar{x}).$$

If $\kappa > 0$, then such a triangle $\overline{\triangle}$ always exists whenever $d(x,y) + d(y,z) + d(z,x)$ is less than $2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$. A geodesic triangle in X is said to satisfy the $\mathrm{CAT}(\kappa)$ *inequality* if for any $p, q \in \triangle(x,y,z)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x},\bar{y},\bar{z})$, we have

$$d(p,q) \le d_{M_\kappa^2}(\bar{p},\bar{q}).$$

Definition 3 Given $\kappa > 0$, a metric space X is a $\mathrm{CAT}(\kappa)$ *space* if X is D_κ-geodesic and any geodesic triangle $\triangle(x,y,z)$ in X with $d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ satisfies the $\mathrm{CAT}(\kappa)$ inequality.

In 1976, Lim [20] introduced the concept of $\triangle$-convergence in a general metric space. Let $\{x_n\}$ be a bounded sequence in a $\mathrm{CAT}(\kappa)$ space X. For $x \in X$, we define

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ *of* $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}.$$

The *asymptotic center* $A(\{x_n\})$ *of* $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(x, \{x_n\}\bigr) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

A sequence $\{x_n\}$ in a $\mathrm{CAT}(\kappa)$ space X is said to $\triangle$*-converge to* $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

It follows from [21] that $\mathrm{CAT}(\kappa)$ spaces are uniquely geodesic spaces. In this paper, we mainly focus on $\mathrm{CAT}(\kappa)$ spaces with $\kappa > 0$, and we now collect some elementary facts about them.

Lemma 2 [15] *Let $\kappa > 0$ and (X,d) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) =: \sup\{d(u,v) : u,v \in X\} < \frac{\pi}{2\sqrt{\kappa}}$. *Then $A(\{x_n\})$ consists of exactly one point.*

Lemma 3 [15] *Let $\kappa > 0$ and (X,d) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then every sequence in X has a $\triangle$-convergent subsequence.*

Lemma 4 [15] *Let $\kappa > 0$ and (X,d) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. D is a closed convex subset of X. If $\{x_n\} \subseteq D$ and $\triangle\text{-}\lim_{n\to\infty} x_n = x$, then $x \in D$.*

Since the asymptotic center is unique by Lemma 2, we can obtain the following lemma.

Lemma 5 [22] *Let $\kappa > 0$ and (X,d) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let $\{x_n\}$ be a sequence in X with $A(\{x_n\}) = \{x\}$. If $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $\{d(x_n,u)\}$ converges, then $x = u$.*

Lemma 6 [21] *Let $\kappa > 0$ and (X,d) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then, for any $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d\bigl((1-t)x \oplus ty, z\bigr) \le (1-t)d(x,z) + td(y,z).$$

Lemma 7 [23] *Let $\kappa > 0$ and (X,d) be a* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Then, for any $x,y,z \in X$ and $t \in [0,1]$, we have*

$$d^2\bigl((1-t)x \oplus ty, z\bigr) \le (1-t)d^2(x,z) + td^2(y,z) - \frac{R}{2}t(1-t)d^2(x,y),$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.

3 Main results

In this section, we prove our main theorems.

Theorem 1 (Demiclosed principle) *Let $\kappa > 0$ and (X,d) be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0,\pi/2)$. Let D be a nonempty closed convex subset of X, and let $T : D \to \mathcal{K}(D)$ be a multivalued mapping satisfying condition (E). If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle\text{-}\lim_{n\to\infty} x_n = x$, then $x \in Tx$, from which we may formally say that $I - T$ is demiclosed at zero.*

Proof Since $\triangle\text{-}\lim_{n\to\infty} x_n = x$, by Lemma 4 we have $x \in D$. For each $n \ge 1$, we choose $z_n \in Tx$ such that

$$d(x_n, z_n) = d(x_n, Tx).$$

By the compactness of Tx, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\to\infty} z_{n_k} = w \in Tx$. It follows from condition (E) that

$$d(x_{n_k}, z_{n_k}) = d(x_{n_k}, Tx) \le \mu d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x)$$

for some $\mu \geq 1$. Note that

$$d(x_{n_k}, w) \leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, w) \leq \mu d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x) + d(z_{n_k}, w).$$

Thus

$$\limsup_{k\to\infty} d(x_{n_k}, w) \leq \limsup_{k\to\infty} d(x_{n_k}, x).$$

By the uniqueness of asymptotic centers, we obtain $x = w \in Tx$. The proof is completed. □

Theorem 2 *Let* $\kappa > 0$ *and* (X, d) *be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0, \pi/2)$. *Let D be a nonempty closed convex subset of X, and let* $T_i : D \to \mathcal{K}(D)$ $(i = 1, \dots, m)$ *be a family of multivalued quasi-nonexpansive mappings satisfying condition* (E). *Suppose that* $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ *and* $T_i(p) = \{p\}$ *for each* $p \in \mathcal{F}$. *Let* $\alpha_{n,i} \in [a, b] \subset (0, 1)$ $(i = 1, \dots, m)$. *Then* $\{x_n\}$ *defined by* (1) $\triangle$*-converges to some point in* $\mathcal{F}$.

Proof We divide our proof into several steps.

Step 1. In the sequel, we shall show that $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in \mathcal{F}$. Since T_1 is quasi-nonexpansive, by Lemma 6 we have

$$\begin{aligned} d(y_{n,1}, p) &= d\big((1-\alpha_{n,1})x_n \oplus \alpha_{n,1}z_{n,1}, p\big) \\ &\leq (1-\alpha_{n,1})d(x_n, p) + \alpha_{n,1}d(z_{n,1}, p) \\ &= (1-\alpha_{n,1})d(x_n, p) + \alpha_{n,1}d\big(z_{n,1}, T_1(p)\big) \\ &\leq (1-\alpha_{n,1})d(x_n, p) + \alpha_{n,1}H\big(T_1(x_n), T_1(p)\big) \\ &\leq (1-\alpha_{n,1})d(x_n, p) + \alpha_{n,1}d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(y_{n,2}, p) &= d\big((1-\alpha_{n,2})x_n \oplus \alpha_{n,2}z_{n,2}, p\big) \\ &\leq (1-\alpha_{n,2})d(x_n, p) + \alpha_{n,2}d(z_{n,2}, p) \\ &= (1-\alpha_{n,2})d(x_n, p) + \alpha_{n,2}d\big(z_{n,2}, T_2(p)\big) \\ &\leq (1-\alpha_{n,2})d(x_n, p) + \alpha_{n,2}H\big(T_2(y_{n,1}), T_2(p)\big) \\ &\leq (1-\alpha_{n,2})d(x_n, p) + \alpha_{n,2}d(y_{n,1}, p) \\ &\leq d(x_n, p). \end{aligned}$$

By continuing this process we have

$$d(x_{n+1}, p) \leq d(x_n, p).$$

It implies that $d(x_n, p)$ is decreasing and bounded below, thus $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in \mathcal{F}$.

Step 2. We shall show that $\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0$ for $i = 1, \dots, m$. In fact, by Lemma 7 we obtain

$$\begin{aligned}
d^2(y_{n,1},p) &= d^2\big((1-\alpha_{n,1})x_n \oplus \alpha_{n,1}z_{n,1},p\big) \\
&\le (1-\alpha_{n,1})d^2(x_n,p) + \alpha_{n,1}d^2(z_{n,1},p) - \frac{R}{2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}) \\
&= (1-\alpha_{n,1})d^2(x_n,p) + \alpha_{n,1}d^2\big(z_{n,1},T_1(p)\big) - \frac{R}{2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}) \\
&\le (1-\alpha_{n,1})d^2(x_n,p) + \alpha_{n,1}H^2\big(T_1(x_n),T_1(p)\big) - \frac{R}{2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}) \\
&\le (1-\alpha_{n,1})d^2(x_n,p) + \alpha_{n,1}d^2(x_n,p) - \frac{R}{2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}) \\
&= d^2(x_n,p) - \frac{R}{2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1})
\end{aligned}$$

and

$$\begin{aligned}
d^2(y_{n,2},p) &= d^2\big((1-\alpha_{n,2})x_n \oplus \alpha_{n,2}z_{n,2},p\big) \\
&\le (1-\alpha_{n,2})d^2(x_n,p) + \alpha_{n,2}d^2(z_{n,2},p) - \frac{R}{2}\alpha_{n,2}(1-\alpha_{n,2})d^2(x_n,z_{n,2}) \\
&= (1-\alpha_{n,2})d^2(x_n,p) + \alpha_{n,2}d^2\big(z_{n,2},T_2(p)\big) - \frac{R}{2}\alpha_{n,2}(1-\alpha_{n,2})d^2(x_n,z_{n,2}) \\
&\le (1-\alpha_{n,2})d^2(x_n,p) + \alpha_{n,2}H^2\big(T_2(y_{n,1}),T_2(p)\big) - \frac{R}{2}\alpha_{n,2}(1-\alpha_{n,2})d^2(x_n,z_{n,2}) \\
&\le (1-\alpha_{n,2})d^2(x_n,p) + \alpha_{n,2}d^2(y_{n,1},p) - \frac{R}{2}\alpha_{n,2}(1-\alpha_{n,2})d^2(x_n,z_{n,2}) \\
&\le d^2(x_n,p) - \frac{R}{2}\alpha_{n,2}\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}) - \frac{R}{2}\alpha_{n,2}(1-\alpha_{n,2})d^2(x_n,z_{n,2}).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2\big((1-\alpha_{n,m})x_n \oplus \alpha_{n,m}z_{n,m},p\big) \\
&\le (1-\alpha_{n,m})d^2(x_n,p) + \alpha_{n,m}d^2(z_{n,m},p) - \frac{R}{2}\alpha_{n,m}(1-\alpha_{n,m})d^2(x_n,z_{n,m}) \\
&= (1-\alpha_{n,m})d^2(x_n,p) + \alpha_{n,m}d^2\big(z_{n,m},T_2(p)\big) - \frac{R}{2}\alpha_{n,m}(1-\alpha_{n,m})d^2(x_n,z_{n,m}) \\
&\le (1-\alpha_{n,m})d^2(x_n,p) + \alpha_{n,m}H^2\big(T_m(y_{n,m-1}),T_m(p)\big) \\
&\quad - \frac{R}{2}\alpha_{n,m}(1-\alpha_{n,m})d^2(x_n,z_{n,m}) \\
&\le (1-\alpha_{n,m})d^2(x_n,p) + \alpha_{n,m}d^2(y_{n,m-1},p) - \frac{R}{2}\alpha_{n,m}(1-\alpha_{n,m})d^2(x_n,z_{n,m}) \\
&\le d^2(x_n,p) - \frac{R}{2}\alpha_{n,m}(1-\alpha_{n,m})d^2(x_n,z_{n,m}) \\
&\quad - \frac{R}{2}\alpha_{n,m}\alpha_{n,m-1}(1-\alpha_{n,m-1})d^2(x_n,z_{n,m-1}) - \cdots \\
&\quad - \frac{R}{2}\alpha_{n,m}\alpha_{n,m-1}\cdots\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1}).
\end{aligned}$$

Then we have

$$\begin{aligned}\frac{R}{2}a^m(1-b)d^2(x_n,z_{n,1}) &\le \frac{R}{2}\alpha_{n,m}\alpha_{n,m-1}\cdots\alpha_{n,1}(1-\alpha_{n,1})d^2(x_n,z_{n,1})\\ &\le d^2(x_n,p)-d^2(x_{n+1},p),\end{aligned}$$

which yields that

$$\sum_{n=1}^{\infty}\frac{R}{2}a^m(1-b)d^2(x_n,z_{n,1}) \le d^2(x_1,p) < \infty,$$

and hence

$$\lim_{n\to\infty} d(x_n,z_{n,1}) = 0.$$

Similarly, we can also have

$$\lim_{n\to\infty} d(x_n,z_{n,k}) = 0 \quad (k=2,\ldots,m).$$

Thus we obtain

$$\lim_{n\to\infty} d\bigl(x_n, T_1(x_n)\bigr) \le \lim_{n\to\infty} d(x_n,z_{n,1}) = 0, \tag{2}$$

$$\lim_{n\to\infty} d\bigl(x_n, T_k(y_{n,k-1})\bigr) \le \lim_{n\to\infty} d(x_n,z_{n,k}) = 0 \tag{3}$$

and

$$\lim_{n\to\infty} d(x_n,y_{n,k-1}) = \alpha_{n,k-1}\lim_{n\to\infty} d(x_n,z_{n,k-1}) = 0 \tag{4}$$

for $k=2,\ldots,m$. Now, by condition (E), (3) and (4), we have, for some $\mu \ge 1$,

$$\begin{aligned}d\bigl(x_n,T_k(x_n)\bigr) &\le d(x_n,y_{n,k-1}) + d\bigl(y_{n,k-1},T_k(x_n)\bigr)\\ &\le d(x_n,y_{n,k-1}) + \mu d\bigl(y_{n,k-1},T_k(y_{n,k-1})\bigr) + d(x_n,y_{n,k-1})\\ &\le d(x_n,y_{n,k-1}) + \mu d(y_{n,k-1},x_n) + \mu d\bigl(x_n,T_k(y_{n,k-1})\bigr)\\ &\quad + d(x_n,y_{n,k-1}) \to 0\end{aligned} \tag{5}$$

as $n\to\infty$ (for $k=2,\ldots,m$). By (2) and (5) we have

$$\lim_{n\to\infty} d\bigl(x_n,T_i(x_n)\bigr) = 0$$

for $i=1,\ldots,m$.

Step 3. Now we are in a position to prove the $\triangle$-convergence of $\{x_n\}$. In fact, let $W_\omega(x_n) := \cup A(\{u_n\})$ for all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_\omega(x_n)\subset\mathcal{F}$. Let $u\in W_\omega(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Lemma 3 and Lemma 4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v\in D$. Since $\lim_{n\to\infty} d(v_n,T_iv_n)=0$ $(i=1,\ldots,m)$, it follows from Theorem 1 that $v\in\mathcal{F}$

and thus $\lim_{n\to\infty} d(x_n, v)$ exists by Step 1. By Lemma 5, $u = v \in \mathcal{F}$, which implies that $W_\omega(x_n) \subset \mathcal{F}$. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and let $A(\{x_n\}) = \{x\}$. Since $u \in W_\omega(x_n) \subset \mathcal{F}$ and $\lim_{n\to\infty} d(x_n, u)$ converges, we get $x = u$ by Lemma 5. It implies that $W_\omega(x_n)$ consists of exactly one point. The proof is completed. □

Remark 2 Theorem 2 improves and extends the corresponding results in Abkar and Eslamian [8, Theorem 3.6].

In the sequel, we make use of *condition* (A) introduced by Senter and Dotson [24]. A mapping $T : D \to D$, where D is a subset of a normed space E, is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r > 0$ such that

$$\|x - Tx\| \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr) \quad \text{for all } x \in D.$$

Theorem 3 *Let* $\kappa > 0$ *and* (X,d) *be a complete* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ *for some* $\varepsilon \in (0,\pi/2)$. *Let* D *be a nonempty closed convex subset of* X, *and let* $T_i : D \to \mathcal{C}(D)$ $(i = 1,\dots,m)$ *be a family of multivalued quasi-nonexpansive mappings satisfying condition* (E). *Suppose that* $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ *and* $T_i p = \{p\}$ *for each* $p \in \mathcal{F}$. *Let* $\alpha_{n,i} \in [a,b] \subset (0,1)$ $(i = 1,\dots,m)$. *Assume that there is a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$, $f(r) > 0$ *for all* $r > 0$ *such that for some* $i = 1,\dots,m$,

$$d\bigl(x_n, T_i(x_n)\bigr) \geq f\bigl(d(x_n,\mathcal{F})\bigr). \tag{6}$$

Then $\{x_n\}$ *defined by* (1) *converges strongly to some point in* $\mathcal{F}$.

Proof As in the proof of Theorem 2, for $i = 1,\dots,m$, we have $\lim_{n\to\infty} d(x_n, T_i(x_n)) = 0$. Hence by assumption (6) we obtain $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$. Now we can choose a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a subsequence $\{p_k\} \subset \mathcal{F}$ such that for all positive integer $k \geq 1$,

$$d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Since for each $p \in \mathcal{F}$ the sequence $\{d(x_n, p)\}$ is decreasing, we get

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Hence

$$d(p_{k+1}, p_k) \leq d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Then $\{p_k\}$ is a Cauchy sequence in D. Without loss of generality, we can assume that $p_k \to p^* \in D$. Since for each $i = 1,\dots,m$

$$d\bigl(p^*, T_i\bigl(p^*\bigr)\bigr) = \lim_{n\to\infty} d\bigl(p_k, T_i\bigl(p^*\bigr)\bigr) \leq \lim_{n\to\infty} H\bigl(T_i(p_k), T_i\bigl(p^*\bigr)\bigr) \leq \lim_{k\to\infty} d\bigl(p_k, p^*\bigr) = 0,$$

then $p^* \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to p^*. Since $\lim_{n\to\infty} d(x_n, p^*)$ exists, it follows that $\{x_n\}$ converges strongly to p^*. The proof is completed. □

Remark 3 Theorem 3 improves and extends the corresponding results in Abkar and Eslamian [8, Theorem 3.9] and Panyanak [19, Theorem 3.2].

Theorem 4 *Let $\kappa > 0$ and (X,d) be a complete $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0,\pi/2)$. D is a nonempty closed convex subset of X. Let $T_i : D \to \mathcal{P}(D)$ $(i = 1,\ldots,m)$ be a family of multivalued mappings with $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ such that P_{T_i} is quasi-nonexpansive satisfying condition (E). For $x_1 \in D$, define the sequence $\{x_n\} \subset D$ as follows*:

$$\begin{cases} y_{n,1} = (1-\beta_{n,1})x_n \oplus \beta_{n,1}z_{n,1}, \\ y_{n,2} = (1-\beta_{n,2})x_n \oplus \beta_{n,2}z_{n,2}, \\ \cdots \\ y_{n,m-1} = (1-\beta_{n,m-1})x_n \oplus \beta_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1-\beta_{n,m})x_n \oplus \beta_{n,m}z_{n,m}, \quad n \geq 1, \end{cases} \tag{7}$$

where $z_{n,1} \in P_{T_1}(x_n)$, $z_{n,k} \in P_{T_k}(y_{n,k-1})$ $(k = 2,\ldots,m)$ and $\beta_{n,i} \in [a,b] \subset (0,1)$ $(i = 1,\ldots,m)$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r > 0$ such that for some $i = 1,\ldots,m$,

$$d\bigl(x_n, T_i(x_n)\bigr) \geq f\bigl(d(x_n,\mathcal{F})\bigr). \tag{8}$$

Then $\{x_n\}$ defined by (7) *converges strongly to some point in $\mathcal{F}$.*

Proof It follows from Lemma 1 and (8) that

$$d\bigl(x_n, P_{T_i}(x_n)\bigr) = d\bigl(x_n, T_i(x_n)\bigr) \geq f\bigl(d(x_n,\mathcal{F})\bigr) = f\left(d\left(x_n, \bigcap_{i=1}^m F(P_{T_i})\right)\right)$$

for some $i = 1,\ldots,m$. Next we show that $P_{T_i}(x)$ is closed for any $i = 1,\ldots,m$ and $x \in D$. In fact, let $\{y_n\} \subset P_{T_i}(x)$ and $\lim_{n\to\infty} y_n = y$ for some $y \in D$. Then

$$d(x,y_n) = d\bigl(x, T_i(x)\bigr) \quad \text{and} \quad \lim_{n\to\infty} d(x,y_n) = d(x,y).$$

It follows that $d(x,y) = d(x,T_i(x))$ and hence $y \in P_{T_i}(x)$. Now applying Theorem 3 to the mappings P_{T_i}, we conclude that the sequence $\{x_n\}$ defined by (7) converges strongly to some point in $\mathcal{F}$. The proof is completed. □

Remark 4 Theorem 4 improves and extends the corresponding results in Abkar and Eslamian [8, Theorem 3.12] and Panyanak [19, Theorem 3.4].

Competing interests

The author declares that they have no competing interests.

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