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A hybrid iterative method for common solutions of variational inequality problems and fixed point problems for single-valued and multi-valued mappings with applications

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Abstract

In this article, we propose a new iterative method for approximating a common element of the set of common fixed points of a finite family of k-strictly pseudononspreading single-valued mappings, the set of common fixed points of a finite family of quasi-nonexpansive multi-valued mappings, and the set of common solutions of a finite family of variational inequality problems in Hilbert spaces. Furthermore, we prove that the proposed iterative method converges strongly to a common element of the above three sets, and we also apply our results to complementarity problems. Finally, we give two numerical examples to support our main result.

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1 Introduction

Let D be a nonempty subset of a real Hilbert space X. Let $CB(D)$ denote the families of nonempty closed bounded subsets of D. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\right\} \quad \text{for } A,B \in CB(D),$$

where $\operatorname{dist}(x,D) = \inf\{\|x-y\| : y \in D\}$. Let $T : D \to CB(D)$ be a multi-valued mapping. An element $x \in D$ is said to be a *fixed point* of T if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$. A multi-valued mapping $T : D \to CB(D)$ is called

(i) *nonexpansive* if

$$H(Tx,Ty) \leq \|x-y\| \quad \text{for all } x,y \in D;$$

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$H(Tx,Tp) \leq \|x-p\| \quad \text{for all } x \in D \text{ and } p \in F(T);$$

(iii) *L-Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \quad \text{for all } x, y \in D.$$

It is clear that every nonexpansive multi-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. It is known that if T is a quasi-nonexpansive multi-valued mapping, then $F(T)$ is closed. In general, the fixed point set of a quasi-nonexpansive multi-valued mapping T is not necessary to be convex. In the next lemma, we show that $F(T)$ is convex under the assumption that $Tp = \{p\}$ for all $p \in F(T)$. The proof of this fact is very easy, therefore we omit it.

Lemma 1.1 *Let D be a nonempty closed convex subset of a real Hilbert space X. Assume that $T : D \to CB(D)$ is a quasi-nonexpansive multi-valued mapping. If $Tp = \{p\}$ for all $p \in F(T)$, then $F(T)$ is convex.*

The fixed point theory of multi-valued mappings is much more complicated and harder than the corresponding theory of single-valued mappings. However, some classical fixed point theorems for single-valued mappings have already been extended to multi-valued mappings; see [1, 2]. The recent fixed point results for multi-valued mappings can be found in [3–11] and the references cited therein.

For a single-valued case, a mapping $t : D \to D$ is called *nonexpansive* if $\|tx - ty\| \leq \|x - y\|$ for all $x, y \in D$. An element $x \in D$ is called a *fixed point* of t if $x = tx$. Recall that a single-valued mapping $t : D \to D$ is said to be *nonspreading* [12, 13] if

$$\|tx - ty\|^2 \leq \|x - y\|^2 + 2\langle x - tx, y - ty\rangle \quad \text{for all } x, y \in D.$$

In 2010, Kurokawa and Takahashi [14] obtained a weak mean ergodic theorem of Baillon's type for nonspreading single-valued mappings in Hilbert spaces. They also proved a strong convergence theorem for this class of single-valued mappings using an idea of mean convergence in Hilbert spaces. Later in 2011, Osilike and Isiogugu [15] introduced a new class of nonspreading type of mappings, which is more general than the class studied in [14], as follows: A single-valued mapping $t : D \to D$ is called *k-strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|tx - ty\|^2 \leq \|x - y\|^2 + k\big\|(I - t)x - (I - t)y\big\|^2 + 2\langle x - tx, y - ty\rangle \quad \text{for all } x, y \in D.$$

Obviously, every nonspreading mapping is k-strictly pseudononspreading. Osilike and Isiogugu proved weak and strong convergence theorems for this mapping in Hilbert spaces. They also provided a property of a k-strictly pseudononspreading mapping as follows.

Lemma 1.2 ([15]) *Let D be a nonempty closed convex subset of a real Hilbert space X, and let $t : D \to D$ be a k-strictly pseudononspreading mapping. If $F(t) \neq \emptyset$, then it is closed and convex.*

Many researchers studied the existence and convergence theorems of those single-valued mappings in both Hilbert spaces and Banach spaces (*e.g.*, see [16–23]).

The problem of finding common fixed points has been extensively studied by mathematicians. To deal with a fixed point problem of a family of nonlinear mappings, several ways have appeared in the literature. For example, in 1999, Atsushiba and Takahashi [24] introduced a new mapping, called *W-mapping*, for finding a common fixed point of a finite family of nonexpansive mappings. This mapping is defined as follows. Let $\{t_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Let $W : D \to D$ be a mapping defined by

$$\begin{aligned}
U_1 &= \beta_1 t_1 + (1-\beta_1)I,\\
U_2 &= \beta_2 t_2 U_1 + (1-\beta_2)I,\\
U_3 &= \beta_3 t_3 U_2 + (1-\beta_3)I,\\
&\vdots\\
U_{N-1} &= \beta_{N-1} t_{N-1} U_{N-2} + (1-\beta_{N-1})I,\\
W = U_N &= \beta_N t_N U_{N-1} + (1-\beta_N)I,
\end{aligned}$$

where I is the identity mapping of D and $\{\beta_i\}_{i=1}^N$ is a sequence in $(0,1)$. This mapping is called the *W-mapping* generated by $t_1, t_2, \ldots, t_N$ and $\beta_1, \beta_2, \ldots, \beta_N$. They also proved that if X is a strictly convex Banach space, then $F(W) = \bigcap_{i=1}^N F(t_i)$.

In 2009, Kangtunyakarn and Suantai [25] introduced a new concept of the S-mapping for finding a common fixed point of a finite family of nonexpansive mappings as follows: Let $\{t_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself. Let $S : D \to D$ be a mapping defined by

$$\begin{aligned}
V_1 &= \delta_1^1 t_1 + \delta_2^1 I + \delta_3^1 I,\\
V_2 &= \delta_1^2 t_2 V_1 + \delta_2^2 V_1 + \delta_3^2 I,\\
V_3 &= \delta_1^3 t_3 V_2 + \delta_2^3 V_2 + \delta_3^3 I,\\
&\vdots\\
V_{N-1} &= \delta_1^{N-1} t_{N-1} V_{N-2} + \delta_2^{N-1} V_{N-2} + \delta_3^{N-1} I,\\
S = V_N &= \delta_1^N t_N V_{N-1} + \delta_2^N V_{N-1} + \delta_3^N I,
\end{aligned}$$

where I is the identity mapping of D and $\delta_j = (\delta_1^j, \delta_2^j, \delta_3^j) \in [0,1] \times [0,1] \times [0,1]$, $j = 1, 2, \ldots, N$, where $\delta_1^j + \delta_2^j + \delta_3^j = 1$ for all $j = 1, 2, \ldots, N$. This mapping is called the *S-mapping* generated by $t_1, t_2, \ldots, t_N$ and $\delta_1, \delta_2, \ldots, \delta_N$. They proved the following lemma important for our results.

Lemma 1.3 *Let D be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of D into itself with $\bigcap_{i=1}^N F(t_i) \neq \emptyset$, and let $\delta_j = (\delta_1^j, \delta_2^j, \delta_3^j) \in [0,1] \times [0,1] \times [0,1]$, $j = 1, 2, \ldots, N$, where $\delta_1^j + \delta_2^j + \delta_3^j = 1$, $\delta_1^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$, $\delta_1^N \in (0,1]$, and $\delta_2^j, \delta_3^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $t_1, t_2, \ldots, t_N$ and $\delta_1, \delta_2, \ldots, \delta_N$. Then S is a nonexpansive mapping and $F(S) = \bigcap_{i=1}^N F(t_i)$.*

Applications of W-mappings and S-mappings for fixed point problems can be found in [26–31].

Let $B : D \to X$ be a nonlinear mapping. The *variational inequality problem* is to find a point $u \in D$ such that

$$\langle Bu, v - u\rangle \geq 0 \quad \text{for all } v \in D. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(D, B)$.

A mapping $B : D \to X$ is called *ϕ-inverse strongly monotone* [32] if there exists a positive real number ϕ such that

$$\langle x - y, Bx - By\rangle \geq \phi \|Bx - By\|^2 \quad \text{for all } x, y \in D.$$

Variational inequality theory, which was first introduced by Stampacchia [33] in 1964, emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in economics, industry, network analysis, optimizations, pure and applied sciences *etc.* In recent years, much attention has been given to developing efficient iterative methods for treating solution problems of variational inequalities (*e.g.*, see [34–39]).

In 2003, Takahashi and Toyoda [40] introduced an iterative method for finding a common element of the set of fixed points of nonexpansive single-valued mappings and the set of solutions of variational inequalities for ϕ-inverse strongly monotone mappings in Hilbert spaces. Recently, by using the concept of S-mapping, Kangtunyakarn [41] introduced a new method for finding a common element of the set of fixed points of k-strictly pseudononspreading single-valued mappings and the set of solutions of variational inequality problems in Hilbert spaces.

Question A How can we construct an iteration process for finding a common element of the set of common fixed points of a finite family of k-strictly pseudononspreading single-valued mappings, the set of common fixed points of a finite family of quasi-nonexpansive multi-valued mappings, and the set of common solutions of a finite family of variational inequality problems?

In the recent years, the problem of finding a common element of the set of fixed points of single-valued mappings and multi-valued mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many researchers. However, no researchers have studied the problem of finding a common element of three sets, *i.e.*, the set of common fixed points of a finite family of single-valued mappings, the set of common fixed points of a finite family of multi-valued mappings, and the set of common solutions of a finite family of variational inequality problems.

In this article, motivated by [41] and the research described above, we propose a new hybrid iterative method for finding a common element of the set of a common fixed point of a finite family of k-strictly pseudononspreading single-valued mappings, the set of common fixed points of a finite family of quasi-nonexpansive multi-valued mappings, and the set of common solutions of a finite family of variational inequality problems in Hilbert spaces and provide an affirmative answer to Question A.

2 Preliminaries

In this section, we give some useful lemmas for proving our main results. Let D be a nonempty closed convex subset of a real Hilbert space X. Let P_D be the metric projection of X onto D, *i.e.*, for $x \in X$, $P_D x$ satisfies the property $\|x - P_D x\| = \min_{y\in D} \|x - y\|$. It is well known that P_D is a nonexpansive mapping of X onto D.

Lemma 2.1 ([42]) *Let X be a Hilbert space, let D be a nonempty closed convex subset of X, and let B be a mapping of D into X. Let $u \in D$. Then, for $\lambda > 0$,*

$$u = P_D(I - \lambda B)u \quad \Longleftrightarrow \quad u \in VI(D, B).$$

Lemma 2.2 ([43]) *Let D be a nonempty closed convex subset of a real Hilbert space X, and let $P_D : X \to D$ be the metric projection. Given $x \in X$ and $z \in D$, then $z = P_D x$ if and only if the following holds*:

$$\langle x - z, y - z\rangle \leq 0 \quad \text{for all } y \in D.$$

Lemma 2.3 ([44]) *Let D be a nonempty closed convex subset of a real Hilbert space X, and let $P_D : X \to D$ be the metric projection. Then the following inequality holds*:

$$\|y - P_D x\|^2 + \|x - P_D x\|^2 \leq \|x - y\|^2 \quad \text{for all } x \in X \text{ and } y \in D.$$

Lemma 2.4 ([43]) *Let X be a real Hilbert space. Then*

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \quad \text{for all } x, y \in X.$$

Lemma 2.5 ([45]) *Let X be a Hilbert space. Let $x_1, x_2, \ldots, x_N \in X$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ be real numbers such that $\sum_{i=1}^N \alpha_i = 1$. Then*

$$\left\| \sum_{i=1}^N \alpha_i x_i \right\|^2 = \sum_{i=1}^N \alpha_i \|x_i\|^2 - \sum_{1\leq i,j\leq N} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.6 ([46]) *Let D be a nonempty closed convex subset of a real Hilbert space X. Given $x, y, z \in X$ and $b \in \mathbb{R}$, the set*

$$\bigl\{u \in D : \|y - u\|^2 \leq \|x - u\|^2 + \langle z, u\rangle + b\bigr\}$$

is closed and convex.

Lemma 2.7 ([42]) *In a strictly convex Banach space X, if*

$$\|x\| = \|y\| = \bigl\|\lambda x + (1 - \lambda)y\bigr\|$$

for all $x, y \in X$ and $\lambda \in (0, 1)$, then $x = y$.

The following lemma obtained by Kangtunyakarn [41] is useful for our results.

Lemma 2.8 *Let D be a nonempty closed convex subset of a Hilbert space X. Let $t : D \to D$ be a k-strictly pseudononspreading mapping with $F(t) \neq \emptyset$. Then $F(t) = VI(D, I - t)$.*

Remark 2.9 ([41]) From Lemmas 2.1 and 2.8, we have

$$F(t) = F\big(P_D\big(I - \lambda(I - t)\big)\big) \quad \text{for all } \lambda > 0.$$

3 Main results

In this section, we prove a strong convergence theorem which solves the problem of finding a common element of the set of common fixed points of a finite family of k-strictly pseudononspreading single-valued mappings, the set of common fixed points of a finite family of quasi-nonexpansive multi-valued mappings, and the set of common solutions of a finite family of variational inequality problems in Hilbert spaces. Before starting the main theorem of this section, we need to prove the following useful lemma in Hilbert spaces.

Lemma 3.1 *Let D be a nonempty closed convex subset of a real Hilbert space X, and let $\{t_i\}_{i=1}^N$ be a finite family of k-strictly pseudononspreading single-valued mappings of D into itself such that $\bigcap_{i=1}^N F(t_i) \neq \emptyset$. Let $R_i : D \to D$ be defined by $R_i x = P_D(I - \lambda(I - t_i))x$ for all $x \in D$, $\lambda \in (0, 1 - k)$, and $i = 1, 2, \ldots, N$. Suppose that $\beta_1, \beta_2, \ldots, \beta_N$ are real numbers such that $0 < \beta_i < 1$ for all $i = 1, 2, \ldots, N - 1$ and $0 < \beta_N \leq 1$. Let W be the W-mapping generated by $R_1, R_2, \ldots, R_N$ and $\beta_1, \beta_2, \ldots, \beta_N$. Then the following hold*:

(i) *W is quasi-nonexpansive*;

(ii) *$F(W) = \bigcap_{i=1}^N F(t_i) = \bigcap_{i=1}^N F(R_i)$.*

Proof (i) For each $x \in D$ and $z \in \bigcap_{i=1}^N F(t_i)$,

$$\begin{aligned} \|R_i x - z\|^2 &= \big\|P_D\big(I - \lambda(I - t_i)\big)x - z\big\|^2 \\ &\leq \big\|\big(I - \lambda(I - t_i)\big)x - z\big\|^2 \\ &= \big\|(x - z) - \lambda(I - t_i)x\big\|^2 \\ &= \|x - z\|^2 - 2\lambda\big\langle x - z, (I - t_i)x\big\rangle + \lambda^2\|I - t_i\|^2. \end{aligned} \tag{3.1}$$

By t_i is k-strictly pseudononspreading, we have

$$\begin{aligned} \|t_i x - t_i z\|^2 &\leq \|x - z\|^2 + k\big\|(I - t_i)x - (I - t_i)z\big\|^2 + 2\big\langle (I - t_i)x, (I - t_i)z\big\rangle \\ &= \|x - z\|^2 + k\big\|(I - t_i)x\big\|^2. \end{aligned} \tag{3.2}$$

Since

$$\begin{aligned} \|t_i x - t_i z\|^2 &= \big\|\big(I - (I - t_i)\big)x - \big(I - (I - t_i)\big)z\big\|^2 \\ &= \big\|(x - z) - \big((I - t_i)x - (I - t_i)z\big)\big\|^2 \\ &= \|x - z\|^2 - 2\big\langle x - z, (I - t_i)x\big\rangle + \big\|(I - t_i)x\big\|^2, \end{aligned} \tag{3.3}$$

it follows by (3.2) that

$$(1 - k)\big\|(I - t_i)x\big\|^2 \leq 2\big\langle x - z, (I - t_i)x\big\rangle.$$

Therefore, by (3.1), we have

$$\begin{aligned}\|R_i x - z\|^2 &\leq \|x - z\|^2 - (1-k)\lambda \big\|(I - t_i)x\big\|^2 + \lambda^2 \big\|(I - t_i)x\big\|^2 \\ &= \|x - z\|^2 - \lambda(1 - k - \lambda)\big\|(I - t_i)x\big\|^2 \\ &\leq \|x - z\|^2.\end{aligned}$$

This implies that

$$\|R_i x - z\| \leq \|x - z\| \quad \text{for all } i = 1, 2, \ldots, N. \tag{3.4}$$

Let $j \in \{1, 2, \ldots, N\}$, we get

$$\begin{aligned}\|U_j x - z\| &= \big\|\beta_j R_j U_{j-1} x + (1 - \beta_j)x - z\big\| \\ &\leq \beta_j \|R_j U_{j-1} x - z\| + (1 - \beta_j)\|x - z\| \\ &\leq \beta_j \|U_{j-1} x - z\| + (1 - \beta_j)\|x - z\|.\end{aligned}$$

So, we have

$$\begin{aligned}\|Wx - z\| &= \|U_N x - z\| \\ &\leq \beta_N \|U_{N-1} x - z\| + (1 - \beta_N)\|x - z\| \\ &\leq \beta_N \big(\beta_{N-1}\|U_{N-2} x - z\| + (1 - \beta_{N-1})\|x - z\|\big) + (1 - \beta_N)\|x - z\| \\ &= \beta_N \beta_{N-1}\|U_{N-2} x - z\| + (1 - \beta_N \beta_{N-1})\|x - z\| \\ &\ \ \vdots \\ &\leq \beta_N \beta_{N-1} \cdots \beta_2 \|U_1 x - z\| + (1 - \beta_N \beta_{N-1} \cdots \beta_2)\|x - z\| \\ &\leq \beta_N \beta_{N-1} \cdots \beta_2 \big(\beta_1 \|x - z\| + (1 - \beta_1)\|x - z\|\big) \\ &\quad + (1 - \beta_N \beta_{N-1} \cdots \beta_2)\|x - z\| \\ &= \|x - z\|.\end{aligned}$$

Thus, W is a quasi-nonexpansive mapping.

(ii) Since $\bigcap_{i=1}^N F(t_i) \subset F(W)$ is trivial, it suffices to show that $F(W) \subset \bigcap_{i=1}^N F(t_i)$. To show this, we suppose that $p \in F(W)$ and $z \in \bigcap_{i=1}^N F(t_i)$. Then we have

$$\begin{aligned}\|p - z\| &= \|Wp - z\| \\ &= \big\|\beta_N (R_N U_{N-1} p - z) + (1 - \beta_N)(p - z)\big\| \\ &\leq \beta_N \|R_N U_{N-1} p - z\| + (1 - \beta_N)\|p - z\| \\ &\leq \beta_N \|U_{N-1} p - z\| + (1 - \beta_N)\|p - z\| \\ &\leq \beta_N \beta_{N-1} \|R_{N-1} U_{N-2} p - z\| + (1 - \beta_N \beta_{N-1})\|p - z\| \\ &\ \ \vdots\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_N\beta_{N-1}\cdots\beta_3\|R_3U_2p-z\| + (1-\beta_N\beta_{N-1}\cdots\beta_3)\|p-z\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_3\|U_2p-z\| + (1-\beta_N\beta_{N-1}\cdots\beta_3)\|p-z\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_3\beta_2\|R_2U_1p-z\| + (1-\beta_N\beta_{N-1}\cdots\beta_3\beta_2)\|p-z\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_3\beta_2\|U_1p-z\| + (1-\beta_N\beta_{N-1}\cdots\beta_3\beta_2)\|p-z\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_3\beta_2\beta_1\|R_1p-z\| + (1-\beta_N\beta_{N-1}\cdots\beta_3\beta_2\beta_1)\|p-z\| \\
&\leq \|p-z\|.
\end{aligned} \tag{3.5}
$$

This shows that

$$\|p-z\| = \beta_N\beta_{N-1}\cdots\beta_2\big\|\beta_1(R_1p-z)+(1-\beta_1)(p-z)\big\| + (1-\beta_N\beta_{N-1}\cdots\beta_2)\|p-z\|.$$

Thus,

$$\|p-z\| = \big\|\beta_1(R_1p-z)+(1-\beta_1)(p-z)\big\|.$$

Again by (3.5), we have

$$\|p-z\| = \|R_1p-z\| = \big\|\beta_1(R_1p-z)+(1-\beta_1)(p-z)\big\|.$$

This implies by Lemma 2.7 that $R_1p = p$ and hence $U_1p = p$.

Again by (3.5), we get

$$\|p-z\| = \beta_N\beta_{N-1}\cdots\beta_3\big\|\beta_2(R_2U_1p-z)+(1-\beta_2)(p-z)\big\| + (1-\beta_N\beta_{N-1}\cdots\beta_3)\|p-z\|,$$

and hence

$$\|p-z\| = \big\|\beta_2(R_2U_1p-z)+(1-\beta_2)(p-z)\big\|. \tag{3.6}$$

By (3.5), we get

$$\|p-z\| = \|R_2U_1p-z\|.$$

From $U_1p = p$ and (3.6), we have

$$\|p-z\| = \|R_2p-z\| = \big\|\beta_2(R_2U_1p-z)+(1-\beta_2)(p-z)\big\|.$$

This implies by Lemma 2.7 that $R_2p = p$ and hence $U_2p = p$.

By continuing this process, we can conclude that $R_ip = p$ and $U_ip = p$ for all $i = 1,2,\dots,N-1$. Since

$$
\begin{aligned}
\|p-R_Np\| &\leq \|p-Wp\| + \|Wp-R_Np\| \\
&= \|p-Wp\| + (1-\beta_N)\|p-R_Np\|,
\end{aligned}
$$

which yields that $p = R_N p$ since $p \in F(W)$. Hence $p = R_i p$ for all $i = 1, 2, \dots, N$ and thus $p \in \bigcap_{i=1}^N F(R_i)$. From Remark 2.9, we have

$$F(t_i) = F\bigl(P_D\bigl(I - \lambda(I - t_i)\bigr)\bigr) = F(R_i) \quad \text{for all } i = 1, 2, \dots, N.$$

This implies that $\bigcap_{i=1}^N F(R_i) = \bigcap_{i=1}^N F(t_i)$, and hence $p \in \bigcap_{i=1}^N F(t_i)$. Therefore, $F(W) = \bigcap_{i=1}^N F(t_i)$. This completes the proof. □

We now prove our main theorem.

Theorem 3.2 *Let D be a nonempty closed convex subset of a real Hilbert space X. Let $\{t_i\}_{i=1}^N$ be a finite family of continuous and k-strictly pseudononspreading mappings of D into itself, let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L-Lipschitzian mappings from D into $CB(D)$ with $T_i p = \{p\}$ for all $i = 1, 2, \dots, N$, $p \in \bigcap_{i=1}^N F(T_i)$, and let $\{B_i\}_{i=1}^N$ be a finite family of ϕ_i-inverse strongly monotone mappings from D into X. Let $R_i : D \to D$ be defined by $R_i x = P_D(I - \lambda(I - t_i))x$ for all $x \in D$, $\lambda \in (0, 1)$, and $i = 1, 2, \dots, N$. Suppose that $\beta_1, \beta_2, \dots, \beta_N$ are real numbers such that $0 < \beta_i < 1$ for all $i = 1, 2, \dots, N-1$ and $0 < \beta_N \le 1$. Let $W : D \to D$ be the W-mapping generated by $R_1, R_2, \dots, R_N$ and $\beta_1, \beta_2, \dots, \beta_N$. Let $G_i : D \to D$ be defined by $G_i x = P_D(I - \eta B_i)x$ for all $x \in D$, $\eta \in (0, 2\phi_i)$, and $i = 1, 2, \dots, N$. Suppose $\delta_j = (\delta_1^j, \delta_2^j, \delta_3^j) \in [0,1] \times [0,1] \times [0,1]$, $j = 1, 2, \dots, N$, where $\delta_1^j + \delta_2^j + \delta_3^j = 1$, $\delta_1^j \in (0, 1)$ for all $j = 1, 2, \dots, N-1$, $\delta_1^N \in (0, 1]$, and $\delta_2^j, \delta_3^j \in [0, 1)$ for all $j = 1, 2, \dots, N$. Let $S : D \to D$ be the S-mapping generated by $G_1, G_2, \dots, G_N$ and $\delta_1, \delta_2, \dots, \delta_N$. Assume that $\mathcal{F} := \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N VI(D, B_i) \neq \emptyset$. Let $x_1 \in D$ with $C_1 = D$, and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences defined by*

$$\begin{aligned}
&y_n = \alpha_n^{(1)} x_n + \alpha_n^{(2)} W x_n + \alpha_n^{(3)} S x_n, \\
&z_n = \gamma_n^{(0)} y_n + \sum_{i=1}^N \gamma_n^{(i)} q_n^{(i)}, \quad q_n^{(i)} \in T_i y_n, \\
&C_{n+1} = \bigl\{p \in C_n : \|z_n - p\| \le \|y_n - p\| \le \|x_n - p\|\bigr\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{aligned} \tag{3.7}$$

where $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}, \{\gamma_n^{(i)}\}$ $(i = 0, 1, \dots, N)$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\lim_{n\to\infty} \alpha_n^{(1)} = 0$, *and* $0 < a \le \alpha_n^{(2)}, \alpha_n^{(3)} < 1$;

(ii) $0 < b \le \gamma_n^{(i)} < 1$ *for all* $i = 0, 1, \dots, N$ *and* $\sum_{i=0}^N \gamma_n^{(i)} = 1$.

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $u = P_{\mathcal{F}} x_1$.

Proof We shall divide our proof into 6 steps.

Step 1. We show that $P_{C_{n+1}} x_1$ is well defined for every $x_1 \in X$.

Let $x, y \in X$. Since B_i is a ϕ_i-inverse strongly monotone mapping and $\eta \in (0, 2\phi_i)$, for $i = 1, 2, \dots, N$, we get that

$$\begin{aligned}
\|G_i x - G_i y\|^2 &= \bigl\|P_C(I - \eta B_i)x - P_C(I - \eta B_i)y\bigr\|^2 \\
&\le \bigl\|(x - y) - \eta(B_i x - B_i y)\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x-y\|^2 - 2\eta\langle x-y, B_ix - B_iy\rangle + \eta^2\|B_ix - B_iy\|^2 \\
&\leq \|x-y\|^2 - 2\eta\phi_i\|B_ix - B_iy\|^2 + \eta^2\|B_ix - B_iy\|^2 \\
&= \|x-y\|^2 - \eta(2\phi_i - \eta)\|B_ix - B_iy\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}
$$

This shows that $G_i = P_D(I - \eta B_i)$ is a nonexpansive mapping for all $i = 1,2,\dots,N$. By Lemma 2.1, the closedness and convexity of $F(G_i)$, we have that $VI(D,B_i) = F(P_D(I-\eta B_i)) = F(G_i)$ is closed and convex for all $i = 1,2,\dots,N$. So, $\bigcap_{i=1}^N VI(D,B_i)$ is closed and convex. By Lemmas 1.1 and 1.2, we also know that $\bigcap_{i=1}^N F(T_i)$ and $\bigcap_{i=1}^N F(t_i)$ are closed and convex. Hence, $\mathcal{F} := \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N VI(D,B_i)$ is also closed and convex. By Lemma 2.6, we observe that C_n is closed and convex. Let $p \in \mathcal{F}$. Since G_i is nonexpansive and t_i is k-strictly pseudononspreading for all $i = 1,2,\dots,N$, it implies by Lemmas 1.3 and 3.1 that $p \in F(S)$ and $p \in F(W)$. So, we have

$$
\begin{aligned}
\|z_n - p\| &= \left\| \gamma_n^{(0)} y_n + \sum_{i=1}^N \gamma_n^{(i)} q_n^{(i)} - p \right\| \\
&\leq \gamma_n^{(0)}\|y_n - p\| + \sum_{i=1}^N \gamma_n^{(i)} \left\| q_n^{(i)} - p \right\| \\
&\leq \gamma_n^{(0)}\|y_n - p\| + \sum_{i=1}^N \gamma_n^{(i)} H(T_iy_n, T_ip) \\
&\leq \gamma_n^{(0)}\|y_n - p\| + \sum_{i=1}^N \gamma_n^{(i)} \|y_n - p\| \\
&= \|y_n - p\| \\
&= \left\| \alpha_n^{(1)} x_n + \alpha_n^{(2)} Wx_n + \alpha_n^{(3)} Sx_n - p \right\| \\
&\leq \alpha_n^{(1)}\|x_n - p\| + \alpha_n^{(2)}\|Wx_n - p\| + \alpha_n^{(3)}\|Sx_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}
$$

This shows that $p \in C_{n+1}$ and hence $\mathcal{F} \subset C_{n+1} \subset C_n$. Therefore, $P_{C_{n+1}}x_1$ is well defined.

Step 2. We show that $\lim_{n\to\infty} x_n = q$ for some $q \in D$.

Since $\mathcal{F}$ is a nonempty closed convex subset of a real Hilbert space X, there exists a unique $\nu \in \mathcal{F}$ such that $\nu = P_{\mathcal{F}}x_1$. From $x_n = P_{C_n}x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, for all $n \in \mathbb{N}$, we get that

$$
\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \quad \text{for all } n \in \mathbb{N}.
$$

On the other hand, by $\mathcal{F} \subset C_n$, we obtain that

$$
\|x_n - x_1\| \leq \|\nu - x_1\| \quad \text{for all } n \in \mathbb{N}.
$$

This implies that $\{x_n\}$ is bounded and nondecreasing. So, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. For $m > n \in \mathbb{N}$, we have $x_m = P_{C_m}x_1 \in C_m \subset C_n$. It implies by Lemma 2.3 that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2 \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, it implies that $\{x_n\}$ is a Cauchy sequence. Hence, there exists an element $q \in D$ such that $\lim_{n\to\infty} x_n = q$.

Step 3. We show that $q \in \bigcap_{i=1}^N F(T_i)$.

From Step 2, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Since $x_{n+1} \in C_{n+1}$, we get that

$$\begin{aligned}\|z_n - x_n\| &\leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &= 2\|x_{n+1} - x_n\|\end{aligned}$$

and

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|.\end{aligned}$$

This implies by (3.8) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.10}$$

Thus, $\lim_{n\to\infty} z_n = q$ and $\lim_{n\to\infty} y_n = q$.

Let $p \in \mathcal{F}$. By Lemma 2.5 and the definition of z_n, for each $j = 1, 2, \ldots, N$, we have

$$\begin{aligned}\|z_n - p\|^2 &= \left\|\gamma_n^{(0)} y_n + \sum_{i=1}^N \gamma_n^{(i)} q_n^{(i)} - p\right\|^2 \\ &= \left\|\gamma_n^{(0)}(y_n - p) + \sum_{i=1}^N \gamma_n^{(i)}\bigl(q_n^{(i)} - p\bigr)\right\|^2 \\ &\leq \gamma_n^{(0)}\|y_n - p\|^2 + \sum_{i=1}^N \gamma_n^{(i)}\bigl\|q_n^{(i)} - p\bigr\|^2 - \gamma_n^{(0)}\gamma_n^{(j)}\bigl\|q_n^{(j)} - y_n\bigr\|^2 \\ &\leq \gamma_n^{(0)}\|y_n - p\|^2 + \sum_{i=1}^N \gamma_n^{(i)}\bigl[H(T_i y_n, T_i p)\bigr]^2 - \gamma_n^{(0)}\gamma_n^{(j)}\bigl\|q_n^{(j)} - y_n\bigr\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_n^{(0)}\|y_n-p\|^2+\sum_{i=1}^{N}\gamma_n^{(i)}\|y_n-p\|^2-\gamma_n^{(0)}\gamma_n^{(j)}\big\|q_n^{(j)}-y_n\big\|^2\\
&=\|y_n-p\|^2-\gamma_n^{(0)}\gamma_n^{(j)}\big\|q_n^{(j)}-y_n\big\|^2\\
&\leq \|x_n-p\|^2-\gamma_n^{(0)}\gamma_n^{(j)}\big\|q_n^{(j)}-y_n\big\|^2.
\end{aligned}
$$

By condition (ii), it implies that

$$
\begin{aligned}
b^2\big\|q_n^{(j)}-y_n\big\|^2 &\leq \gamma_n^{(0)}\gamma_n^{(j)}\big\|q_n^{(j)}-y_n\big\|^2\\
&\leq \|x_n-p\|^2-\|z_n-p\|^2\\
&\leq \|x_n-z_n\|\big(\|x_n-z_n\|+\|z_n-p\|\big).
\end{aligned}
$$

Thus, by (3.9), we have

$$
\lim_{n\to\infty}\big\|q_n^{(j)}-y_n\big\|=0 \quad \text{for all } j=1,2,\ldots,N. \tag{3.11}
$$

For each $i=1,2,\ldots,N$, we get

$$
\begin{aligned}
\operatorname{dist}(q,T_iq) &\leq \|q-y_n\|+\big\|y_n-q_n^{(i)}\big\|+\operatorname{dist}\big(q_n^{(i)},T_iq\big)\\
&\leq \|q-y_n\|+\big\|y_n-q_n^{(i)}\big\|+H(T_iy_n,T_iq)\\
&\leq \|q-y_n\|+\big\|y_n-q_n^{(i)}\big\|+L\|y_n-q\|\\
&=(1+L)\|q-y_n\|+\big\|y_n-q_n^{(i)}\big\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}y_n=q$, it implies by (3.11) that

$$
\operatorname{dist}(q,T_iq)=0 \quad \text{for all } i=1,2,\ldots,N.
$$

This shows that $q\in T_iq$ for all $i=1,2,\ldots,N$, and hence $q\in\bigcap_{i=1}^{N}F(T_i)$.

Step 4. We show that $q\in\bigcap_{i=1}^{N}VI(D,B_i)$.

For $p\in\mathcal{F}$, we have

$$
\begin{aligned}
\|y_n-p\|^2 &\leq \alpha_n^{(1)}\|x_n-p\|^2+\alpha_n^{(2)}\|Wx_n-p\|^2+\alpha_n^{(3)}\|Sx_n-p\|^2\\
&\quad -\alpha_n^{(2)}\alpha_n^{(3)}\|Wx_n-Sx_n\|^2\\
&\leq \|x_n-p\|^2-\alpha_n^{(2)}\alpha_n^{(3)}\|Wx_n-Sx_n\|^2.
\end{aligned}
$$

This implies by condition (i) that

$$
\begin{aligned}
a^2\|Wx_n-Sx_n\|^2 &\leq \alpha_n^{(2)}\alpha_n^{(3)}\|Wx_n-Sx_n\|^2\\
&\leq \|x_n-p\|^2-\|y_n-p\|^2\\
&\leq \|x_n-y_n\|\big(\|x_n-p\|+\|y_n-p\|\big).
\end{aligned}
$$

Then, by (3.10), we have

$$
\lim_{n\to\infty}\|Wx_n-Sx_n\|=0. \tag{3.12}
$$

Since

$$\begin{aligned}\|y_n - Wx_n\| &= \left\|\alpha_n^{(1)}x_n + \alpha_n^{(2)}Wx_n + \alpha_n^{(3)}Sx_n - Wx_n\right\| \\ &= \left\|\alpha_n^{(1)}(x_n - Wx_n) + \alpha_n^{(3)}(Sx_n - Wx_n)\right\| \\ &\le \alpha_n^{(1)}\|x_n - Wx_n\| + \alpha_n^{(3)}\|Sx_n - Wx_n\|,\end{aligned}$$

it follows by condition (i) and (3.12) that

$$\lim_{n\to\infty}\|y_n - Wx_n\| = 0. \tag{3.13}$$

From (3.10) and (3.13), we obtain that

$$\|x_n - Wx_n\| \le \|x_n - y_n\| + \|y_n - Wx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Since $y_n - x_n = \alpha_n^{(2)}(Wx_n - x_n) + \alpha_n^{(3)}(Sx_n - x_n)$ and $0 < a < \alpha_n^{(3)} < 1$, we get

$$a\|Sx_n - x_n\| \le \alpha_n^{(3)}\|Sx_n - x_n\| \le \|y_n - x_n\| + \alpha_n^{(2)}\|Wx_n - x_n\|.$$

This implies by (3.10) and (3.14) that

$$\lim_{n\to\infty}\|Sx_n - x_n\| = 0. \tag{3.15}$$

Since $x_n \to q \in D$ as $n \to \infty$, it follows by (3.15) and the nonexpansiveness of S that

$$\begin{aligned}\|Sq - q\| &\le \|Sq - Sx_n\| + \|Sx_n - x_n\| + \|x_n - q\| \\ &\le 2\|x_n - q\| + \|Sx_n - x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

This shows that $q \in F(S)$. Since $P_D(I - \eta B_i)x = G_ix$ for all $x \in D$ and $i = 1, 2, \ldots, N$, by Lemma 2.1, we have $VI(D, B_i) = F(P_D(I - \eta B_i)) = F(G_i)$ for all $i = 1, 2, \ldots, N$. By Lemma 1.3, we obtain

$$F(S) = \bigcap_{i=1}^{N} F(G_i) = \bigcap_{i=1}^{N} VI(D, B_i).$$

Thus, $q \in \bigcap_{i=1}^{N} VI(D, B_i)$.

Step 5. We show that $q \in \bigcap_{i=1}^{N} F(t_i)$.

Since t_i is continuous for all $i = 1, 2, \ldots, N$, it follows that $P_D(I - \lambda(I - t_i))$ is continuous for all $i = 1, 2, \ldots, N$. So, W is continuous. This implies by $x_n \to q$ that $Wx_n \to Wq$ as $n \to \infty$. Then, by (3.14), we have

$$\|Wq - q\| \le \|Wq - Wx_n\| + \|Wx_n - x_n\| + \|x_n - q\| \to 0 \quad \text{as } n \to \infty.$$

This shows that $q \in F(W)$. By Lemma 3.1, we have $q \in \bigcap_{i=1}^{N} F(t_i)$.

Step 6. Finally, we show that $q = u = P_{\mathcal{F}}x_1$.

Since $x_n = P_{C_n}x_1$ and $\mathcal{F} \subset C_n$, we obtain

$$\langle x_1 - x_n, x_n - p\rangle \geq 0 \quad \text{for all } p \in \mathcal{F}. \tag{3.16}$$

Taking limits in the above inequality, we get

$$\langle x_1 - q, q - p\rangle \geq 0 \quad \text{for all } p \in \mathcal{F}.$$

This shows that $q = P_{\mathcal{F}}x_1 = u$.

By Step 1 to Step 6, we conclude that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$. This completes the proof. □

As a direct consequence of Theorem 3.2, we have the following two corollaries.

Corollary 3.3 *Let D be a nonempty closed convex subset of a real Hilbert space X. Let $\{t_i\}_{i=1}^N$ be a finite family of continuous and k-strictly pseudononspreading mappings of D into itself, and let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L-Lipschitzian mappings from D into $CB(D)$ with $T_ip = \{p\}$ for all $i = 1,2,\dots,N$, $p \in \bigcap_{i=1}^N F(T_i)$. Let $R_i : D \to D$ be defined by $R_ix = P_D(I - \lambda(I - t_i))x$ for all $x \in D$, $\lambda \in (0,1)$, and $i = 1,2,\dots,N$. Suppose that $\beta_1, \beta_2, \dots, \beta_N$ are real numbers such that $0 < \beta_i < 1$ for all $i = 1,2,\dots,N-1$ and $0 < \beta_N \leq 1$. Let $W : D \to D$ be the W-mapping generated by $R_1, R_2, \dots, R_N$ and $\beta_1, \beta_2, \dots, \beta_N$. Assume that $\mathcal{F} := \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_1 \in D$ with $C_1 = D$, and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences defined by*

$$\begin{aligned}
&y_n = \bigl(\alpha_n^{(1)} + \alpha_n^{(3)}\bigr)x_n + \alpha_n^{(2)}Wx_n, \\
&z_n = \gamma_n^{(0)}y_n + \sum_{i=1}^N \gamma_n^{(i)}q_n^{(i)}, \quad q_n^{(i)} \in T_iy_n, \\
&C_{n+1} = \bigl\{p \in C_n : \|z_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|\bigr\}, \\
&x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N},
\end{aligned}$$

where $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\gamma_n^{(i)}\}$ $(i = 0,1,\dots,N)$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\lim_{n\to\infty}\alpha_n^{(1)} = 0$, *and* $0 < a \leq \alpha_n^{(2)}, \alpha_n^{(3)} < 1$;

(ii) $0 < b \leq \gamma_n^{(i)} < 1$ *for all* $i = 0,1,\dots,N$ *and* $\sum_{i=0}^N \gamma_n^{(i)} = 1$.

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $u = P_{\mathcal{F}}x_1$.

Proof Let $B_ix = 0$ for all $x \in D$ and $i = 1,2,\dots,N$ in Theorem 3.2. Then we obtain that $Sx_n = x_n$ for all $n \in \mathbb{N}$. Therefore the conclusion follows. □

Corollary 3.4 *Let D be a nonempty closed convex subset of a real Hilbert space X. Let $t : D \to D$ be a continuous and k-strictly pseudononspreading mapping, let $T : D \to CB(D)$ be a quasi-nonexpansive and L-Lipschitzian mapping with $Tp = \{p\}$ for all $p \in F(T)$, and let $B : D \to X$ be a ϕ-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(t) \cap F(T) \cap VI(D,B) \neq \emptyset$. Let $x_1 \in D$ with $C_1 = D$, and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences defined by*

$$y_n = \alpha_n^{(1)}x_n + \alpha_n^{(2)}P_D\bigl(I - \lambda(I - t)\bigr)x_n + \alpha_n^{(3)}P_D(I - \eta B)x_n,$$

$$z_n = \gamma_n^{(0)} y_n + \gamma_n^{(1)} q_n, \quad q_n \in T y_n,$$
$$C_{n+1} = \{p \in C_n : \|z_n - p\| \le \|y_n - p\| \le \|x_n - p\|\},$$
$$x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N},$$

where $\lambda \in (0,1)$, $\eta \in (0,2\phi)$, *and* $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\gamma_n^{(0)}\}$, $\{\gamma_n^{(1)}\}$ *are sequences in* $(0,1)$ *satisfying the following conditions*:

(i) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\lim_{n\to\infty} \alpha_n^{(1)} = 0$, *and* $0 < a \le \alpha_n^{(2)}, \alpha_n^{(3)} < 1$;

(ii) $0 < b \le \gamma_n^{(0)}, \gamma_n^{(1)} < 1$ *and* $\gamma_n^{(0)} + \gamma_n^{(1)} = 1$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to* $u = P_{\mathcal{F}} x_1$.

Proof In Theorem 3.2, put $N = 1$, $t_1 = t$, $T_1 = T$, $B_1 = B$, $\beta_1 = 1$, and $\delta_1^1 = 1$. Then $W = P_D(I - \lambda(I - t))$ and $S = P_D(I - \eta B)$. Hence, we obtain the desired result from Theorem 3.2. □

Remark 3.5 It is known that the class of k-strictly pseudononspreading mappings contains the classes of nonexpansive mappings and nonspreading mappings. Thus, Lemma 3.1, Theorem 3.2, Corollaries 3.3 and 3.4 can be applied to these classes of mappings.

4 Applications to complementarity problems

In this section, we apply our results to complementarity problems in Hilbert spaces. Let D be a nonempty *closed convex cone* in a real Hilbert space X, *i.e.*, a nonempty closed set with $rD + sD \subset D$ for all $r, s \in [0,\infty)$. The *polar* of D is the set

$$D^* = \{y \in X : \langle x, y\rangle \ge 0, \forall x \in D\}.$$

Let $B : D \to X$ be a nonlinear mapping. The *complementarity problem* is to find an element $u \in D$ such that

$$Bu \in D^* \quad \text{and} \quad \langle u, Bu\rangle = 0. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $CP(D,B)$.

A complementarity problem is a special case of a variational inequality problem. The following lemma indicates the equivalence between the complementarity problem and the variational inequality problem. The proof of this fact can be found in [42]; for convenience of the readers, we include the details.

Lemma 4.1 *Let D be a nonempty closed convex cone in a real Hilbert space X, and let D^* be the polar of D. Let B be a mapping of D into X. Then $VI(D,B) = CP(D,B)$.*

Proof Let $x \in VI(D,B)$. Then we have

$$\langle u - x, Bx\rangle \ge 0 \quad \text{for all } u \in D. \tag{4.2}$$

In particular, if $u = 0$, we have $\langle x, Bx\rangle \le 0$. If $u = \lambda x$ with $\lambda > 1$, we have $\langle \lambda x - x, Bx\rangle = (\lambda - 1)\langle x, Bx\rangle \ge 0$ and hence $\langle x, Bx\rangle \ge 0$. Therefore, $\langle x, Bx\rangle = 0$. Next, we show that $Bx \in D^*$.

To show this, suppose not. Then there exists $u_0 \in D$ such that $\langle u_0, Bx\rangle < 0$. By (4.2), we obtain

$$\langle u_0 - x, Bx\rangle \geq 0.$$

So, we get

$$0 > \langle u_0, Bx\rangle \geq \langle x, Bx\rangle = 0.$$

This is a contradiction. Thus, $Bx \in D^*$. So, $x \in CP(D,B)$.

Conversely, let $x \in CP(D,B)$. Then we have

$$Bx \in D^* \quad \text{and} \quad \langle x, Bx\rangle = 0.$$

For any $u \in D$, we get

$$\begin{aligned}\langle u - x, Bx\rangle &= \langle u, Bx\rangle - \langle x, Bx\rangle \\ &= \langle u, Bx\rangle \geq 0.\end{aligned}$$

Thus, $x \in VI(D,B)$. This completes the proof. □

Theorem 4.2 *Let D be a nonempty closed convex cone in a real Hilbert space X. Let $\{t_i\}_{i=1}^N$ be a finite family of continuous and k-strictly pseudononspreading mappings of D into itself, let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L-Lipschitzian mappings from D into $CB(D)$ with $T_ip = \{p\}$ for all $i = 1,2,\dots,N$, $p \in \bigcap_{i=1}^N F(T_i)$, and let $\{B_i\}_{i=1}^N$ be a finite family of ϕ_i-inverse strongly monotone mappings from D into X. Let $R_i : D \to D$ be defined by $R_ix = P_D(I - \lambda(I - t_i))x$ for all $x \in D$, $\lambda \in (0,1)$, and $i = 1,2,\dots,N$. Suppose that $\beta_1, \beta_2, \dots, \beta_N$ are real numbers such that $0 < \beta_i < 1$ for all $i = 1,2,\dots,N-1$ and $0 < \beta_N \leq 1$. Let $W : D \to D$ be the W-mapping generated by $R_1, R_2, \dots, R_N$ and $\beta_1, \beta_2, \dots, \beta_N$. Let $G_i : D \to D$ be defined by $G_ix = P_D(I - \eta B_i)x$ for all $x \in D$, $\eta \in (0, 2\phi_i)$, and $i = 1,2,\dots,N$. Suppose $\delta_j = (\delta_1^j, \delta_2^j, \delta_3^j) \in [0,1] \times [0,1] \times [0,1]$, $j = 1,2,\dots,N$, where $\delta_1^j + \delta_2^j + \delta_3^j = 1$, $\delta_1^j \in (0,1)$ for all $j = 1,2,\dots,N-1$, $\delta_1^N \in (0,1]$, and $\delta_2^j, \delta_3^j \in [0,1)$ for all $j = 1,2,\dots,N$. Let $S : D \to D$ be the S-mapping generated by $G_1, G_2, \dots, G_N$ and $\delta_1, \delta_2, \dots, \delta_N$. Assume that $\mathcal{F} := \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N CP(D, B_i) \neq \emptyset$. Let $x_1 \in D$ with $C_1 = D$, and let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences defined by*

$$\begin{aligned}
&y_n = \alpha_n^{(1)} x_n + \alpha_n^{(2)} W x_n + \alpha_n^{(3)} S x_n, \\
&z_n = \gamma_n^{(0)} y_n + \sum_{i=1}^N \gamma_n^{(i)} q_n^{(i)}, \quad q_n^{(i)} \in T_i y_n, \\
&C_{n+1} = \bigl\{p \in C_n : \|z_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|\bigr\}, \\
&x_{n+1} = P_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{aligned}$$

where $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\gamma_n^{(i)}\}$ $(i = 0,1,\dots,N)$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n^{(1)} + \alpha_n^{(2)} + \alpha_n^{(3)} = 1$, $\lim_{n\to\infty} \alpha_n^{(1)} = 0$, *and* $0 < a \le \alpha_n^{(2)}, \alpha_n^{(3)} < 1$;
(ii) $0 < b \le \gamma_n^{(i)} < 1$ *for all* $i = 0, 1, \ldots, N$ *and* $\sum_{i=0}^{N} \gamma_n^{(i)} = 1$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to* $u = P_{\mathcal{F}} x_1$.

Proof By Lemma 4.1, we get that

$$VI(D, B_i) = CP(D, B_i) \quad \text{for all } i = 1, 2, \ldots, N.$$

Then we obtain the result. □

5 Numerical results

In this section, we give two numerical examples to support our main result.

Example 5.1 We consider the nonempty closed convex subset $D = [0, 5]$ of the real Hilbert space $\mathbb{R}$. Define two single-valued mappings t_1 and t_2 on $[0, 5]$ as follows:

$$t_1 x = -\frac{5}{7} x, \qquad t_2 x = -\frac{9}{11} x.$$

Also we define two multi-valued mappings T_1 and T_2 on $[0, 5]$ as follows:

$$T_1 x = \left[\frac{x}{6}, \frac{x}{2}\right], \qquad T_2 x = \left[0, \frac{x}{5}\right].$$

For $i = 1, 2$, let $B_i : [0, 5] \to [0, 5] \subset \mathbb{R}$ be defined by

$$B_i x = \frac{x}{15} i.$$

Let W be the W-mapping generated by R_1, R_2 and β_1, β_2, where $R_i x = P_{[0,5]}(I - \frac{1}{2}(I - t_i))x$ for all $i = 1, 2$, and $\beta_1 = \beta_2 = \frac{1}{2}$. Let S be the S-mapping generated by G_1, G_2 and δ_1, δ_2, where $G_i x = P_{[0,5]}(I - \frac{1}{2} B_i)x$ for all $i = 1, 2$, and $\delta_1 = \delta_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by (3.7), where $\alpha_n^{(1)} = \frac{1}{10n}$, $\alpha_n^{(2)} = \frac{10n-1}{30n}$, $\alpha_n^{(3)} = \frac{10n-1}{15n}$, $\gamma_n^{(0)} = \frac{1}{5} + \frac{4}{75n}$, $\gamma_n^{(1)} = \frac{15n-1}{75n}$, $\gamma_n^{(2)} = \frac{15n-1}{25n}$ for all $n \in \mathbb{N}$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(T_i) \cap \bigcap_{i=1}^{2} VI([0,5], B_i)$.

Solution. It is easy to see that t_1, t_2 are k-strictly pseudononspreading, T_1, T_2 are quasi-nonexpansive and Lipschitzian, and B_1, B_2 are inverse strongly monotone. Obviously, $\{T_i\}_{i=1}^{2}$ satisfies the condition $T_i p = \{p\}$ for all $i = 1, 2$, $p \in \bigcap_{i=1}^{2} F(T_i)$ since $\bigcap_{i=1}^{2} F(T_i) = \{0\}$. From the definitions of these mappings, we get that

$$\bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(T_i) \cap \bigcap_{i=1}^{2} VI\bigl([0,5], B_i\bigr) = \{0\}.$$

For every $n \in \mathbb{N}$, $\alpha_n^{(1)} = \frac{1}{10n}$, $\alpha_n^{(2)} = \frac{10n-1}{30n}$, $\alpha_n^{(3)} = \frac{10n-1}{15n}$, $\gamma_n^{(0)} = \frac{1}{5} + \frac{4}{75n}$, $\gamma_n^{(1)} = \frac{15n-1}{75n}$, $\gamma_n^{(2)} = \frac{15n-1}{25n}$. Then the sequences $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\gamma_n^{(0)}\}$, $\{\gamma_n^{(1)}\}$, and $\{\gamma_n^{(2)}\}$ satisfy all the conditions of Theorem 3.2. For any arbitrary $x_1 \in C_1 = [0, 5]$, it follows by (3.7) that $0 \le z_1 \le y_1 \le x_1 \le 5$. Then we have

$$C_2 = \bigl\{p \in C_1 : |z_1 - p| \le |y_1 - p| \le |x_1 - p|\bigr\} = \left[0, \frac{z_1 + y_1}{2}\right].$$

Since $\frac{z_1+y_1}{2} \le x_1$, we get

$$x_2 = P_{C_2}x_1 = \frac{z_1+y_1}{2}.$$

By continuing this process, we obtain that

$$C_{n+1} = \{p \in C_n : |z_n - p| \le |y_n - p| \le |x_n - p|\} = \left[0, \frac{z_n+y_n}{2}\right],$$

and hence

$$x_{n+1} = P_{C_{n+1}}x_1 = \frac{z_n+y_n}{2}.$$

Now, we have the following algorithm:

$$\begin{aligned} &x_1 \in [0,5], \\ &y_n = \frac{1}{10n}x_n + \frac{10n-1}{30n}Wx_n + \frac{10n-1}{15n}Sx_n, \\ &z_n = \left(\frac{1}{5} + \frac{4}{75n}\right)y_n + \frac{15n-1}{75n}q_n^{(1)} + \frac{15n-1}{25n}q_n^{(2)}, \quad q_n^{(i)} \in T_iy_n, i = 1,2, \\ &x_{n+1} = \frac{z_n+y_n}{2}, \quad n \in \mathbb{N}. \end{aligned} \tag{5.1}$$

Since W is the W-mapping generated by R_1, R_2 and β_1, β_2, where $R_ix = P_{[0,5]}(I - \frac{1}{2}(I - t_i))x$ for all $i = 1,2$ and $\beta_1 = \beta_2 = \frac{1}{2}$, and S is the S-mapping generated by G_1, G_2 and δ_1, δ_2, where $G_ix = P_{[0,5]}(I - \frac{1}{2}B_i)x$ for all $i = 1,2$ and $\delta_1 = \delta_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we obtain that

$$Wx = \frac{81}{154}x, \qquad Sx = \frac{3{,}931}{4{,}050}x \quad \text{for all } x \in [0,5].$$

Put $q_n^{(1)} = \frac{y_n}{6}$ and $q_n^{(2)} = \frac{y_n}{7}$. Then we rewrite (5.1) as follows:

$$\begin{aligned} &x_1 \in [0,5], \\ &y_n = \left(\frac{769{,}399}{935{,}550} + \frac{166{,}151}{9{,}355{,}500n}\right)x_n, \\ &z_n = \left(\frac{67}{210} + \frac{143}{3{,}150n}\right)y_n, \\ &x_{n+1} = \frac{z_n+y_n}{2}, \quad n \in \mathbb{N}. \end{aligned} \tag{5.2}$$

Using algorithm (5.2) with the initial point $x_1 = 4.5$, we have numerical results in Table 1.

From Table 1, we see that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge to 0. We observe that $x_{27} = 0.0000007$ is an approximation of the common element in $\bigcap_{i=1}^2 F(t_i) \cap \bigcap_{i=1}^2 F(T_i) \cap \bigcap_{i=1}^2 VI([0,5], B_i)$ with accuracy at 6 significant digits.

Next, we give the numerical example to support our main theorem in a two-dimensional space of real numbers.

Table 1 The values of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in Example 5.1

n	x_n	y_n	z_n
1	4.5000000	3.7807316	1.3778666
2	2.5792991	2.1441268	0.7327468
3	1.4384368	1.1914900	0.3981720
4	0.7948310	0.6572003	0.2171369
5	0.4371686	0.3610815	0.1184806
⋮	⋮	⋮	⋮
16	0.0005599	0.0004611	0.0001484
⋮	⋮	⋮	⋮
23	0.0000079	0.0000065	0.0000021
24	0.0000043	0.0000035	0.0000011
25	0.0000023	0.0000019	0.0000006
26	0.0000013	0.0000010	0.0000003
27	0.0000007	0.0000006	0.0000002

Example 5.2 Let $\mathbf{x} = (x^{(1)}, x^{(2)})$, $\mathbf{y} = (y^{(1)}, y^{(2)}) \in \mathbb{R}^2$, and let the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle \mathbf{x}, \mathbf{y} \rangle = x^{(1)}y^{(1)} + x^{(2)}y^{(2)}$ and the usual norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{(x^{(1)})^2 + (x^{(2)})^2}$. We consider the nonempty closed convex subset $D = [0, 100] \times [0, 100]$ of the real Hilbert space $\mathbb{R}^2$. Define three single-valued mappings t_1, t_2, and t_3 on D as follows:

$$t_1\mathbf{x} = \left(-\frac{3}{5}x^{(1)}, -\frac{5}{7}x^{(2)}\right), \qquad t_2\mathbf{x} = \left(-\frac{5}{7}x^{(1)}, -\frac{9}{11}x^{(2)}\right), \qquad t_3\mathbf{x} = \left(-\frac{7}{9}x^{(1)}, -\frac{15}{17}x^{(2)}\right).$$

Define three multi-valued mappings T_1, T_2, and T_3 on D as follows:

$$T_1\mathbf{x} = \left[\frac{\mathbf{x}}{12}, \frac{\mathbf{x}}{4}\right], \qquad T_2\mathbf{x} = \left[0, \frac{\mathbf{x}}{5}\right], \qquad T_3\mathbf{x} = \left[\frac{\mathbf{x}}{2}, \frac{15\mathbf{x}}{17}\right].$$

For $i = 1, 2, 3$, let $B_i : D \to D \subset \mathbb{R}^2$ be defined by

$$B_i\mathbf{x} = \left(\frac{ix^{(1)}}{30}, \frac{ix^{(2)}}{30}\right).$$

Let W be the W-mapping generated by R_1, R_2, R_3 and β_1, β_2, β_3, where $R_i\mathbf{x} = P_D(I - \frac{1}{2}(I - t_i))\mathbf{x}$ for all $i = 1, 2, 3$, and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{2}$. Let S be the S-mapping generated by G_1, G_2, G_3 and δ_1, δ_2, δ_3, where $G_i\mathbf{x} = P_D(I - \frac{1}{2}B_i)\mathbf{x}$ for all $i = 1, 2, 3$, and $\delta_1 = \delta_2 = \delta_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let $\mathbf{x}_n = (x_n^{(1)}, x_n^{(2)})$, $\mathbf{y}_n = (y_n^{(1)}, y_n^{(2)})$, and $\mathbf{z}_n = (z_n^{(1)}, z_n^{(2)})$ and the sequences $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$, and $\{\mathbf{z}_n\}$ be generated by (3.7), where $\alpha_n^{(1)} = \frac{1}{12n}$, $\alpha_n^{(2)} = \frac{12n-1}{36n}$, $\alpha_n^{(3)} = \frac{12n-1}{18n}$, $\gamma_n^{(0)} = \frac{1}{5} + \frac{7}{80n}$, $\gamma_n^{(1)} = \frac{16n-1}{80n}$, $\gamma_n^{(2)} = \gamma_n^{(3)} = \frac{24n-3}{80n}$ for all $n \in \mathbb{N}$. Then the sequences $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$, and $\{\mathbf{z}_n\}$ converge strongly to $(0, 0)$, where $\{(0, 0)\} = \bigcap_{i=1}^{3} F(t_i) \cap \bigcap_{i=1}^{3} F(T_i) \cap \bigcap_{i=1}^{3} VI(D, B_i)$.

Solution. It is obvious that t_1, t_2, t_3 are k-strictly pseudononspreading, T_1, T_2, T_3 are quasi-nonexpansive and Lipschitzian, and B_1, B_2, B_3 are inverse strongly monotone. Also, $\{T_i\}_{i=1}^{3}$ satisfies the condition $T_i\mathbf{p} = \{\mathbf{p}\}$ for all $i = 1, 2, 3$, $\mathbf{p} \in \bigcap_{i=1}^{3} F(T_i)$ since $\bigcap_{i=1}^{3} F(T_i) = \{(0, 0)\}$. Obviously, $\bigcap_{i=1}^{3} F(t_i) \cap \bigcap_{i=1}^{3} F(T_i) \cap \bigcap_{i=1}^{3} VI(D, B_i) = \{(0, 0)\}$. For every $n \in \mathbb{N}$, $\alpha_n^{(1)} = \frac{1}{12n}$, $\alpha_n^{(2)} = \frac{12n-1}{36n}$, $\alpha_n^{(3)} = \frac{12n-1}{18n}$, $\gamma_n^{(0)} = \frac{1}{5} + \frac{7}{80n}$, $\gamma_n^{(1)} = \frac{16n-1}{80n}$, $\gamma_n^{(2)} = \gamma_n^{(3)} = \frac{24n-3}{80n}$. Then the sequences $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$, $\{\gamma_n^{(0)}\}$, $\{\gamma_n^{(1)}\}$, $\{\gamma_n^{(2)}\}$ and $\{\gamma_n^{(3)}\}$ satisfy all the conditions of Theorem 3.2. Now, we get the following algorithm:

$$\begin{aligned}
&\mathbf{x}_1 \in D,\\
&\mathbf{y}_n = \frac{1}{12n}\mathbf{x}_n + \frac{12n-1}{36n}W\mathbf{x}_n + \frac{12n-1}{18n}S\mathbf{x}_n,\\
&\mathbf{z}_n = \left(\frac{1}{5} + \frac{7}{80n}\right)\mathbf{y}_n + \frac{16n-1}{80n}\mathbf{q}_n^{(1)} + \frac{24n-3}{80n}\mathbf{q}_n^{(2)} + \frac{24n-3}{80n}\mathbf{q}_n^{(3)},\\
&\quad \mathbf{q}_n^{(i)} \in T_i\mathbf{y}_n, i = 1,2,3,\\
&C_{n+1} = \bigl\{\mathbf{p} = \bigl(p^{(1)},p^{(2)}\bigr) \in C_n : 2\bigl(z_n^{(1)} - y_n^{(1)}\bigr)p^{(1)} + 2\bigl(z_n^{(2)} - y_n^{(2)}\bigr)p^{(2)} + \bigl(y_n^{(1)}\bigr)^2\\
&\qquad + \bigl(y_n^{(2)}\bigr)^2 - \bigl(z_n^{(1)}\bigr)^2 - \bigl(z_n^{(2)}\bigr)^2 \le 0, 2\bigl(z_n^{(1)} - x_n^{(1)}\bigr)p^{(1)} + 2\bigl(z_n^{(2)} - x_n^{(2)}\bigr)p^{(2)}\\
&\qquad + \bigl(x_n^{(1)}\bigr)^2 + \bigl(x_n^{(2)}\bigr)^2 - \bigl(z_n^{(1)}\bigr)^2 - \bigl(z_n^{(2)}\bigr)^2 \le 0, 2\bigl(y_n^{(1)} - x_n^{(1)}\bigr)p^{(1)}\\
&\qquad + 2\bigl(y_n^{(2)} - x_n^{(2)}\bigr)p^{(2)} + \bigl(x_n^{(1)}\bigr)^2 + \bigl(x_n^{(2)}\bigr)^2 - \bigl(y_n^{(1)}\bigr)^2 - \bigl(y_n^{(2)}\bigr)^2 \le 0\bigr\},\\
&\mathbf{x}_{n+1} = P_{C_{n+1}}\mathbf{x}_1, \quad n \in \mathbb{N}.
\end{aligned} \tag{5.3}$$

Since W is the W-mapping generated by R_1, R_2, R_3 and β_1, β_2, β_3, where $R_i\mathbf{x} = P_D(I - \frac{1}{2}(I - t_i))\mathbf{x}$ for all $i = 1,2,3$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{2}$, and S is the S-mapping generated by G_1, G_2, G_3 and δ_1, δ_2, δ_3, where $G_i\mathbf{x} = P_D(I - \frac{1}{2}B_i)\mathbf{x}$ for all $i = 1,2,3$ and $\delta_1 = \delta_2 = \delta_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we obtain that

$$W\mathbf{x} = \left(\frac{167}{315}x^{(1)}, \frac{2{,}699}{5{,}236}x^{(2)}\right), \qquad S\mathbf{x} = \left(\frac{315{,}493}{324{,}000}x^{(1)}, \frac{315{,}493}{324{,}000}x^{(2)}\right) \quad \text{for all } \mathbf{x} \in D.$$

Put $\mathbf{q}_n^{(1)} = \frac{\mathbf{y}_n}{12}$, $\mathbf{q}_n^{(2)} = \frac{\mathbf{y}_n}{5}$, and $\mathbf{q}_n^{(3)} = \frac{\mathbf{y}_n}{2}$. Then algorithm (5.3) becomes

$$\begin{aligned}
&\mathbf{x}_1 \in D,\\
&\mathbf{y}_n = \left(\left(\frac{2{,}809{,}651}{3{,}402{,}000} + \frac{592{,}349}{40{,}824{,}000n}\right)x_n^{(1)},\right.\\
&\qquad \left.\left(\frac{522{,}289{,}837}{636{,}174{,}000} + \frac{113{,}884{,}163}{7{,}634{,}088{,}000n}\right)x_n^{(2)}\right),\\
&\mathbf{z}_n = \left(\left(\frac{32}{75} + \frac{289}{4{,}800n}\right)y_n^{(1)}, \left(\frac{32}{75} + \frac{289}{4{,}800n}\right)y_n^{(2)}\right),\\
&C_{n+1} = \bigl\{\mathbf{p} = \bigl(p^{(1)},p^{(2)}\bigr) \in C_n : 2\bigl(z_n^{(1)} - y_n^{(1)}\bigr)p^{(1)} + 2\bigl(z_n^{(2)} - y_n^{(2)}\bigr)p^{(2)} + \bigl(y_n^{(1)}\bigr)^2\\
&\qquad + \bigl(y_n^{(2)}\bigr)^2 - \bigl(z_n^{(1)}\bigr)^2 - \bigl(z_n^{(2)}\bigr)^2 \le 0, 2\bigl(z_n^{(1)} - x_n^{(1)}\bigr)p^{(1)} + 2\bigl(z_n^{(2)} - x_n^{(2)}\bigr)p^{(2)}\\
&\qquad + \bigl(x_n^{(1)}\bigr)^2 + \bigl(x_n^{(2)}\bigr)^2 - \bigl(z_n^{(1)}\bigr)^2 - \bigl(z_n^{(2)}\bigr)^2 \le 0, 2\bigl(y_n^{(1)} - x_n^{(1)}\bigr)p^{(1)}\\
&\qquad + 2\bigl(y_n^{(2)} - x_n^{(2)}\bigr)p^{(2)} + \bigl(x_n^{(1)}\bigr)^2 + \bigl(x_n^{(2)}\bigr)^2 - \bigl(y_n^{(1)}\bigr)^2 - \bigl(y_n^{(2)}\bigr)^2 \le 0\bigr\},\\
&\mathbf{x}_{n+1} = P_{C_{n+1}}\mathbf{x}_1, \quad n \in \mathbb{N}.
\end{aligned} \tag{5.4}$$

For any arbitrary $(x_n^{(1)}, x_n^{(2)}) \in D$, by algorithm (5.4), we see that

$$0 \le z_n^{(i)} \le y_n^{(i)} \le x_n^{(i)} \le x_1^{(i)} \le 100 \quad \text{for all } i = 1,2, \text{ and } n \in \mathbb{N}.$$

The numerical results for the sequences $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$, and $\{\mathbf{z}_n\}$ are shown in Table 2 and Table 3.

Table 2 The values of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ with the initial point $x_1 = (7.5, 9.1)$ in Example 5.2

n	x_n	y_n	z_n
1	(7.5000000,9.1000000)	(6.3029396,7.6067234)	(3.0687437,3.7035235)
2	(4.6619648,5.6749079)	(3.8840555,4.7013476)	(1.7741233,2.1474385)
3	(2.8183693,3.4332496)	(2.3412722,2.8357215)	(1.0459308,1.2668192)
4	(1.6857031,2.0577916)	(1.3983068,1.6970922)	(0.6176583,0.7496374)
5	(1.0092457,1.2223240)	(0.8364468,1.0071576)	(0.3669562,0.4418484)
6	(0.5676300,0.7527995)	(0.4701681,0.6199094)	(0.2053231,0.2707153)
7	(0.7174567,0.1573218)	(0.5940218,0.1294942)	(0.2585586,0.0563647)
8	(0.3517952,0.4346565)	(0.2911794,0.3576574)	(0.1264280,0.1552922)
9	(0.4298726,0.0764960)	(0.3557172,0.0629290)	(0.1541523,0.0272707)
⋮	⋮	⋮	⋮
45	(0.0000963,0.0000292)	(0.0000795,0.0000240)	(0.0000340,0.0000103)
⋮	⋮	⋮	⋮
55	(0.0000088,0.0000042)	(0.0000072,0.0000034)	(0.0000031,0.0000015)
56	(0.0000033,0.0000065)	(0.0000027,0.0000053)	(0.0000012,0.0000023)
57	(0.0000054,0.0000020)	(0.0000044,0.0000017)	(0.0000019,0.0000007)
58	(0.0000022,0.0000036)	(0.0000018,0.0000030)	(0.0000008,0.0000013)
59	(0.0000033,0.0000009)	(0.0000030,0.0000008)	(0.0000012,0.0000003)
60	(0.0000015,0.0000020)	(0.0000012,0.0000017)	(0.0000005,0.0000007)
61	(0.0000020,0.0000004)	(0.0000016,0.0000003)	(0.0000007,0.0000001)
62	(0.0000010,0.0000011)	(0.0000008,0.0000009)	(0.0000004,0.0000004)

Table 3 The values of the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ with the initial point $x_1 = (0, 75.6)$ in Example 5.2

n	x_n	y_n	z_n
1	(0.0000000,75.6000000)	(0.0000000,63.1943178)	(0.0000000,30.7677335)
2	(0.0000000,46.9810256)	(0.0000000,38.9211840)	(0.0000000,17.7780617)
3	(0.0000000,28.3496228)	(0.0000000,23.4156103)	(0.0000000,10.4605987)
4	(0.0000000,16.9381045)	(0.0000000,13.9691137)	(0.0000000,6.1704195)
5	(0.0000000,10.0697666)	(0.0000000,8.2971795)	(0.0000000,3.6400418)
6	(0.0000000,5.9686106)	(0.0000000,4.9149845)	(0.0000000,2.1463806)
7	(0.0000000,3.5306826)	(0.0000000,2.9061647)	(0.0000000,1.2649601)
8	(0.0000000,2.0855624)	(0.0000000,1.7161061)	(0.0000000,0.7451208)
9	(0.0000000,1.2306135)	(0.0000000,1.0123560)	(0.0000000,0.4387110)
⋮	⋮	⋮	⋮
15	(0.0000000,0.0512698)	(0.0000000,0.0421428)	(0.0000000,0.0181501)
⋮	⋮	⋮	⋮
30	(0.0000000,0.0000175)	(0.0000000,0.0000144)	(0.0000000,0.0000062)
31	(0.0000000,0.0000103)	(0.0000000,0.0000084)	(0.0000000,0.0000036)
32	(0.0000000,0.0000060)	(0.0000000,0.0000049)	(0.0000000,0.0000021)
33	(0.0000000,0.0000035)	(0.0000000,0.0000029)	(0.0000000,0.0000012)
34	(0.0000000,0.0000021)	(0.0000000,0.0000017)	(0.0000000,0.0000007)
35	(0.0000000,0.0000012)	(0.0000000,0.0000010)	(0.0000000,0.0000004)
36	(0.0000000,0.0000007)	(0.0000000,0.0000006)	(0.0000000,0.0000003)
37	(0.0000000,0.0000004)	(0.0000000,0.0000003)	(0.0000000,0.0000001)

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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