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Common fixed points of Caristi's type mappings via w-distance

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Abstract

Motivated by fixed point theorems of Obama and Kuroiwa (Sci. Math. Jpn. 72(1), 41-48, 2010), we discuss their results with another, weaker assumption and obtain some estimating expressions. Furthermore, we also discuss the results with the lower semicontinuities of the dominated functions in place of the original orbital continuities of the mappings.

1 Introduction

In 1981, Bhakta and Basu [1] proved the following common fixed point theorem.

Theorem BB *Let (X,d) be a complete metric space and $S,T:X\to X$ be two given mappings. Let $\varphi,\psi:X\to[0,\infty)$ be any functions. Suppose that the following conditions are satisfied*:

(BB) $d(Sx,Ty)\leq\varphi(x)-\varphi(Sx)+\psi(y)-\psi(Ty)$, *for all* $x,y\in X$.

(oc) *S and T are orbitally continuous.*

Then S and T have a unique common fixed point.

Recall that $T:X\to X$ is *orbitally continuous* if for every $x,w\in X$, the following implication holds:

$$T^{n_j}x\to w \quad\Longrightarrow\quad T\bigl(T^{n_j}x\bigr)\to Tw.$$

Let (X,d) be a metric space. A function $p:X\times X\to[0,\infty)$ is called a *w-distance* on X if the following conditions are satisfied:

(w1) $p(x,y)\leq p(x,z)+p(z,y)$, for all $x,y,z\in X$.

(w2) For each $x\in X$, the function $y\mapsto p(x,y)$ is lower semicontinuous.

(w3) For each $\varepsilon>0$ there exists $\delta>0$ such that $d(y,z)\leq\varepsilon$ whenever $p(x,y)\leq\delta$ and $p(x,z)\leq\delta$.

Using this notion, the following interesting result was proved.

Theorem KST ([2]) *Let (X,d) be a complete metric space and p be a w-distance on X. Let $T:X\to X$ be a mapping. Let $\psi:X\to[0,\infty)$ be any function. Suppose that the following conditions hold*:

(KST) *$p(x, Tx) \le \psi(x) - \psi(Tx)$, for all $x \in X$.*

(lsc) *ψ is lower semicontinuous.*

Then there exists $u \in X$ such that $u = Tu$ and $p(u,u) = 0$.

It is clear that Theorem KST includes Caristi's theorem as a special case.

Obama and Kuroiwa [3] used the concept of w-distance to prove the following two theorems. The first one is a generalization of Theorem BB.

Theorem OK1 *Let (X,d) be a complete metric space and p be a w-distance on X. Let $S, T : X \to X$ be two given mappings. Let $\varphi, \psi : X \to [0,\infty)$ be any functions. Suppose that the following conditions are satisfied*:

(OK1) *$\max\{p(Sx, Ty), p(Ty, Sx)\} \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, for all $x, y \in X$.*

(oc) *S and T are orbitally continuous.*

Then S and T have a unique common fixed point.

Theorem OK2 *Let (X,d) be a complete metric space and p be a w-distance on X. Let $S, T : X \to X$ be two given mappings. Let $\varphi, \psi : X \to [0,\infty)$ be any functions. Suppose that the following conditions are satisfied*:

(OK2) *$\max\{p(x,y), p(y,x)\} + p(x, Sx) + p(y, Ty) \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, for all $x, y \in X$.*

(oc) *S and T are orbitally continuous.*

Then S and T have a unique common fixed point.

The main tool of our result via the w-distance is based on the following lemma.

Lemma KST ([2]) *Let (X,d) be a complete metric space and p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x, y, z \in X$. Then the following statements hold*:

(1) *If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$, for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.*

(2) *If $p(x_n, x_m) \le \alpha_n$, for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

2 Discussions on Theorems OK1 and OK2

In the appearance of the condition (OK1) or (SS) of Lemma 1 below, the problem of finding a common fixed point of S and T reduces to that of finding a fixed point of each mapping individually.

Lemma 1 *Let (X,d) be a metric space and p be a w-distance on X. Let $S, T : X \to X$ be two given mappings. Let $\varphi, \psi : X \to [0,\infty)$ be any functions. Let $x_0, y_0 \in X$. Suppose that one of the following conditions is satisfied*:

(OK1) *$\max\{p(Sx, Ty), p(Ty, Sx)\} \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, for all $x, y \in X$.*

(SS) *$m(x,y) + p(x, Sx) + p(y, Ty) \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, for all $x, y \in X$ where*

$$m(x,y) := \min\left\{\begin{matrix} p(x,y), p(y,x), p(x,Ty), p(Ty,x), \\ p(y,Sx), p(Sx,y), p(Sx,Ty), p(Ty,Sx) \end{matrix}\right\}.$$

If there are $\widehat{x}, \widehat{y} \in X$ such that $\widehat{x} = S\widehat{x}$ and $\widehat{y} = T\widehat{y}$, then $\widehat{x} = \widehat{y}$ and $\widehat{x}$ is a unique common fixed point of S and T. Moreover, $p(\widehat{x}, \widehat{x}) = 0$.

Proof Let $\widehat{x},\widehat{y} \in X$ be such that $\widehat{x} = S\widehat{x}$ and $\widehat{y} = T\widehat{y}$.

Suppose that (OK1) holds. Then

$$\begin{aligned}\max\bigl\{p(\widehat{x},\widehat{y}),p(\widehat{y},\widehat{x})\bigr\} &= \max\bigl\{p(S\widehat{x},T\widehat{y}),p(T\widehat{y},S\widehat{x})\bigr\}\\ &\le \varphi(\widehat{x}) - \varphi(S\widehat{x}) + \psi(\widehat{y}) - \psi(T\widehat{y}) = 0.\end{aligned}$$

It follows that $p(\widehat{x},\widehat{y}) = p(\widehat{y},\widehat{x}) = 0$ and so

$$p(\widehat{x},\widehat{x}) \le p(\widehat{x},\widehat{y}) + p(\widehat{y},\widehat{x}) = 0.$$

Hence $\widehat{x} = \widehat{y}$, and we have $\widehat{x} = S\widehat{x} = T\widehat{x}$. The uniqueness is obvious.

Suppose that (SS) holds. Then

$$\begin{aligned}m(\widehat{x},\widehat{y}) + p(\widehat{x},\widehat{x}) + p(\widehat{y},\widehat{y}) &= m(\widehat{x},\widehat{y}) + p(\widehat{x},S\widehat{x}) + p(\widehat{y},T\widehat{y})\\ &\le \varphi(\widehat{x}) - \varphi(S\widehat{x}) + \psi(\widehat{y}) - \psi(T\widehat{y}) = 0.\end{aligned}$$

If $m(\widehat{x},\widehat{y}) = p(\widehat{x},\widehat{y})$, then

$$p(\widehat{x},\widehat{x}) = p(\widehat{x},\widehat{y}) = 0.$$

So $\widehat{x} = \widehat{y}$ and hence $\widehat{x} = S\widehat{x} = T\widehat{x}$. The same conclusion holds if $m(\widehat{x},\widehat{y}) = p(\widehat{y},\widehat{x})$. The uniqueness is obvious. □

Remark 2 It is clear that (OK2) ⇒ (SS).

2.1 On Theorem OK1

Let (X,d) be a metric space and $S : X \to X$ be a mapping. Recall that $G : X \to \mathbb{R}$ is *S-orbitally lower semicontinuous* at $x_0 \in X$ if $G(\widehat{x}) \le \liminf_{n\to\infty} G(x_n)$ whenever $\{x_n\}$ is a sequence in $O(x_0,S) := \{x_0,Sx_0,S^2x_0,\ldots\}$ and $\lim_{n\to\infty} x_n = \widehat{x}$.

The following result is motivated by the one proved by Bollenbacher and Hicks [4].

Theorem 3 *Let (X,d) be a metric space and p be a w-distance on X. Let $S,T : X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Let $x_0,y_0 \in X$. Suppose that the following conditions are satisfied*:

(BH) $\max\{p(Sx,Ty),p(Ty,Sx)\} \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all $x \in O(x_0,S)$ and $y \in O(y_0,T)$.*

(cc) *Every Cauchy sequence in $O(x_0,S)$ converges to a point in X and every Cauchy sequence in $O(y_0,T)$ converges to a point in X.*

Then the following statements are true:

(i) *There exists $z \in X$ such that $\lim_{n\to\infty} S^nx_0 = \lim_{n\to\infty} T^ny_0 = z$. Moreover,*

$$\lim_{n\to\infty} p\bigl(S^nx_0,z\bigr) = \lim_{n\to\infty} p\bigl(T^ny_0,z\bigr) = 0.$$

(ii) $p(S^{n+1}x_0,z) \le \varphi(S^nx_0) + \varphi(S^{n+1}x_0) + 2\psi(T^ny_0)$ *and* $p(T^{n+1}y_0,z) \le 2\varphi(S^{n+1}x_0) + \psi(T^ny_0) + \psi(T^{n+1}y_0)$, *for all $n \ge 0$.*

(iii) *Define* $G(x) := p(x, Sx)$, *for all* $x \in X$, *and* $H(y) := p(y, Ty)$, *for all* $y \in X$. *Then the following statements are equivalent*:

(a) $z = Sz = Tz$ *and* $p(z, z) = 0$.

(b) G *is* S*-orbitally lower semicontinuous at* x_0 *and* H *is* T*-orbitally lower semicontinuous at* y_0.

Proof Let $x_0, y_0 \in X$. Define $x_n = S^n x_0$ and $y_n = T^n y_0$, for all $n \in \mathbb{N}$. Then, for all $i \in \mathbb{N}$, we have

$$p(x_i, y_i) \leq \varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i)$$

and

$$p(y_i, x_{i+1}) \leq \varphi(x_i) - \varphi(x_{i+1}) + \psi(y_{i-1}) - \psi(y_i).$$

For all integers $n \geq 1$ and $k \geq 0$, we have

$$\sum_{i=n}^{n+k} p(x_i, y_i) \leq \varphi(x_{n-1}) - \varphi(x_{n+k}) + \psi(y_{n-1}) - \psi(y_{n+k}) \tag{1}$$

and

$$\sum_{i=n}^{n+k} p(y_i, x_{i+1}) \leq \varphi(x_n) - \varphi(x_{n+k+1}) + \psi(y_{n-1}) - \psi(y_{n+k}). \tag{2}$$

In particular,

$$\sum_{n=1}^{\infty} p(x_n, y_n) \leq \varphi(x_0) + \psi(y_0) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} p(y_n, x_{n+1}) \leq \varphi(x_1) + \psi(y_0) < \infty.$$

Thus $\sum_{n=0}^{\infty} p(x_n, x_{n+1}) < \infty$. It follows that

$$p(x_n, x_{n+k+1}) \leq \sum_{i=n}^{n+k} p(x_i, x_{i+1}) \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $O(x_0, S)$ and hence $x_n \to z$ for some $z \in X$. So

$$p(x_n, z) \leq \liminf_{k\to\infty} p(x_n, x_{n+k}) \leq \sum_{i=n}^{\infty} p(x_i, x_{i+1}) \to 0 \quad \text{as } n \to \infty.$$

From $\sum_{n=1}^{\infty} p(x_n, y_n) < \infty$, we have $p(x_n, y_n) \to 0$. It follows from Lemma KST that $y_n \to z$. It is not difficult to show that $p(y_n, z) \to 0$. Hence (i) holds.

To see (ii), from (1) and (2), we have

$$\sum_{i=n}^{n+k} p(x_i, x_{i+1}) \leq \varphi(x_{n-1}) - \varphi(x_{n+k}) + 2\psi(y_{n-1}) - 2\psi(y_{n+k}) + \varphi(x_n) - \varphi(x_{n+k+1}).$$

It follows that

$$\begin{aligned}
&p(x_n, x_{n+k+1}) \\
&\quad \le \sum_{i=n}^{n+k} p(x_i, x_{i+1}) \\
&\quad \le \varphi(x_{n-1}) - \varphi(x_{n+k}) + 2\psi(y_{n-1}) - 2\psi(y_{n+k}) + \varphi(x_n) - \psi(y_{n+k+1}) \\
&\quad \le \varphi(x_{n-1}) + \varphi(x_n) + 2\psi(y_{n-1}).
\end{aligned}$$

Thus

$$p(x_n, z) \le \liminf_{k\to\infty} p(x_n, x_{n+k+1}) \le \varphi(x_{n-1}) + \varphi(x_n) + 2\psi(y_{n-1}).$$

Hence, $p(S^{n+1}x_0, z) \le \varphi(S^n x_0) + \varphi(S^n x_0) + 2\psi(T^n y_0)$. Similarly, we have

$$p\big(T^{n+1}y_0, z\big) \le 2\varphi\big(S^{n+1}x_0\big) + \psi\big(T^n y_0\big) + \psi\big(T^{n+1}y_0\big).$$

This proves (ii).

Finally, we prove (iii). (a) $\Rightarrow$ (b) Assume that $z = Sz = Tz$ and $p(z,z) = 0$. Let $\{z_n\}$ be a sequence in $O(x_0, S)$. We may assume that $z_n \to z$. It follows that $G(z) = 0 \le \liminf_{n\to\infty} G(z_n)$. Hence G is S-orbitally lower semicontinuous at x_0. Similarly, H is T-orbitally lower semicontinuous at y_0.

(b) $\Rightarrow$ (a) Assume that G is S-orbitally lower semicontinuous at x_0 and H is T-orbitally lower semicontinuous at y_0. By the proof of (i), we have $x_n = S^n x_0 \to z$, $y_n = T^n y_0 \to z$, $\lim_{n\to\infty} G(x_n) = \lim_{n\to\infty} p(S^n x_0, S^{n+1}x_0) = 0$, and $\lim_{n\to\infty} H(y_n) = \lim_{n\to\infty} p(T^n y_0, T^{n+1}y_0) = 0$. By the assumption, we have $G(z) = H(z) = 0$, that is,

$$p(z, Sz) = p(z, Tz) = 0.$$

Hence $Sz = Tz$. Thus

$$p\big(S^n x_0, Sz\big) \le p\big(S^n x_0, z\big) + p(z, Sz) \to 0 \quad \text{as } n \to \infty.$$

By (i) and Lemma KST, we have $z = Sz$. Hence $z = Sz = Tz$ and $p(z,z) = 0$. □

Remark 4 Our Theorem 3 improves Theorem OK1 in the following ways:

(1) We replace the completeness of X with the weaker assumption (cc).

(2) A necessary and sufficient condition for the existence of a common fixed point of S and T is given in terms of the orbitally lower semicontinuities of $G(x) := p(x, Sx)$ and $H(y) := p(y, Ty)$.

(3) We obtain some estimating expression for the iterative sequences.

(4) In the presence of the conditions (BH) and (cc) of Theorem 3, it is clear that if S and T are orbitally continuous, then $G(x) := p(x, Sx)$ is S-orbitally lower semicontinuous at x_0 and $H(y) := p(y, Ty)$ is T-orbitally lower semicontinuous at y_0.

2.2 On Theorem OK2

We can follow the proof of Bollenbacher and Hicks' result [4] and the proof of Theorem 3 to obtain the following result in terms of a w-distance. This result is related to Theorem KST.

Theorem 5 *Let (X,d) be a metric space and p be a w-distance on X. Let $T : X \to X$ be a given mapping and $\Phi : X \to [0,\infty)$ be any function. Let $x_0 \in X$. Suppose that the following conditions are satisfied*:

(BH) $p(x,Tx) \le \Phi(x) - \Phi(Tx)$, *for all* $x \in O(x_0,T)$.

(cc) *Every Cauchy sequence in $O(x_0,T)$ converges to a point in X.*

Then the following statements are true:

(i) *There exists $z \in X$ such that* $\lim_{n\to\infty} d(T^n x_0, z) = \lim_{n\to\infty} p(T^n x_0, z) = 0$.

(ii) $p(T^n x_0, z) \le \Phi(T^n x_0)$, *for all* $n \ge 1$.

(iii) $z = Tz$ *and* $p(z,z) = 0$ *if and only if* $H(x) := p(x,Tx)$ *is T-orbitally lower semicontinuous at* x_0.

In this subsection, we give another proof of Theorem OK2 via Theorem 5 and the following lemmas. We obtain the same conclusion as Theorem 3.

Lemma 6 *Let (X,d) be a metric space and $p : X \times X \to [0,\infty)$ be a mapping. Let $S,T : X \to X$ be two given mappings and $\boldsymbol{X} := X \times X$. Then the following statements are true*:

(i) *Define* $\boldsymbol{d} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty)$ *by*

$$\boldsymbol{d}\big((x,y),(z,w)\big) = d(x,z) + d(y,w),$$

for all $x,y,z,w \in X$. Then $\boldsymbol{d}$ is a metric on $\boldsymbol{X}$.

(ii) *Define* $\boldsymbol{p} : \boldsymbol{X} \times \boldsymbol{X} \to [0,\infty)$ *by*

$$\boldsymbol{p}\big((x,y),(z,w)\big) = p(x,z) + p(y,w),$$

for all $x,y,z,w \in X$. If p is a w-distance (w.r.t. d), then $\boldsymbol{p}$ is a w-distance (w.r.t. $\boldsymbol{d}$).

Lemma 7 *Let (X,d) be a metric space and p be a w-distance on X. Let $S,T : X \to X$ be two given mappings. Let $\boldsymbol{X}$, $\boldsymbol{d}$, and $\boldsymbol{p}$ be the same as in Lemma 6. Let $\boldsymbol{x}_0 = (x_0,y_0) \in \boldsymbol{X}$. Define $\boldsymbol{T} : \boldsymbol{X} \to \boldsymbol{X}$ by*

$$\boldsymbol{T}(\boldsymbol{x}) = (Sx, Ty),$$

for all $\boldsymbol{x} = (x,y) \in \boldsymbol{X}$. If $G(x) := p(x,Sx)$ is S-orbitally lower semicontinuous at x_0 and $H(y) := p(y,Ty)$ is T-orbitally lower semicontinuous at y_0, then $\boldsymbol{H}(\boldsymbol{z}) := \boldsymbol{p}(\boldsymbol{z},\boldsymbol{T}\boldsymbol{z})$ is $\boldsymbol{T}$-orbitally lower semicontinuous at $\boldsymbol{x}_0$.

Proof Let $\{\boldsymbol{z}_n\}$ be a sequence in $O(\boldsymbol{x}_0,\boldsymbol{T})$ $(:= \{\boldsymbol{x}_0, \boldsymbol{T}\boldsymbol{x}_0, \boldsymbol{T}^2\boldsymbol{x}_0, \ldots\})$ such that $\boldsymbol{z}_n \to \boldsymbol{z}$ for some $\boldsymbol{z} = (z,w) \in \boldsymbol{X}$. We write $\boldsymbol{z}_n = (x_n,y_n)$ where $x_n, y_n \in X$, for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ are sequences in $O(x_0,S)$ and $O(y_0,T)$, respectively. Moreover, $x_n \to z$ and $y_n \to w$. By the assumption, we have

$$p(z,Sz) = G(z) \le \liminf_{n\to\infty} G(x_n) = \liminf_{n\to\infty} p(x_n,Sx_n)$$

and

$$p(w,Tw) = H(w) \leq \liminf_{n\to\infty} H(y_n) = \liminf_{n\to\infty} p(y_n, Ty_n).$$

It follows that

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{z}) &= \boldsymbol{p}(\boldsymbol{z},\boldsymbol{T}\boldsymbol{z}) \\ &= p(z,Sz) + p(w,Tw) \\ &\leq \liminf_{n\to\infty} p(x_n,Sx_n) + \liminf_{n\to\infty} p(y_n,Ty_n) \\ &\leq \liminf_{n\to\infty} \boldsymbol{H}(\boldsymbol{z}_n).\end{aligned}$$

Hence $\boldsymbol{H}$ is $\boldsymbol{T}$-orbitally lower semicontinuous at $\boldsymbol{x}_0$. □

The following result follows directly from Theorem 5, Lemmas 6, and 7.

Lemma 8 *Let (X,d) be a metric space and p be a w-distance on X. Let $S,T : X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Let $x_0, y_0 \in X$. Suppose that the following conditions are satisfied*:

(OK2*) $p(x,Sx) + p(y,Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all $x \in O(x_0,S)$ and $y \in O(y_0,T)$.*

(cc) *Every Cauchy sequence in $O(x_0,S)$ converges to a point in X and every Cauchy sequence in $O(y_0,T)$ converges to a point in X.*

Then the following statements are true:

(i) *There exist $z,w \in X$ such that $\lim_{n\to\infty} S^n x_0 = z$ and $\lim_{n\to\infty} T^n y_0 = w$. Moreover,*

$$\lim_{n\to\infty} p\bigl(S^n x_0, z\bigr) = \lim_{n\to\infty} p\bigl(T^n y_0, w\bigr) = 0.$$

(ii) $p(S^n x_0, z) + p(T^n y_0, w) \leq \varphi(S^n x_0) + \psi(T^n y_0)$, *for all $n \geq 1$.*

(iii) *Define $G(x) := p(x,Sx)$, for all $x \in X$, and $H(y) := p(y,Ty)$, for all $y \in X$. Then the following statements are equivalent*:

  (a) *$z = Sz$ and $w = Tw$ and $p(z,z) = p(w,w) = 0$.*

  (b) *G is S-orbitally lower semicontinuous at x_0 and H is T-orbitally lower semicontinuous at y_0.*

Proof Let $\boldsymbol{X}$, $\boldsymbol{d}$ and $\boldsymbol{p}$ be the same as in Lemma 6 and $\boldsymbol{T}$ be the same as in Lemma 7. Define $\boldsymbol{\Phi} : \boldsymbol{X} \to [0,\infty)$ by

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \varphi(x) + \psi(y),$$

for all $\boldsymbol{x} = (x,y) \in \boldsymbol{X}$. Note that the condition (OK2*) is equivalent to

$$\boldsymbol{p}(\boldsymbol{x},\boldsymbol{T}\boldsymbol{x}) \leq \boldsymbol{\Phi}(\boldsymbol{x}) - \boldsymbol{\Phi}(\boldsymbol{T}\boldsymbol{x}),$$

for all $\boldsymbol{x} \in O(\boldsymbol{x}_0,\boldsymbol{T})$ and $\boldsymbol{x}_0 = (x_0,y_0) \in \boldsymbol{X}$. Define $\boldsymbol{x}_n = \boldsymbol{T}^n\boldsymbol{x}_0$, for all $n \in \mathbb{N}$. Then by Theorem 5, we obtain the statements (i) and (ii).

We finally prove the statement (iii).

(b) $\Rightarrow$ (a) It follows from Theorem 5 and Lemma 7.

(a) $\Rightarrow$ (b) It is similar to the proof of (iii) of Theorem 3 so we omit the proof. □

We now obtain our result related to Theorem OK2.

Theorem 9 *Let (X,d) be a metric space and p be a w-distance on X. Let $S,T : X \to X$ be two given mappings. Let $\varphi, \psi : X \to [0,\infty)$ be any functions. Let $x_0, y_0 \in X$. Suppose that the following conditions are satisfied*:

(OK2) $\max\{p(x,y), p(y,x)\} + p(x,Sx) + p(y,Ty) \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all* $x \in O(x_0,S)$ *and* $y \in O(y_0,T)$.

(cc) *Every Cauchy sequence in $O(x_0,S)$ converges to a point in X and every Cauchy sequence in $O(y_0,T)$ converges to a point in X.*

Then the following statements are true:

(i) *There exists $z \in X$ such that $\lim_{n\to\infty} S^n x_0 = \lim_{n\to\infty} T^n y_0 = z$. Moreover,*

$$\lim_{n\to\infty} p(S^n x_0, z) = \lim_{n\to\infty} p(T^n y_0, z) = 0.$$

(ii) $p(S^n x_0, z) + p(T^n y_0, z) \le \varphi(S^n x_0) + \psi(T^n y_0)$, *for all* $n \ge 1$.

(iii) *Define $G(x) := p(x,Sx)$, for all $x \in X$ and $H(y) := p(y,Ty)$, for all $y \in X$. Then the following statements are equivalent*:

  (a) $z = Sz = Tz$ *and* $p(z,z) = 0$.

  (b) *G is S-orbitally lower semicontinuous at x_0 and H is T-orbitally lower semicontinuous at y_0.*

Proof Let $x_0, y_0 \in X$. Define $x_n = S^n x_0$ and $y_n = T^n y_0$, for all $n \ge 1$. Note that (OK2) $\Rightarrow$ (OK2*). It follows from Lemma 8(i) that there exist $z, w \in X$ such that $\lim_{n\to\infty} x_n = z$, $\lim_{n\to\infty} y_n = w$, and $\lim_{n\to\infty} p(x_n, z) = \lim_{n\to\infty} p(y_n, w) = 0$. By the condition (OK2), we have

$$\begin{aligned} p(x_n,y_n) &\le \max\bigl\{p(x_n,y_n), p(y_n,x_n)\bigr\} + p(x_n,x_{n+1}) + p(y_n,y_{n+1}) \\ &\le \varphi(x_n) - \varphi(x_{n+1}) + \psi(y_n) - \psi(y_{n+1}), \end{aligned}$$

for all $n \ge 0$. So $\sum_{n=0}^{\infty} p(x_n,y_n) \le \varphi(x_0) + \psi(y_0) < \infty$, that is, $p(x_n,y_n) \to 0$. Hence $y_n \to z$, that is, $z = w$. The statements (ii) and (iii) follow trivially. □

Remark 10 Our Theorem 9 improves Theorem OK2 in the following ways:

(1) We replace the completeness of X with the weaker assumption (cc).

(2) A necessary and sufficient condition for the existence of a common fixed point of S and T is given in terms of the orbitally lower semicontinuities of $G(x) := p(x,Sx)$ and $H(y) := p(y,Ty)$.

(3) We obtain some estimating expression for the iterative sequences.

(4) In the presence of the conditions (OK2) and (cc) of Theorem 9, it is clear that if S and T are orbitally continuous, then $G(x) := p(x,Sx)$ is S-orbitally lower semicontinuous at x_0 and $H(y) := p(y,Ty)$ is T-orbitally lower semicontinuous at y_0.

(5) It is easy to see that the condition (OK2) can be replaced by the weaker condition (SS):

$$m(x,y) + p(x,Sx) + p(y,Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty),$$

for all $x \in O(x_0,S)$ and $y \in O(y_0,T)$. In fact, if $m(x_n,y_n) \to 0$, then there is a strictly increasing sequence $\{n_k\}$ on $\mathbb{N}$ such that one of the following sequences converges to zero:

$$\{p(x_{n_k},y_{n_k})\}, \quad \{p(y_{n_k},x_{n_k})\}, \quad \{p(x_{n_k},y_{n_k+1})\}, \quad \{p(y_{n_k+1},x_{n_k})\},$$
$$\{p(x_{n_k+1},y_{n_k})\}, \quad \{p(y_{n_k},x_{n_k+1})\}, \quad \{p(x_{n_k+1},y_{n_k+1})\}, \quad \{p(y_{n_k+1},x_{n_k+1})\}.$$

Before moving to the next section, we give an example which satisfies our conditions in Theorems 3 and 9 but cannot be concluded from Theorems OK1 and OK2.

Example 11 Let $X = [0,1/3)$ be equipped with the usual metric d. Define $S,T : X \to X$ by $Sx = 0$ and $Tx = x^2$, for all $x \in X$. Also, define $\varphi,\psi : X \to [0,\infty)$ by $\varphi(x) = 2x$ and $\psi(x) = 5x/2$, for all $x \in X$. It is clear that X is not complete. Moreover, it is not hard to see that

$$d(Sx,Ty) \leq d(x,y) + d(x,Sx) + d(y,Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty),$$

for all $x,y \in X$.

3 Results on Theorems OK1 and OK2 with the lower semicontinuities of φ and ψ

As studied in Theorem KST, it is more practical to put an assumption on the dominated function ψ than to put one on the mapping T itself. In this section, we discuss Theorems OK1 and OK2 where the functions φ and ψ are assumed to be lower semicontinuous.

3.1 Results related to Theorem OK1

The following observation is obvious.

Lemma 12 *Let (X,d) be a metric space and p be a w-distance on X. Let $T : X \to X$ be a given mapping. Let $\varphi : X \to [0,\infty)$ be any function. Let $\widehat{y} \in X$. Define $\widehat{p} : X \times X \to [0,\infty)$ and $\widehat{\varphi} : X \to [0,\infty)$ by*

$$\widehat{p}(x,y) := \frac{1}{2}p(T\widehat{y},x) + \frac{1}{2}p(T\widehat{y},y), \quad \textit{for all } x,y \in X,$$

and

$$\widehat{\varphi}(x) := \varphi(x) + \frac{1}{2}p(T\widehat{y},x), \quad \textit{for all } x \in X.$$

Then the following statements hold:

(i) *$\widehat{p}$ is a w-distance on X.*
(ii) *If φ is lower semicontinuous, then so is $\widehat{\varphi}$.*

In the setting of Theorem OK1 with the appearance of the lower semicontinuities of φ and ψ in place of the orbital continuities of S and T, we get a partial result with some additional assumption.

Theorem 13 *Let (X,d) be a complete metric space and p be a w-distance on X. Let $S,T: X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Suppose that the following conditions are satisfied*:

(OK1) $\max\{p(Sx,Ty),p(Ty,Sx)\} \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all* $x,y \in X$.

(lsc) φ *and* ψ *are lower semicontinuous.*

Then the following statements hold:

- S *has a fixed point if and only if there exists an element* $\widehat{y} \in X$ *such that* $\psi(\widehat{y}) \le \psi(T\widehat{y})$.
- T *has a fixed point if and only if there exists an element* $\widehat{x} \in X$ *such that* $\varphi(\widehat{x}) \le \varphi(S\widehat{x})$.

In each case above, we find that S and T have a unique common fixed point.

Proof We first prove the following two statements.

(i) If there exists an element $\widehat{y} \in X$ such that $\psi(\widehat{y}) \le \psi(T\widehat{y})$, then S has a fixed point.

(ii) If there exists an element $\widehat{x} \in X$ such that $\varphi(\widehat{x}) \le \varphi(S\widehat{x})$, then T has a fixed point.

Since the proof of (ii) is similar to that of (i), we prove only (i). Assume that there exists an element $\widehat{y} \in X$ such that $\psi(\widehat{y}) \le \psi(T\widehat{y})$. Then we have

$$\max\{p(Sx,T\widehat{y}),p(T\widehat{y},Sx)\} \le \varphi(x) - \varphi(Sx), \quad \text{for all } x \in X. \tag{3}$$

Let $\widehat{p}$ and $\widehat{\varphi}$ be the same as in Lemma 12. Then it follows from (3) that

$$\begin{aligned}
\widehat{p}(x,Sx) &= \frac{1}{2}p(T\widehat{y},x) + \frac{1}{2}p(T\widehat{y},Sx) \\
&= \frac{1}{2}p(T\widehat{y},x) + p(T\widehat{y},Sx) - \frac{1}{2}p(T\widehat{y},Sx) \\
&\le \frac{1}{2}p(T\widehat{y},x) + \max\{p(Sx,T\widehat{y}),p(T\widehat{y},Sx)\} - \frac{1}{2}p(T\widehat{y},Sx) \\
&\le \frac{1}{2}p(T\widehat{y},x) + \varphi(x) - \varphi(Sx) - \frac{1}{2}p(T\widehat{y},Sx) \\
&= \widehat{\varphi}(x) - \widehat{\varphi}(Sx),
\end{aligned}$$

for all $x \in X$. By Lemma 12 and Theorem KST, there exists $\widehat{x} \in X$ such that $\widehat{x} = S\widehat{x}$.

Next, we prove the following statement:

(iii) If S has a fixed point, then there exists an element $\widehat{y} \in X$ such that $\psi(\widehat{y}) \le \psi(T\widehat{y})$.

In fact, if $\widehat{x} = S\widehat{x}$, then $\varphi(\widehat{x}) = \varphi(S\widehat{x})$. It follows from (ii) that T has a fixed point, that is, there exists $\widehat{y} \in X$ such that $\widehat{y} = T\widehat{y}$, so we obtain (iii). The uniqueness follows immediately from Lemma 1. □

Remark 14 In (i) of the proof of Theorem 13, we also have $ST\widehat{y} = T\widehat{y}$. To see this, we note that $p(\widehat{x},\widehat{x}) = 0$ and $p(\widehat{x},\widehat{y}) \le \psi(\widehat{y}) - \psi(T\widehat{y}) \le 0$. This gives $\widehat{x} = \widehat{y}$ and hence $ST\widehat{y} = S\widehat{x} = \widehat{x} = T\widehat{y}$.

Since d is a w-distance, we immediately obtain this corollary which is related to Theorem BB where the condition (oc) is replaced by the condition (lsc).

Corollary 15 *Let (X,d) be a complete metric space and $S,T: X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Suppose that the following conditions are* *satisfied*:

(BB) $d(Sx, Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all* $x, y \in X$.

(lsc) φ *and* ψ *are lower semicontinuous.*

Then the following statements hold:

- *S has a fixed point if and only if there exists an element* $\widehat{y} \in X$ *such that* $\psi(\widehat{y}) \leq \psi(T\widehat{y})$.
- *T has a fixed point if and only if there exists an element* $\widehat{x} \in X$ *such that* $\varphi(\widehat{x}) \leq \varphi(S\widehat{x})$.

In each case above, S and T have a unique common fixed point.

3.2 Results related to Theorem OK2

Lemma 16 *Let* (X, d) *be a complete metric space and p be a w-distance on X. Let* $S, T : X \to X$ *be two given mappings. Let* $\varphi, \psi : X \to [0, \infty)$ *be any functions. Suppose that the following conditions hold*:

(OK2*) $p(x, Sx) + p(y, Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all* $x, y \in X$.

(lsc) φ *and* ψ *are lower semicontinuous.*

Then there exists $(\widehat{x}, \widehat{y}) \in X \times X$ *such that* $\widehat{x} = S\widehat{x}$ *and* $\widehat{y} = T\widehat{y}$.

Proof Let $\boldsymbol{X}$, $\boldsymbol{d}$, and $\boldsymbol{p}$ be the same as in Lemma 6 and $\boldsymbol{T}$ be the same as in Lemma 7. It is clear that $(\boldsymbol{X}, \boldsymbol{d})$ is complete. Define $\boldsymbol{\Phi} : \boldsymbol{X} \to [0, \infty)$ by

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \varphi(x) + \psi(y),$$

for all $\boldsymbol{x} = (x, y) \in \boldsymbol{X}$. Since φ and ψ are lower semicontinuous, we conclude that $\boldsymbol{\Phi}$ is lower semicontinuous. Note that (OK2*) is equivalent to

$$\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{T}\boldsymbol{x}) \leq \boldsymbol{\Phi}(\boldsymbol{x}) - \boldsymbol{\Phi}(\boldsymbol{T}\boldsymbol{x}), \quad \text{for all } \boldsymbol{x} \in \boldsymbol{X}.$$

By Theorem KST, there exists $\widehat{\boldsymbol{x}} = (\widehat{x}, \widehat{y}) \in \boldsymbol{X}$ such that $\widehat{\boldsymbol{x}} = \boldsymbol{T}\widehat{\boldsymbol{x}}$, that is, $\widehat{x} = S\widehat{x}$ and $\widehat{y} = T\widehat{y}$. □

We now obtain a result related to Theorem OK2 where the condition (oc) is replaced by the condition (lsc).

Theorem 17 *Let* (X, d) *be a complete metric space and p be a w-distance on X. Let* $S, T : X \to X$ *be two given mappings. Let* $\varphi, \psi : X \to [0, \infty)$ *be any functions. Suppose that the following conditions hold*:

(OK2) $\max\{p(x, y), p(y, x)\} + p(x, Sx) + p(y, Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$, *for all* $x, y \in X$.

(lsc) φ *and* ψ *are lower semicontinuous.*

Then S and T have a unique common fixed point.

Proof It follows immediately from Lemmas 1 and 16. □

Remark 18 It is easy to see that the condition (OK2) can be replaced by the weaker condition (SS):

$$m(x, y) + p(x, Sx) + p(y, Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty),$$

for all $x, y \in X$.

With a slight modification of the condition (OK2*), we can conclude a common fixed point even if we assume that either φ or ψ is lower semicontinuous. However, the uniqueness is not guaranteed.

Theorem 19 *Let (X,d) be a complete metric space and p be a w-distance on X. Let $S,T: X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Suppose that the following conditions are satisfied*:

(SS*) $p(x,Sx) + p(y,Ty) \le \varphi(x) - \psi(Sx) + \psi(y) - \varphi(Ty)$, *for all $x,y \in X$.*

(lsc*) *Either φ or ψ is lower semicontinuous.*

Then S and T have a common fixed point.

Proof We may assume that φ is lower semicontinuous. Let $y = Sx$ in the condition (SS*) and we have

$$\begin{aligned} p(x,TSx) &\le p(x,Sx) + p(Sx,TSx) \\ &\le \varphi(x) - \psi(Sx) + \psi(Sx) - \varphi(TSx) \\ &\le \varphi(x) - \varphi(TSx). \end{aligned}$$

By Theorem KST, there exists $\widehat{x} \in X$ such that $\widehat{x} = TS\widehat{x}$. It follows that

$$p(\widehat{x},\widehat{x}) \le p(\widehat{x},S\widehat{x}) + p(S\widehat{x},TS\widehat{x}) \le \varphi(\widehat{x}) - \psi(S\widehat{x}) + \psi(S\widehat{x}) - \varphi(TS\widehat{x}) = 0.$$

Thus $p(\widehat{x},\widehat{x}) = p(\widehat{x},S\widehat{x}) = p(S\widehat{x},TS\widehat{x}) = 0$. It follows from the first equality that $\widehat{x} = S\widehat{x}$. Then $p(\widehat{x},T\widehat{x}) = 0$ and hence $\widehat{x} = T\widehat{x}$. This completes the proof. □

Remark 20 The condition (SS*) in Theorem 19 is motivated by Lemma 2.2 of [5]. More precisely, let (X,d) be a metric space and let $S,T : X \to X$ be two mappings with two nonnegative real numbers λ and μ such that $\lambda + \mu < 1$ and

$$d(Sx,TSx) \le \frac{\lambda}{1-\mu} d(x,Sx)$$

and

$$d(Tx,STx) \le \frac{\lambda}{1-\mu} d(x,Tx),$$

for all $x \in X$. Set $\varphi(x) = \frac{1-\mu}{1-\lambda-\mu} d(x,Sx)$ and $\psi(x) = \frac{1-\mu}{1-\lambda-\mu} d(x,Tx)$. It follows then that

$$d(x,Sx) \le \varphi(x) - \psi(Sx)$$

and

$$d(x,Tx) \le \psi(x) - \varphi(Tx)$$

for all $x \in X$. In particular, S and T satisfy the condition (SS*).

3.3 The existence of a common fixed point of S and T is equivalent to their orbital continuities

First, let us start with the following easy observation.

Lemma 21 *Let (X,d) be a metric space and p be a w-distance. Let $\varphi : X \to [0,\infty)$ be any function. Suppose that $S : X \to X$ is a mapping satisfying the following condition*:

There exists $z \in X$ such that

$$p(z,Sx) \le \varphi(x) - \varphi(Sx)$$

for all $x \in X$.

Then the following statements are true:

(i) $\lim_{n\to\infty} p(z,S^n x) = 0$ *for all $x \in X$.*

(ii) $z = Sz$ *if and only if S is orbitally continuous and $p(z,z) = 0$.*

Proof (i) Let $x \in X$ be given. It follows that

$$0 \le p\big(z,S^n x\big) \le \varphi\big(S^n x\big) - \varphi\big(S^{n+1} x\big)$$

for all integers $n \ge 0$. Then the sequence $\{\varphi(S^n x)\}$ is nonincreasing and hence it is convergent. In particular, $\lim_{n\to\infty} p(z,S^n x) = 0$.

(ii) ($\Rightarrow$) Assume that $z = Sz$. In particular, $p(z,z) = p(z,Sz) \le \varphi(z) - \varphi(Sz) = \varphi(z) - \varphi(z) = 0$. To show that S is orbitally continuous, let $x \in X$ be such that $S^{n_j} x \to w$ for some $w \in X$. It follows from (i) and Lemma KST that $S^n x \to z$. So, we have $w = z$. Now, $S(S^{n_j} x) \to z = Sz$. ($\Leftarrow$) Assume that S is orbitally continuous and $p(z,z) = 0$. It follows from (i) and Lemma KST that $S^n z \to z$. Since S is orbitally continuous, we have $S(S^n z) \to Sz$. It follows then that $z = Sz$. □

The following result shows that the condition (oc) is not only sufficient but also necessary for the existence of a common fixed point in Theorems OK1 and OK2.

Theorem 22 *Let (X,d) be a metric space and p be a w-distance on X. Let $S,T : X \to X$ be two given mappings. Let $\varphi,\psi : X \to [0,\infty)$ be any functions. Suppose that one of the following conditions holds*:

(OK1) $\max\{p(Sx,Ty),p(Ty,Sx)\} \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$ *for all $x,y \in X$.*

(OK2) $\max\{p(x,y),p(y,x)\} + p(x,Sx) + p(y,Ty) \le \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty)$ *for all $x,y \in X$.*

If S and T have a (unique) common fixed point, then:

(oc) *S and T are orbitally continuous.*

Proof We assume that S, T satisfy the condition (OK1) and $z = Sz = Tz$. Letting $y = z$ in the condition (OK1) gives $p(z,Sx) \le \varphi(x) - \varphi(Sx)$ for all $x \in X$. It follows from the preceding lemma that S is orbitally continuous. Similarly, interchanging the role of S and T ensures that T is orbitally continuous as well.

We assume that S, T satisfy the condition (OK2) and $z = Sz = Tz$. It is clear that $p(z,z) = 0$. Because of this, letting $y = z$ in the condition (OK2) gives $p(z,Sx) \le p(z,x) + p(x,Sx) +$

$p(z, Tz) \leq \varphi(x) - \varphi(Sx) + \psi(z) - \psi(Tz) = \varphi(x) - \varphi(Sx)$ for all $x \in X$. As proved in the first part, we conclude that S and T are orbitally continuous. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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